CAMBRIDGE

Brighter Thinking

A Level Mathematics for OCR A

Student Book 1 (AS/Year 1)

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Acknowledgements

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Introduction

You have probably been told that mathematics is very useful, yet it can often seem like a lot of techniques that just have to be learnt to answer examination questions. You are now getting to the point where you will start to see how some of these techniques can be applied in solving real problems. However, as well as seeing how maths can be useful we hope that anyone working through this resource will realise that it can also be incredibly frustrating, surprising and ultimately beautiful.

The resource is woven around three key themes from the new curriculum.

Proof

Maths is valued because it trains you to think logically and communicate precisely. At a high level, maths is far less concerned about answers and more about the clear communication of ideas. It is not about being neat – although that might help! It is about creating a coherent argument that other people can easily follow but find difficult to refute. Have you ever tried looking at your own work? If you cannot follow it yourself it is unlikely anybody else will be able to understand it. In maths we communicate using a variety of means – feel free to use combinations of diagrams, words and algebra to aid your argument. And once you have attempted a proof, try presenting it to your peers. Look critically (but positively) at some other people's attempts. It is only through having your own attempts evaluated and trying to find flaws in other proofs that you will develop sophisticated mathematical thinking. This is why we have included lots of common errors in our 'work it out' boxes – just in case your friends don't make any mistakes!

Problem solving

Maths is valued because it trains you to look at situations in unusual, creative ways, to persevere and to evaluate solutions along the way. We have been heavily influenced by a great mathematician and maths educator George Polya, who believed that students were not just born with problem solving skills – they were developed by seeing problems being solved and reflecting on their solutions before trying similar problems. You may not realise it but good mathematicians spend most of their time being stuck. You need to spend some time on problems you can't do, trying out different possibilities. If after a while you have not cracked it then look at the solution and try a similar problem. Don't be disheartened if you cannot get it immediately – in fact, the longer you spend puzzling over a problem the more you will learn from the solution. You may never need to integrate a rational function in future, but we firmly believe that the problem-solving skills you will develop by trying it can be applied to many other situations.

Modelling

Maths is valued because it helps us solve real-world problems. However maths describes ideal situations and the real world is messy! Modelling is about deciding on the important features needed to describe the essence of a situation and turning that into a mathematical form, then using it to make predictions, compare to reality and possibly improve the model. In many situations the technical maths is actually the easy part – especially with modern technology. Deciding which features of reality to include or ignore and anticipating the consequences of these decisions is the hard part. Yet it is amazing how some fairly drastic assumptions – such as pretending a car is a single point or that people's votes are independent – can result in models which are surprisingly accurate.

More than anything else, this resource is about making links. Links between the different chapters, the topics covered and the themes above, links to other subjects and links to the real world. We hope that you will grow to see maths as one great complex but beautiful web of interlinking ideas.

Maths is about so much more than examinations, but we hope that if you take on board these ideas (and do plenty of practice!) you will find maths examinations a much more approachable and possibly even enjoyable experience. However, always remember that the result of what you write down in a few hours by yourself in silence under exam conditions is not the only measure you should consider when judging your mathematical ability – it is only one variable in a much more complicated mathematical model!

How to use this resource

Throughout this resource you will notice particular features that are designed to aid your learning. This section provides a brief overview of these features.

In this chapter you will learn how to:

- use terms such as identity and equation to describe mathematical objects
- disprove a mathematical idea using a counter example
- use deduction and exhaustion to prove a mathematical idea.

Learning objectives

A short summary of the content that you will learn in each chapter.

Before you start...

GCSE	You should know how to use the definition of the square root function.	1 What is $\sqrt{9}$?
GCSE	You should know how to manipulate	2 Factorise $4x^2 - 1$.

Before you start

Points you should know from your previous learning and questions to check that you're ready to start the chapter.

WORKED EXAMPLE

The left-hand side shows you how to set out your working. The right-hand side explains the more difficult steps and helps you understand why a particular method was chosen.

PROOF

Step-by-step walkthroughs of standard proofs and methods of proof.

WORK IT OUT

Can you identify the correct solution and find the mistakes in the two incorrect solutions?

Key point

A summary of the most important methods, facts and formulae.

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Explore

Ideas for activities and investigations to extend your understanding of the topic.

Tip

Useful guidance, including on ways of calculating or checking and use of technology.

Each chapter ends with a **Checklist of learning and understanding** and a **Mixed practice exercise**, which includes past paper questions marked with the icon .

In between chapters, you will find extra sections that bring together topics in a more synoptic way.

Focus on ...

Unique sections relating to the preceding chapters that develop your skills in proof, problem solving and modelling.

CROSS-TOPIC REVIEW EXERCISE

Questions covering topics from across the preceding chapters, testing your ability to apply what you have learnt.

You will find **Paper 1 and Paper 2 practice questions** towards the end of the resource, as well as a **glossary** of key terms (picked out in colour within the chapters), **answers** and full **worked solutions**.

Maths is all about making links, which is why throughout this resource you will find signposts emphasising connections between different topics, applications and suggestions for further research.

Rewind

Reminders of where to find useful information from earlier in your study.

Fast forward

Links to topics that you may cover in greater detail later in your study.

Focus on ...

Links to problem solving, modelling or proof exercises that relate to the topic currently being studied.

Did you know?

Interesting or historical information and links with other subjects to improve your awareness about how mathematics contributes to society.

Worksheet

A support sheet for each chapter contains further worked examples and exercises on the most common question types. Extension sheets provide further challenge for the more ambitious.

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Gateway to A Level

GCSE transition material that provides a summary of facts and methods you need to know before you start a new topic, with worked examples and practice questions.

Colour-coding of exercises

The questions in the exercises are designed to provide careful progression, ranging from basic fluency to practice questions. They are uniquely colour-coded, as shown here.

1 By factorising, solve the following equations:

a i $3x^2 + 2x = x^2 + 3x + 6$

ii $2x^2 + 3 = 17x - 7 - x^2$

b i $9x^2 = 24x - 16$

ii $18x^2 = 2x^2 - 40x - 25$

c i $(x - 3)(x + 2) = 14$

ii $(2x + 3)(x - 1) = 12$

9 Do two lines that never meet have to be parallel?

3 Use an appropriate substitution to solve $x^2 + \frac{9}{x^2} = 10$.

8 The gradient of the graph of $y = e^{ax}$ at the point where $y = 4.6$ equals -1.2. Find the value of x at the point where the gradient is -5.

9 **a** Find the value of k so that $8^x = e^{kx}$.

b Hence find the gradient of the graph of $y = 8^x$ at the point where $x = -0.5$.

Black – drill questions. These come in several parts, each with subparts **i** and **ii**. You only need attempt subpart **i** at first; subpart **ii** is essentially the same question, which you can use for further practice if you got part **i** wrong, for homework, or when you revisit the exercise during revision.

Green – practice questions at a basic level.

Yellow – designed to encourage reflection and discussion.

Blue – practice questions at an intermediate level.

Red – practice questions at an advanced level.

A – indicates content that is for A Level students only

AS – indicates content that is for AS Level students only

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Working with the large data set

As part of your course you are expected to work with the large data set covering different methods of transport and age distributions in different parts of the country and in different years. This large data set is an opportunity to explore statistics in real life. As well as supporting the ideas introduced in Chapters 16 and 18 we shall be using the large data set to guide you through four key themes. All of these themes will be explored with examples and questions in the large data set section in the Cambridge Elevate edition. You will not have to work with the full data set in the final examination, but familiarity with it will help you as many examination questions will be set in the context of this data set.

Practical difficulties with data

Unlike most textbook or examination problems, the real world is messy. Often there are difficulties with being overwhelmed by too much data, or perhaps there are errors, missing items or labels which are ambiguous. For example, how do you deal with the fact that in 2001 Cornwall was made up of separate districts that were later combined into a single unitary authority, if you want to compare areas over time? If you are grouping data for a histogram, how big a difference does it make where you choose to put the class boundaries?

Using technology

Modern statistics is heavily based on familiarity with technology. We will be encouraging you to use spreadsheets and graphing packages, looking at the common tools available to help simplify calculations and present data effectively.

One important technique we can employ with modern technology is simulation. We will try to gain a better understanding of hypothesis testing by using the data set to simulate the effect of sampling on making inferences about the population.

Thinking critically about statistics

Why might someone want to use a pie chart rather than a histogram? Whenever statistics are calculated or data sets are represented graphically, some information is lost and some information is highlighted. An important part of modern statistics is to ask critical questions about the way evidence provided by statistics is used to support arguments.

One important part of this is the idea of validating statistics. For example, with the information presented it is not clear which category or categories a person would be included in if they travel to work by bicycle on some days and take the bus on others. We will look at ways in which we can interrogate the data to try to understand it more.

Statistical problem solving

Technology can often do calculations for us. However the art of modern statistics is deciding what calculations to do on what data. One of the big difficulties is that we rarely have exactly the data we want, so we have to make indirect inferences from the data we have. For example, you will probably not see newspaper headlines saying ‘the correlation coefficient between median age and percentage of people cycling to work is 0.64’, but you might see something saying ‘Pensioners promote pedalling!’ Deciding on an appropriate statistical technique to determine whether older people are more likely to use a bicycle and then interpreting results is the type of thing which is hard to examine but very valuable in real-world statistics.

There are lots of decisions to be made. Should you use the total number of cyclists in an area? Or the percentage of people who cycle? Or the percentage of people who travel to work who cycle? We shall see how the answer to your main question depends on decisions like these.

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1 Proof and mathematical communication



In this chapter you will learn how to:

- use terms such as identity and equation to describe mathematical objects
- disprove a mathematical idea using a counter example
- use deduction and exhaustion to prove a mathematical idea.

Before you start...

GCSE	You should know how to use the definition of the square root function.	1 What is $\sqrt{9}$?
GCSE	You should know how to manipulate algebraic expressions.	2 Factorise $4x^2 - 1$.
GCSE	You should know basic angle facts.	3 a. What is the sum of the angles in a triangle? b. What is the sum of the exterior angles of any polygon?
GCSE	You should be able to define rational and irrational numbers.	4 Which of these numbers are irrational? π, $0.\dot{3}$, 0.5, $\sqrt{2}$
GCSE	You should be able to work with function notation.	5 If $\mathrm{f}(x) = 2x^2 - 3$ find $\mathrm{f}(3)$.

Why is proof important?

One thing for which mathematicians are valued is their ability to communicate their ideas precisely and to make very convincing arguments, called proofs. In this chapter we will look at the language used by mathematicians and some of the ways they prove their ideas.

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Section 1: Mathematical structures and arguments

We can represent mathematical ideas in many different ways such as tables, diagrams, graphs or words. One of the most fundamental representations is an **equation**: a mathematical statement involving an '=' sign. For example:

$$x^2 - 1 = 8$$

An equation is only true for some values of x (or perhaps none); in this case it is true for $x = \pm 3$.

Another similar mathematical structure is called an **identity**. An identity is a relation which is true for **all** values of the unknown. It is given the $\equiv$ symbol.

Did you know?

The first recorded use of the equals sign occurs in Robert Recorde's 1557 book *The Whetstone of Witte*. He explains that he used two parallel lines 'because no two things can be more equal'.

For example, $x^2 - 1 \equiv (x - 1)(x + 1)$.

Two statements connected by the identity symbol are called **congruent expressions**. They are equal for **all** values of the variable(s).

There are some rules which only apply to identities. For example, if two polynomials are identically equal then their coefficients must be the same.

Tip

Whenever you are simplifying an expression, technically you should use an identity symbol $(\equiv)$. However, it is common in mathematics to use an equals sign $(=)$ instead.

Tip

A **polynomial** is a function that is a sum of terms containing non-negative (positive or zero) integer powers of x.

A **coefficient** is the constant in front of (multiplying) a variable. For example, in the quadratic $2x^2 - 3x$, 2 is the coefficient of x^2 and -3 is the coefficient of x.

The word **integer** just means 'whole number'.

Fast forward

In Chapter 3 you will see that you often have to write quadratics in the form shown here.

WORKED EXAMPLE 1.1

$2x^2 + 12x - 3 \equiv a(x + p)^2 + q$

Find the value of a, p and q.

$$\begin{aligned} 2x^2 + 12x - 3 &\equiv a(x + p)^2 + q \\ &\equiv a(x^2 + 2px + p^2) + q \\ &\equiv ax^2 + 2apx + ap^2 + q \end{aligned}$$

Multiply out the brackets to allow coefficients to be compared.

Coefficient of x^2: $2 = a$

Coefficient of x: $12 = 2ap$

Constant term: $-3 = ap^2 + q$

Compare coefficients. The two expressions are equal for all values of x, so all the coefficients must be equal.

$4p = 12$

$p = 3$

Substitute $a = 2$ into the second equation.

$2 \times 3^2 + q = -3$

$q = -21$

Substitute $a = 2$, $p = 3$ into the third equation.

Another common structure in mathematics is called a **function**. This is just a rule for changing an input into an output.

For example, if you have a function called f that transforms the number x into $\sqrt{x^2 + 9}$ you would write that as $\mathrm{f}(x) = \sqrt{x^2 + 9}$.

So $\mathrm{f}(4) = \sqrt{4^2 + 9} = \sqrt{25} = 5$ and $\mathrm{f}(y) = \sqrt{y^2 + 9}$.

Gateway to A Level

See Gateway to A Level section A for a reminder of expanding brackets.

Implication and equivalence

You can manipulate both equations and inequalities by doing the same thing to both sides. You often structure your solution by writing lines of working underneath each other. In more formal work, you can emphasise the logic of the argument by using special symbols.

Key point 1.1

- The symbol $\Rightarrow$ means that a subsequent equation follows from the previous one.
  $P \Rightarrow Q$ means 'P implies Q' or 'if P is true then Q is true' or 'P is sufficient for Q'.
- The symbol $\Leftarrow$ means that the previous statement follows from the subsequent statement.
  $P \Leftarrow Q$ means 'P is implied by Q' or 'if Q is true then P is true' or 'P is necessary for Q'
- The symbol $\Leftrightarrow$ means that a subsequent equation is equivalent to the previous one.
  $P \Leftrightarrow Q$ means 'P is equivalent to Q' or 'Q is true if and only if P is true'.
  This can also be written in the shorthand 'Q iff P'.

You will also sometimes see the symbol $\therefore$ for 'therefore,' which means we are drawing a conclusion from previous lines of working.

WORKED EXAMPLE 1.2

Insert either a $\Rightarrow$ or a $\Leftrightarrow$ symbol on each line of working:

a
$2x + 1 = 9$

$2x = 8$

$x = 4$

b
$x = 4$

$x^2 = 16$

a
$2x + 1 = 9$

$\Leftrightarrow 2x = 8$

These statements are equivalent: the logic flows both ways.

$\Leftrightarrow x = 4$

Again $2x = 8$ and $x = 4$ are equivalent.

b $x = 4$

$\Rightarrow x^2 = 16$

$x = 4$ implies that $x^2 = 16$ but the reverse is not true as $x^2 = 16$ implies $x = \pm 4$ (not just $x = 4$).

When solving equations, it is important to know whether or not each line of working is equivalent to the previous one. If it is, then you can be sure that you have found the complete set of solutions. For example:

$$x^2 - x = 2$$

$$\Leftrightarrow x^2 - x - 2 = 0$$

$$\Leftrightarrow (x - 2)(x + 1) = 0$$

$$\Leftrightarrow x = 2 \text{ or } -1$$

From this chain of equivalences you can be certain that both 2 and -1 are solutions of the equation, and that there are no others.

However, if some of the lines are only connected by implications, it is possible to find 'solutions' which don't actually work, or to miss some of the solutions.

Gateway to A Level

See Gateway to A Level sections B and C for a reminder of solving quadratic equations by factorising.

WORKED EXAMPLE 1.3

A student is attempting to solve the equation $\sqrt{x + 6} = x$.

a Find the error with the following:

$\sqrt{x + 6} = x$

$x + 6 = x^2$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$x = 3$ or $x = -2$

b Solve the equation correctly.

a $\sqrt{x + 6} = x$

$\Rightarrow x + 6 = x^2$

Look at each line in turn to see whether a $\Leftrightarrow$ symbol is valid.

They are not equivalent since:

$x + 6 = x^2$

$\Rightarrow \pm\sqrt{x + 6} = x$

The first line implies the second, but the second does not imply the first, so they are not equivalent.

This leads to one incorrect solution,

coming from $-\sqrt{x+6} = x$.

All subsequent lines are equivalent, so one of the solutions is correct.

b Check both "solutions":

$x = 3$: $\sqrt{(3+6)} = 3$, so this is a solution.

$x = -2$: $\sqrt{(-2+6)} = 2 \neq -2$, so this is not a solution.

$\therefore$ The correct solution is $x = 3$.

You need to check both solutions: substitute the x values into the LHS and the RHS and check whether they are equal.

Tip

Squaring an equation is a common way of introducing incorrect solutions since it prevents lines of working being equivalent.

Tip

LHS and RHS are standard abbreviations for the left-hand side and right-hand side of an equation.

In practice it is often easier not to worry about whether every line is equivalent, but to be aware that the 'solutions' you get need to be checked by substituting them back into the original equation. Any that are not correct can then just be deleted.

Dividing by zero can remove solutions in the same way that squaring can introduce them.

Fast forward

You will also see this problem arise when you solve equations involving logarithms in Chapter 7.

WORKED EXAMPLE 1.4

Insert an appropriate $\Rightarrow$, $\Leftarrow$ or a $\Leftrightarrow$ symbol in the space marked $\square$. Hence explain why the solution is incomplete.

$$x^2 = 6x$$

Dividing by x:

$$\square \; x = 6$$

$\square$ is $a \Leftarrow$

If $x = 6$ then $x^2 = 6x$, but the reverse is not always true...

So $x = 6$ may not be the only solution – in this case there is also the possibility that $x = 0$.

EXERCISE 1A

1 If $\mathrm{f}(x) = x^2 + 1$, find and simplify where possible:

a i $\mathrm{f}(2)$

ii $\mathrm{f}(3)$

b i $2\mathrm{f}(1) + \mathrm{f}(4)$

ii $\mathrm{f}(5) - \mathrm{f}(3)$

c i $\mathrm{f}(x)$

ii $\mathrm{f}(y)$

d i $\mathrm{f}(2x)$

ii $\mathrm{f}(3y)$

e i $\mathrm{f}(x + 1)$

ii $\mathrm{f}(x - 2)$

2 Where appropriate insert a $\Rightarrow$, $\Leftarrow$ or a $\Leftrightarrow$ symbol in the space marked $\square$.

a i Shape P is a rectangle. $\square$ Shape P is a square.

ii Shape Q is a quadrilateral. $\square$ Shape Q is a rhombus.

b i n is even. $\square$ n is a whole number.

ii n is a prime number. $\square$ n is a whole number.

c i A triangle has two equal sides. $\square$ A triangle has two equal angles.

ii Two circles have the same area $\square$ Two circles have the same radius.

d i $x^2 - 2x - 3 = 0$ $\square$ $x = 3$

ii $x^2 - 2x + 1 = 0$ $\square$ $x = 1$

e i Sam can do 10 press-ups. $\square$ Sam can do 100 press-ups.

ii Niamh is over 21. $\square$ Niamh is over 18.

f i Neither A nor B is true. $\square$ A is false and B is false.

ii A and B are not both true. $\square$ A and B are both not true.

g i Chris is a boy. $\square$ Chris is a footballer.

ii Shape X is a right-angled triangle. $\square$ Shape X is an isosceles triangle.

Fast forward

A You will learn more about functions in Student Book 2, Chapter 5.

3 $x^3 - 4x^2 - 3x + 18 = (x + a)(x - b)^2$ for all x. Find the values of a and b.

4 $x^4 + 8x^3 + 2x + 16 = (x^3 + a)(x + b)$ for all x. Find the values of a and b.

Did you know?

In all of these examples, we are assuming that each statement is either true or false. The study of this type of logic is called Boolean algebra.

5 What is the flaw in the following working?

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Question: For $x = 4$, find the value of $2x + 2$

Working: $2 \times 4 = 8 + 2 = 10$

6 Where is the flaw in the following argument?

Suppose	$1 = 3$
Subtract 2:	$-1 = 1$
Square:	$1 = 1$

Therefore the first line is true.

7 Consider the equation $\sqrt{x^2 + 9} = 3x - 7$.

a Add appropriate symbols ($\Leftrightarrow$ or $\Rightarrow$) to each line of working in the solution shown.

$\sqrt{x^2 + 9} = 3x - 7$

Square: $x^2 + 9 = 9x^2 - 42x + 49$

Subtract $x^2 + 9$: $0 = 8x^2 - 42x + 40$

Divide by 2: $0 = 4x^2 - 21x + 20$

Factorise: $0 = (4x - 5)(x - 4)$

$x = \frac{5}{4}$ or $x = 4$

b Hence explain the flaw in the solution shown.

8 **a** Insert appropriate symbols ($\Rightarrow, \Leftarrow$ or $\Leftrightarrow$) in the spaces marked $\square$.

$\frac{1}{x^2} = \frac{2}{x}$

$\square$ $x = 2x^2$

$\square$ $0 = 2x^2 - x$

$\square$ $0 = x(2x - 1)$

$\square$ $x = 0$ or $x = \frac{1}{2}$

b Hence explain the error in the working.

9 **a** Insert appropriate symbols ($\Rightarrow, \Leftarrow$ or $\Leftarrow$) in the spaces marked $\square$.

$x^2 + 3x = 4x + 12$

$\square$ $x(x + 3) = 4(x + 3)$

$\square$ $x = 4$

b Hence explain the error in the working.

10 Do you agree with the following statement?

Either A or B is true $\Leftrightarrow$ A and B are not both true.

11 Where is the flaw in the following argument?

Suppose two numbers a and b are equal.

$a = b$

Multiply by a: $a^2 = ab$

Subtract b^2:	$a^2 - b^2 = ab - b^2$
Factorise:	$(a - b)(a + b) = b(a - b)$
Cancel $(a - b)$:	$(a + b) = b$
Use the fact that $a = b$:	$2b = b$
Divide by b:	$2 = 1$

Did you know?

The word ‘or’ can be ambiguous. In formal logical we use the terms OR and XOR to have two different meanings.

Section 2: Inequality notation

You know from your previous study that solving a linear inequality is just like solving an equation, as long as you don't multiply or divide by a negative number. Your answer is written as an inequality.

For example:

$$2x - 5 \geqslant 9$$

$$\Leftrightarrow 2x \geqslant 14$$

$$\Leftrightarrow x \geqslant 7$$

This solution can be written in **set notation**: $\{x : x \geqslant 7\}$.

This is read as 'x such that x is greater than or equal to 7'.

It can also be written in **interval notation**: $[7, \infty)$. This means that the solution lies in the interval from 7 (included) to infinity (not included).

It is assumed that x is a real number ($x \in \mathbb{R}$) unless stated otherwise. So the interval $[7, \infty)$ includes all the real numbers greater than or equal to 7, and the interval $(-\infty, 7)$ all the real numbers smaller than 7. Note that infinity (∞) is not a real number, so can never be included in the interval.

Gateway to A Level

For a reminder and more practice of solving linear inequalities, see Gateway to A Level section D.

Tip

The $\in$ symbol in set notation means 'is in the set...' or 'belongs to the set...'

Key point 1.2

- $x \in (a, b)$ means $a < x < b$
- $x \in [a, b]$ means $a \leqslant x \leqslant b$
- $x \in [a, b)$ means $a \leqslant x < b$
- $x \in (a, b]$ means $a < x \leqslant b$

Two different intervals can be combined using notation taken from set theory:

Key point 1.3

$x \in A \cup B$ means that x can be in either A or B (or both). $A \cup B$ is called the **union** of A and B.

$x \in A \cap B$ means that x is in both A and B. $A \cap B$ is called the **intersection** of A and B.

If there are no solutions to the inequality we can write $x \in \emptyset$, where $\emptyset$ is the symbol for the empty set.

WORKED EXAMPLE 1.5

Write the following in set notation:

a $1 \leqslant x < 7$

b $x > 1$ or $x < -2$

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c $x < 2$ and $x > 4$

a $\{x : x \geqslant 1\} \cap \{x : x < 7\}$

x is **both** greater than or equal to 1, **and** less than 7 so you need the intersection.

b $\{x : x > 1\} \cup \{x : x < -2\}$

x is greater than 1 **or** less than -2 so you need the union.

c $x \in \emptyset$.

No values of x are both smaller than 2 and greater than 4.

EXERCISE 1B

1 Write the following inequalities in set notation and interval notation.

a i $x > 7$

ii $x < 6$

b i $x \leqslant 10$

ii $x \geqslant 5$

c i $0 < x \leqslant 1$

ii $5 < x < 7$

d i $x > 5$ or $x \leqslant 0$

ii $x \geqslant 10$ or $x < 2$

2 Write the following statements as inequalities in x.

a i $x \in [1, 4)$

ii $x \in (2, 8]$

b i $x \in [1, 3]$

ii $x \in (2, 4)$

c i $x \in (-\infty, 5)$

ii $x \in [12, \infty)$

d i $\{x : 0 < x < 10\} \cap \{x : x \geqslant 8\}$

ii $\{x : 1 < x < 4\} \cap \{x : x \geqslant 3\}$

3 Represent the following intervals on a number line.

a i $(1, 5]$

ii $[-2, 1]$

b i $(-\infty, 4) \cap (-2, \infty)$

ii $(-\infty, 0) \cap [-5, \infty)$

c i $(-\infty, 3] \cap (0, 3)$

ii $[-2, 5] \cap (2, 5)$

d i $[2, 5) \cup (7, \infty)$

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ii $(-\infty, -3] \cup (0, 5]$

4 Write the following statements using intervals combined using set notation. Rewrite each as a single interval if possible.

a i $1 \leqslant x \leqslant 3$ and $x > 2$

b ii $x \leqslant -1$ and $1 < x < 3$

c i $x \leqslant -2$ or $x \geqslant 2$

d ii $1 \leqslant x < 5$ or $x \geqslant 7$

5 Solve the following inequalities and express your solution using interval notation.

a i $\frac{x}{2} + 1 \leqslant \frac{3x}{5}$

ii $2x - 1 > \frac{4x}{3} + 3$

b i $-5 \leqslant \frac{x}{3} - 1 < 3$

ii $4 < \frac{3x}{2} - 2 \leqslant 7$

c i $\frac{3x - 2}{5} > \frac{x + 1}{2}$

ii $\frac{x - 3}{7} \geqslant \frac{2x + 1}{3}$

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Section 3: Disproof by counter example

It is usually not possible to prove that something is always true by looking at examples. However, it is possible to use examples to prove that something is **not** always true. This is called a **counter example**.

WORKED EXAMPLE 1.6

Disprove by counter example that $(x+1)^2 = x^2 + 1$ for all x.

When $x = 2$:

LHS: $(2+1)^2 = 9$

RHS: $2^2 + 1 = 5$

So $x = 2$ is a counter example.

When searching for a counter example, try different types of numbers.

EXERCISE 1C

1. Disprove the statement $\sqrt{x^2+9} \equiv x + 3$.
2. Use a counter example to prove that $2x \nparallel 2\sin x$.
3. Use a counter example to prove that $\sqrt{x^2}$ is not always x.
4. Prove that the product of two prime numbers is not always odd.

Gateway to A Level

For a reminder of rational and irrational numbers, see Gateway to A Level section E.

5. Prove that the number of factors of a number is not always even.
6. Prove that the sum of two irrational numbers is not always irrational.
7. Use a counter example to disprove the following statement:

$$x < 3 \Rightarrow x^2 < 9$$

8. A student claims that $n^2 + n + 41$ takes prime values for all positive integers. Use a counter example to disprove this claim.
9. Do two lines that never meet have to be parallel?

Section 4: Proof by deduction

Proving a result is usually much harder than disproving it. You need to start with what is given in the question and form a series of logical steps to reach the required conclusion.

Algebra is a useful tool that allows you to express ideas in general terms. You will often need to use algebraic expressions for even and odd numbers. For example, it is common to express:

- an even number as $2n$, for some integer n
- an odd number as $2n + 1$, for some integer n.

Focus on ...

The Focus on ... sections in this book show you proofs of some important results you will meet in this course.

WORKED EXAMPLE 1.7

Prove that the product of an even and an odd number is always even.

Let the even number be $2n$, for some integer n.	Define a general even number.
Let the odd number be $2m + 1$, for some integer m.	And define a general odd number. Note that you mustn't use $2n + 1$ as that would be the next integer up from $2n$, which would be a specific odd number, not a general one.
$2n(2m + 1) = 2(2mn + n)$ $= 2k$ for some integer k.	Aim to write the product in the form $2k$ to show that it is even.
So this is even.	Make a conclusion.

WORKED EXAMPLE 1.8

Prove that the difference between the squares of consecutive odd numbers is always a multiple of 8.

Let the smaller odd number be $2n - 1$ Let the larger odd number be $2n + 1$	Define two consecutive odd numbers. This time you **do** want n in both, as the two numbers are related.
$(2n + 1)^2 - (2n - 1)^2 = (4n^2 + 4n + 1)$ $-(4n^2 - 4n + 1)$ $= 4n + 4n$ $= 8n$	Square each, and subtract the smaller from the larger.
So this is a multiple of 8.	Make a conclusion.

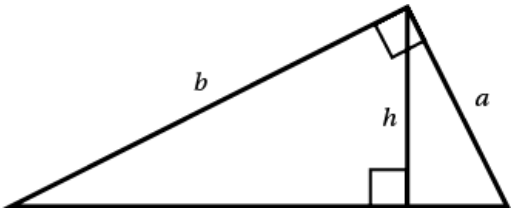
EXERCISE 1D

1. Prove that if n is odd then n^2 is also odd.
2. Prove that the sum of an even number and an odd number is odd.
3. Prove that the sum of any three consecutive integers is always a multiple of three.
4. Prove that:
   - **a** the sum of two consecutive multiples of 5 is always odd
   - **b** the product of two consecutive multiples of 5 is always even.

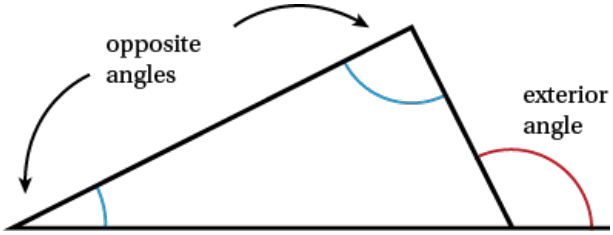
Worksheet

For a further example on algebraic proof and more practice questions, see Support sheet 1.

5. Prove that the height, h, in the following diagram is given by $h = \dfrac{ab}{\sqrt{a^2 + b^2}}$.



6. Prove that the sum of the interior angles of a hexagon is 720°.
7. Prove that if a number leaves a remainder 2 when it is divided by 3 then its square leaves a remainder 1 when divided by 3.
8. **a** Expand $(x + 2)^2$.
   - **b** Prove the statement: $y = x^2 + 4x + 10 \Rightarrow y > 0$.
9. Prove that an exterior angle in a triangle is the sum of the two opposite angles.



10. Prove that $n^2 + 3n + 2$ is never prime if n is a positive integer.
11. **a** Let n be a four-digit whole number '*abcd*'. Explain why $n = 1000a + 100b + 10c + d$.
    - **b** Prove that n is divisible by 9 if and only if $a + b + c + d$ is a multiple of 9.
    - **c** Prove that n is divisible by 11 if and only if $a - b + c - d$ is divisible by 11.
12. By considering $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$, prove that an irrational number raised to an irrational power can be rational.

Section 5: Proof by exhaustion

You should be aware that simply considering some examples does not constitute a mathematical proof. However, in some situations it is possible to check **all** possibilities and this can lead to a valid proof. This is called **proof by exhaustion**.

WORKED EXAMPLE 1.9

Prove that 89 is a prime number.

We only need to check prime numbers smaller than the square root of 89, since any factor above this would have to be paired with a factor below the square root.

89 is not divisible by 2, 3, 5 or 7.

Therefore it must be a prime number.

A prime number does not have any prime factors smaller than itself.

WORKED EXAMPLE 1.10

A whole number is squared and divided by 3. Prove that the remainder can only be 0 or 1.

You cannot check all whole numbers, but you can split them into three groups when considering division by 3: those that give no remainder, those that give remainder 1 and those that give remainder 2. You can then check what squaring does to numbers from each group.

Let n be a whole number.

Then n is either a multiple of $3 (n = 3k)$

or one more than a multiple of $3 (n = 3k + 1)$

or two more than a multiple of $3 (n = 3k + 2)$

Now use algebra to write each type of number.

If $n = 3k$ then $n^2 = (3k)^2 = 9k^2$

which is a multiple of 3.

If $n = 3k + 1$ then

$$n^2 = (3k + 1)^2 = 9k^2 + 6k + 1$$
$$= 3(3k^2 + 2k) + 1$$

which is one more than a multiple of 3.

If $n = 3k + 2$ then

$$n^2 = (3k + 2)^2 = 9k^2 + 12k + 4$$
$$= 3(3k^2 + 4k + 1) + 1$$

Now check what happens when you square each type of number.

which is one more than a multiple of 3.

So there is either no remainder or the remainder is 1.

You have now checked each possible whole number, so you can write the conclusion.

Explore

Proof by exhaustion can only be used when there is a relatively small number of possibilities to check. The use of computers has made it possible to apply this method to a wider variety of problems (although some mathematicians question whether we can always trust a computer check). Find out about the Four Colour Theorem – one of the most famous problems that has been solved in this way.

EXERCISE 1E

1. Prove that 11 is a prime number.
2. Prove that 83 is a prime number.
3. Prove that all regular polygons with fewer than 7 sides have angles with a whole number of degrees.
4. Prove that no square number less than 100 ends in a 7.
5. Let $f(x)$ be the function that gives the number of factors of x. For example, $f(10) = 4$ because it has factors 1, 2, 5 and 10.

   Prove that for any single digit positive number $f(n) \leqslant n$.
6. Prove that $n^2 + 2$ is not divisible by 4 for integers between 1 and 5 inclusive.

Gateway to A Level

See Gateway to A Level section F for a reminder of function notation.

7. Prove that $n^2 + n$ is always even if $n \in \mathbb{Z}$.
8. Prove that when the square of a whole number is divided by 5, the remainder is either 0, 1 or 4.
9. Prove that $2x^3 + 3x^2 + x$ is always divisible by 6 if x is an integer.
10. The modulus function, $|x|$, is defined as x if x is positive and $-x$ if x is negative so, for example, $|-2| = 2$ and $|5| = 5$.$|0|$ is defined to be 0.

    Prove the triangle inequality: $|a + b| \leqslant |a| + |b|$.

Fast forward

You will learn more about the modulus function in Student Book 2.

Checklist of learning and understanding

- You can express mathematical ideas using descriptions such as diagrams, equations and identities.
- You can communicate a mathematical argument using a series of equations or identities put together in a logical order.
  These can be connected using implication symbols: $\Rightarrow$ or $\Leftrightarrow$.
  - The symbol $\Rightarrow$ means that a subsequent statement follows from the previous one.
  - The symbol $\Leftrightarrow$ means that a subsequent statement is equivalent to the previous one.
- An identity is a relation that is true for **all** values of the unknown. It is given the $\equiv$ symbol.
- You can represent solutions of inequalities using set notation or interval notation.
  - In interval notation, the square bracket [or] means that the endpoint is included, and the round bracket (or) means that the endpoint is **not** included.
- One counter example is sufficient to prove that a statement is not always true.
- An algebraic proof is often required to show that a statement is always true.
- Proof by exhaustion involves checking all possibilities. This can only be done if there is a small number of options, or the options can be split up into a small number of cases.

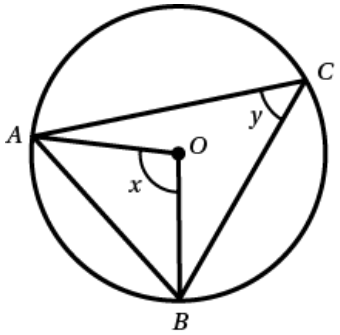
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Mixed practice 1

1. Prove that the product of any two odd numbers is always odd.
2. Prove that if n is even then n^2 is divisible by 4.
3. Prove that if $\frac{a}{b} = \frac{c}{d}$ it does not follow that $a = c$ and $b = d$.
4. Prove the following statement or disprove it with a counter example:

   'The sum of two numbers is always larger than their difference.'
5. Prove that the product of two rational numbers is always rational.
6. Prove that the sum of the interior angles in an n-sided shape is $(180n - 360)^\circ$.
7. Given that $x^3 + y^3 \equiv (x + y)(ax^2 + bxy + cy^2)$ find the values of a, b and c.
8. Prove the following statement:

   n is odd $\Rightarrow n^2 + 4n + 3$ is a multiple of 4
9. Prove that the angle from a chord to the centre of a circle is twice the angle to a point on the circumference in the major sector.



10. Prove that all cube numbers are either multiples of 9 or within one of a multiple of 9.
11. Prove the following statements, or disprove them with a counter example:

    a ab is an integer $\Leftrightarrow$ a is an integer and b is an integer

    b a is irrational and b is irrational $\Leftrightarrow$ ab is irrational.
12. Prove that the product of any three consecutive positive integers is a multiple of 6.
13. Prove that the difference between the squares of any two odd numbers is a multiple of 8.
14. **a** Prove that $n^2 - 79n + 1601$ is not always prime when n is a positive whole number.

    b Prove that $n^2 - 1$ is never prime when n is a whole number greater than 2.
15. $x = a^2 - b^2$ where a and b are both whole numbers. Prove that x is either odd or a multiple of 4.

Worksheet

For questions on another principle used in proof, see Extension sheet 1.

2 Indices and surds

In this chapter you will learn how to:

- use laws of indices
- work with expressions involving square roots (called surds).

Before you start...

GCSE	You should know how to evaluate expressions involving powers, including working with the order of operations.	1 Evaluate 3×2^3.
GCSE	You should know how to evaluate expressions involving roots.	2 Evaluate $\sqrt[3]{27}$.
GCSE	You should know how to work with negative, fractional and zero indices.	3 a Write $\frac{2x^0}{x^3}$ in the form ax^b. b Write $\frac{1}{\sqrt[3]{x^2}}$ in the form x^d.
GCSE	You should know how to multiply out two brackets.	4 Expand $(1+x)(2-y)$.
GCSE	You should be able to recognise the difference of two squares.	5 Expand and simplify $(2a+b)(2a-b)$.

Why study indices and surds?

Powers and roots are needed to write equations describing many situations, both within pure mathematics and in applications. For example, the volume of a sphere is proportional to the cube of its radius; the magnitude of the gravitational force between two objects is inversely proportional to the square of the distance between them; and standard deviation (a measure of the spread of data in statistics) is the square root of the average square deviation from the mean.

In this chapter you will review algebraic rules for working with powers and roots and use them to simplify more complex expressions.

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Section 1: Using the laws of indices

From your previous study you should know the following laws of indices:

Key point 2.1

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{m \times n}$
- $a^0 = 1$
- $a^{-n} = \dfrac{1}{a^n}$
- $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

You can use these rules to evaluate negative and fractional powers of numbers. For example:

$$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4.$$

The rules of indices must be combined accurately with the other rules of algebra you already know.

Gateway to A Level

For a reminder and more practice of the laws of indices, see Gateway to A Level section G.

Tip

Make sure that you can actively use these rules in **both** directions – i.e. if you see 2^6 you can rewrite it as $(2^3)^2$ and if you see $(2^3)^2$ you can rewrite it as 2^6. Both ways will be important!

Fast forward

To formally prove these rules requires a method called **mathematical induction**, which you will meet if you study Further Mathematics, in Pure Core Student Book 1.

WORKED EXAMPLE 2.1

Simplify $3xy \times 8xy^{-2}$.

$3xy \times 8xy^{-2} = 3 \times 8 \times x \times x \times y \times y^{-2}$ — You can rearrange multiplication into any convenient order.

$= 24x^2y^{-1}$ — Apply $a^m \times a^n = a^{m+n}$ (Key point 2.1) to x and y.

WORKED EXAMPLE 2.2

Simplify $\dfrac{12ab^{-2} - 16a}{8ab}$.

$\dfrac{12ab^{-2} - 16a}{8ab} = \dfrac{12ab^{-2}}{8ab} - \dfrac{16a}{8ab}$ — You can split a fraction up if the top is a sum or a difference.

Turn each fraction into a convenient product.

$= \frac{12}{8} \times \frac{a}{a} \times \frac{b^{-2}}{b} - \frac{16}{8} \times \frac{a}{a} \times \frac{1}{b}$

$= \frac{3}{2}b^{-3} - 2b^{-1}$

Then use $a^m \div a^n = a^{m-n}$ (Key point 2.1).

WORKED EXAMPLE 2.3

Write $\frac{\sqrt{x}}{5\sqrt[3]{x}}$ in the form kx^p

$\frac{\sqrt{x}}{5\sqrt[3]{x}} = \frac{1}{5} \times x^{\frac{1}{2}} \div x^{\frac{1}{3}}$

Dividing by 5 is the same as multiplying by $\frac{1}{5}$.

Re-write the roots using $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ (Key point 2.1).

$= \frac{1}{5}x^{\frac{1}{6}}$

Now use $a^m \div a^n = a^{m-n}$ (Key point 2.1).

You also need to be able to manipulate indices to solve equations.

WORKED EXAMPLE 2.4

Solve $x^{\frac{4}{3}} = \frac{1}{81}$.

$x^{\frac{4}{3}} = \frac{1}{81}$

$\left(x^{\frac{4}{3}}\right)^{\frac{3}{4}} = \left(\frac{1}{81}\right)^{\frac{3}{4}}$

$x = \frac{1}{27}$

Using $(a^m)^n = a^{m \times n}$ (Key point 2.1),

$\left(x^{\frac{4}{3}}\right)^{\frac{3}{4}} = x^{\frac{4}{3} \times \frac{3}{4}} = x^1 = x$ so raise both sides of the equation to the power $\frac{3}{4}$.

WORKED EXAMPLE 2.5

Solve $2^x \times 8^{x-1} = \frac{1}{4^{2x}}$.

$2^x \times 8^{x-1} = \frac{1}{4^{2x}}$

$2^x \times \left(2^3\right)^{x-1} = \frac{1}{\left(2^2\right)^{2x}}$

Express each term in the same base (2 is easiest) so that the laws of indices can be applied.

$2^x \times 2^{3x-3} = \frac{1}{2^{4x}}$

Use $(a^m)^n = a^{m \times n}$ (Key point 2.1).

$2^{4x-3} = 2^{-4x}$

And then $a^m \times a^n = a^{m+n}$ on the LHS and $a^{-n} = \frac{1}{a^n}$ on the RHS.

Equate the powers and solve.

$$4x - 3 = -4x$$
$$8x = 3$$
$$x = \frac{3}{8}$$

An equation like this with the unknown (x) in the power is called an **exponential equation**. In Chapter 7 you will see how to solve more complicated examples using logarithms.

Be careful when you are combining expressions with different bases by multiplication or division. Just remembering 'multiplication means add the exponents together' is too simplistic because it is **only** true when the **bases are the same**. There is another rule that works when the bases are different, but then the exponents have to be the same. Consider the following:

$$3^2 \times 5^2 = 3 \times 3 \times 5 \times 5 = 3 \times 5 \times 3 \times 5 = 15 \times 15 = 15^2$$

This suggests the following rules:

Key point 2.2

- $a^n \times b^n = (ab)^n$
- $a^n \div b^n = \left(\frac{a}{b}\right)^n$

WORKED EXAMPLE 2.6

Simplify $\left(\frac{x^6}{8}\right)^{\frac{1}{3}}$.

$$\left(\frac{x^6}{8}\right)^{\frac{1}{3}} = \frac{\left(x^6\right)^{\frac{1}{3}}}{8^{\frac{1}{3}}}$$

Use $a^n \div b^n = \left(\frac{a}{b}\right)^n$ (Key point 2.2) to apply the power to each part of the fraction.

$$= \frac{x^2}{2}$$

Use $(a^m)^n = a^{m \times n}$ (Key point 2.1) on the top of the fraction and recognise the cube root of 8 on the bottom.

WORKED EXAMPLE 2.7

Simplify $\frac{\left(16a^2b^8\right)^{\frac{1}{2}}}{ab^3}$.

$$\frac{\left(16a^2b^8\right)^{\frac{1}{2}}}{ab^3} = \frac{16^{\frac{1}{2}}\left(a^2\right)^{\frac{1}{2}}\left(b^8\right)^{\frac{1}{2}}}{ab^3}$$

Use $a^n \times b^n = (ab)^n$ (Key point 2.2).

$$= \frac{4ab^4}{ab^3}$$

$$= 4b$$

Apply $a^m \div a^n = a^{m-n}$ (Key point 2.1) to the a's and b's.

WORK IT OUT 2.1

Simplify $\left(16x^2 + 16y^2\right)^{\frac{1}{2}}$.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$$\begin{aligned}\left(16x^2+16y^2\right)^{\frac{1}{2}} &= 16^{\frac{1}{2}}\left(x^2+y^2\right)^{\frac{1}{2}} \\ &= 4\left(x^2+y^2\right)^{\frac{1}{2}}\end{aligned}$$

Solution 2

$$\begin{aligned}\left(16x^2+16y^2\right)^{\frac{1}{2}} &= 16^{\frac{1}{2}}\left(x^2+y^2\right)^{\frac{1}{2}} \\ &= 8\sqrt{x^2+y^2}\end{aligned}$$

Solution 3

$$\begin{aligned}\left(16x^2+16y^2\right)^{\frac{1}{2}} &= \left(16x^2\right)^{\frac{1}{2}}+\left(16y^2\right)^{\frac{1}{2}} \\ &= 4x+4y\end{aligned}$$

EXERCISE 2A

1 Evaluate without a calculator:

a **i** $16^{\frac{1}{2}}$

ii $27^{\frac{1}{3}}$

b **i** 2^{-3}

ii 5^{-2}

c **i** $\left(\frac{2}{3}\right)^{-2}$

ii $\left(\frac{1}{4}\right)^{-3}$

d **i** $27^{\frac{2}{3}}$

ii $9^{\frac{3}{2}}$

e **i** $125^{-\frac{2}{3}}$

ii $16^{-\frac{3}{4}}$

f **i** $\left(\frac{4}{9}\right)^{-\frac{3}{2}}$

ii $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$

g **i** $0.25^{\frac{1}{2}}$

ii 0.5^{-2}

h **i** $32^{0.2}$

ii $81^{-0.25}$

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Gateway to A Level

For more practice with indices, see Gateway to A Level section G.

2 Write the following in the form 4^p:

a i $\sqrt{4}$

ii $\sqrt[3]{4}$

b i 16

ii 1

c i 2

ii $\frac{1}{2}$

d i 0.25

ii 0.5

e i 8

ii 32

f i $4\sqrt[3]{4}$

ii $2\sqrt[5]{4}$

g i $\sqrt{2}$

ii $\sqrt{8}$

3 Simplify the following:

a i $\left(x^6\right)^{\frac{1}{2}}$

ii $\left(x^9\right)^{\frac{4}{3}}$

b i $\left(4x^{10}\right)^{0.5}$

ii $\left(8x^{12}\right)^{-\frac{1}{3}}$

c i $\left(\frac{27x^9}{64}\right)^{\frac{1}{3}}$

ii $\left(\frac{x^4}{y^8}\right)^{-1.5}$

4 Solve the following:

a i $x^{-\frac{5}{2}} = 32$

ii $x^{\frac{3}{2}} = \frac{8}{27}$

b i $2^{x-1} = \frac{4^x}{8}$

ii $27^x = \sqrt{3} \times 9^{x+2}$

5 Write in the form ax^p:

a i $3\sqrt{x}$

ii $5\sqrt[3]{x}$

b i $x\sqrt[3]{x}$

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ii $x^2\sqrt{x}$

c i $\dfrac{3}{\sqrt{x}}$

ii $\dfrac{5}{\sqrt[3]{x}}$

d i $\dfrac{1}{(2x^2)}$

ii $\dfrac{1}{(3x^2)}$

e i $\dfrac{2}{3\sqrt{x}}$

ii $\dfrac{5}{2\sqrt[3]{x}}$

6 Write in the form $ax^p + bx^q$:

a i $\dfrac{(3+2x)}{\sqrt{x}}$

ii $\dfrac{(4-3\sqrt{x})}{x}$

b i $\dfrac{\left(x^2-3\right)}{(2x)}$

ii $\dfrac{(4x-3)}{(2x^2)}$

c i $\dfrac{\left(2x^2+1\right)}{4\sqrt{x}}$

ii $\dfrac{(2-9\sqrt{x})}{(3x)}$

7 Simplify $\left(100x^4\right)^{-\frac{1}{2}}$.

8 Simplify $\dfrac{x^{\frac{2}{3}} \times x^{\frac{1}{2}}}{x^{\frac{1}{6}}}$.

9 Simplify $\dfrac{\sqrt[3]{64p^6q^4}}{2q}$.

10 Write $\dfrac{1-x^3}{\sqrt[3]{x}}$ in the form $x^a - x^b$.

Gateway to A Level

For a reminder of basic manipulations with surds, see Gateway to A Level section H.

11 An elementary computer program is known to be able to sort n input values in $k \times n^{1.5}$ microseconds. Observations show that it sorts a million values in half a second. Find the value of k.

12 The volume and surface area of a family of regular solid shapes are related by the formula $V = kA^{1.5}$ where V is given in cubic units and A in square units.

a For one such shape, $A = 81$ and $V = 243$. Find k.

b Hence determine the surface area of a shape with volume $\dfrac{64}{3}$ cm^3.

13 A square-ended cuboid has volume xy^2, where x and y are lengths. A cuboid for which $x = 2y$ has volume 128 cm^3. Find x.

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14 Simplify $\left(3x^9+\frac{3}{8}x^9\right)^{-\frac{1}{3}}$.

15 If $\sqrt{3^{2a+b}}=\frac{27^a}{3^b}$ express a in terms of b.

16 Express $\left(3x^{\frac{3}{4}}-x^{-\frac{3}{4}}\right)^2$ in the form $ax^n+\frac{b}{x^n}+c$ where a, b, c and n are to be found.

17 Make x the subject of the equation $4^{ax}=b\times 8^x$ where a and b are constants. Leave your answer in a simplified form.

18 Anything raised to the power of zero is 1, but zero raised to any power is zero, so what is the value of 0^0?

19 What is the value of 2^{2^2}?

Section 2: Working with surds

A **surd** is any number that can only be expressed using roots. You need to be able to work with surd expressions, for example:

$$2+3\sqrt{5},\ \sqrt{3}-\sqrt{7},\ \sqrt[3]{2}+\sqrt{2}$$

The most important thing to know when working with surds is that square rooting is just another way of writing 'raise to the power of a half' so all the rules for indices apply to surds.

Fast forward

A In Student Book 2, Chapter 1, you will see a method for proving that surds **cannot** be written as rational numbers.

WORKED EXAMPLE 2.8

Write $\sqrt{8}+\sqrt{2}$ in the form $\sqrt{a}$.

$\sqrt{8}=\sqrt{4\times 2}$
$=\sqrt{4}\times\sqrt{2}$
$=2\sqrt{2}$

Square root is equivalent to the power $\frac{1}{2}$. You can apply $(ab)^{\frac{1}{2}}=a^{\frac{1}{2}}\times b^{\frac{1}{2}}$ (Key point 2.2) to $\sqrt{8}$ to split it up into a product. Choose to split 8 into a square number and something else.

$\sqrt{8}+\sqrt{2}=2\sqrt{2}+\sqrt{2}$
$=3\sqrt{2}$

$2\sqrt{2}$ plus another $\sqrt{2}$ is just 3 lots of $\sqrt{2}$.

$=\sqrt{9}\sqrt{2}$
$=\sqrt{18}$

If you want to put it all under one square root, you can use Key point 2.2 again. However, you need to write 3 as the square root of another number.

WORKED EXAMPLE 2.9

Simplify $\left(1+\sqrt{2}\right)^2$.

$\left(1+\sqrt{2}\right)^2=\left(1+\sqrt{2}\right)\left(1+\sqrt{2}\right)$
$=1+\sqrt{2}+\sqrt{2}+\sqrt{2}\times\sqrt{2}$

Expand the brackets as normal.

$=1+2\sqrt{2}+2$
$=3+2\sqrt{2}$

Note that $\sqrt{2}\times\sqrt{2}=\left(\sqrt{2}\right)^2=\left(2^{\frac{1}{2}}\right)^2=2^1=2$.

One important method used when dealing with surds in fractions is called **rationalising the denominator**. This is the process when surds are removed from the denominator.

WORKED EXAMPLE 2.10

Rationalise the denominator of $\frac{2}{\sqrt{3}}$.

$$\frac{2}{\sqrt{3}} = \frac{2 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{2\sqrt{3}}{3}$$

One obvious thing that would make the bottom rational is to multiply it by $\sqrt{3}$. If you do this you have to multiply the top by $\sqrt{3}$ also.

It is also possible to rationalise the denominator of more complicated expressions. The trick to do this uses the difference of two squares:

$$a^2 - b^2 = (a - b)(a + b)$$

If you have an expression such as $5 + \sqrt{3}$ you know that by multiplying it by $5 - \sqrt{3}$ the result will be $5^2 - \left(\sqrt{3}\right)^2 = 22$. Importantly, this product is a rational number.

Gateway to A Level

For revision of the difference of two squares factorisation, see Gateway to A Level section B.

Key point 2.3

To rationalise the denominator of a fraction, multiply top and bottom by the appropriate expression to create a 'difference of two squares'.

WORKED EXAMPLE 2.11

Rationalise the denominator of $\frac{3}{8 - 2\sqrt{3}}$.

$$\frac{3}{8 - 2\sqrt{3}} = \frac{3 \times (8 + 2\sqrt{3})}{(8 - 2\sqrt{3}) \times (8 + 2\sqrt{3})}$$

The appropriate term to multiply top and bottom by is $8 + 2\sqrt{3}$

$$= \frac{24 + 6\sqrt{3}}{64 - 4 \times 3}$$

You do not need to multiply the bottom out. You can use the difference of two squares identity.

$$= \frac{24 + 6\sqrt{3}}{52}$$

There is a factor of 2 on top and bottom.

$$= \frac{12 + 3\sqrt{3}}{26}$$

WORK IT OUT 2.2

Rationalise the denominator of $\frac{1}{\sqrt{5} - \sqrt{2}}$.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$$\frac{1}{\sqrt{5} - \sqrt{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Solution 2

$$\frac{1}{\sqrt{5}-\sqrt{2}} = \frac{1 \times (\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2}) \times (\sqrt{5}+\sqrt{2})}$$
$$= \frac{\sqrt{5}+\sqrt{2}}{3}$$

Solution 3

$$\frac{1}{\sqrt{5}-\sqrt{2}} = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{2}}$$
$$= \frac{\sqrt{5}}{5} - \frac{\sqrt{2}}{2}$$
$$= \frac{2\sqrt{5}-5\sqrt{2}}{10}$$

EXERCISE 2B

1 Write the following in the form $k\sqrt{5}$.

a i $\sqrt{125}$

ii $\sqrt{20}$

b i $7\sqrt{5}-2\sqrt{5}$

ii $\sqrt{5}+9\sqrt{5}-3\sqrt{5}$

c i $3\sqrt{80}-5\sqrt{20}$

ii $\sqrt{125}+7\sqrt{45}$

2 Write the following in the form $\sqrt{a}$.

a i $4\sqrt{2}$

ii $10\sqrt{3}$

b i $\sqrt{7}+2\sqrt{7}$

ii $3\sqrt{5}+\sqrt{5}$

c i $\sqrt{3}+\sqrt{75}$

ii $\sqrt{32}+\sqrt{8}$

3 Write in the form $a+b\sqrt{3}$.

a i $2\left(3-\sqrt{3}\right)-3\left(1-\sqrt{3}\right)$

ii $\left(1+\sqrt{3}\right)-\left(1-\sqrt{3}\right)$

b i $\left(1+2\sqrt{3}\right)\left(2-\sqrt{3}\right)$

ii $\left(1+\sqrt{3}\right)\left(2+\sqrt{3}\right)$

c i $\left(1+\sqrt{3}\right)^2$

ii $\left(2-\sqrt{3}\right)^2$

4 Rationalise the denominator of:

a i $\frac{7}{\sqrt{7}}$

ii $\frac{2}{\sqrt{5}}$

b **i** $\dfrac{3-\sqrt{6}}{\sqrt{6}}$

ii $\dfrac{\sqrt{2}+\sqrt{6}}{\sqrt{3}}$

c **i** $\dfrac{1}{\sqrt{2}-1}$

ii $\dfrac{1+\sqrt{5}}{1+\sqrt{7}}$

Gateway to A Level

For a reminder and more practice of questions like this, see Gateway to A Level section I.

5 Simplify $\dfrac{1}{1+\sqrt{n}}+\dfrac{1}{1-\sqrt{n}}$.

6 Show that $\dfrac{4}{\sqrt{20}-\sqrt{12}}$ can be written in the form $\sqrt{a}+\sqrt{b}$ where a and b are whole numbers.

7 Show that $\dfrac{5+\sqrt{2}}{3-2\sqrt{2}}$ can be written in the form $a+b\sqrt{2}$ where a and b are constants to be found.

8 Explain without using decimal approximations why $3\sqrt{2}$ is larger than $2\sqrt{3}$.

9 Solve the equation $x\sqrt{27}=5x\sqrt{3}+2\sqrt{48}$.

10 Rationalise the denominator of $\dfrac{1}{2\sqrt{n}-3}$.

Worksheet

For more practice at solving equations involving surds, see Support sheet 2.

11 If n is a positive whole number write $\left(n\sqrt{15}-\sqrt{5}\right)^2$ in the form $a+b\sqrt{3}$.

12 A rectangle has length $a+b\sqrt{2}$ and width $b-a\sqrt{2}$.

a Find the area of the rectangle in the form $m+n\sqrt{2}$.

b Find and simplify an expression for the length of the diagonal of the rectangle.

13 **a** Write $\sqrt{27}+\sqrt{3}$ in the form $\sqrt{a}$.

b Explain without using decimal approximations whether $\sqrt{27}-\sqrt{20}$ is bigger or smaller than $\sqrt{5}-\sqrt{3}$.

14 **a** Find and simplify an expression for $(a+b\sqrt{2})^2$.

b By considering $(1-\sqrt{2})^4$ prove that $\sqrt{2}<\dfrac{17}{12}$.

15 **a** Show that $a^3-b^3=(a-b)(a^2+ab+b^2)$.

b Hence rationalise the denominator of $\dfrac{1}{\sqrt[3]{3}-\sqrt[3]{2}}$.

16 Is it always true that $\sqrt{x^2}$ equals x?

Checklist of learning and understanding

- You need to learn these laws of indices:
  - $a^m \times a^n = a^{m+n}$

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- $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$
  - $a^0 = 1$
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $(a^m)^n = a^{m \times n}$
  - $a^n \div b^n = \left(\frac{a}{b}\right)^n$
  - $a^m \div a^n = a^{m-n}$
  - $a^n \times b^n = (ab)^n$
  - $a^{-n} = \frac{1}{a^n}$
- The second point above means that you can write roots as powers; in particular, $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}}$.
- Surds are numbers that can only be expressed using roots. To rationalise the denominator of a fraction you can multiply top and bottom by the appropriate expression to create a 'difference of two squares'.

Mixed practice 2

1. Express $\left(n+\sqrt{5}\right)^2$ in the form $a+b\sqrt{5}$.
2. If $z=xy^2$ and $y=3x$ express z in terms of x only.
3. Show that $\dfrac{10}{\sqrt{28}-\sqrt{8}}$ can be written in the form $\sqrt{a}+\sqrt{b}$.
4. If $y=\dfrac{2}{\sqrt{x}}$, write y^{-4} in the form kx^a.
5. If $3x\sqrt{8}=x\sqrt{2}+\sqrt{32}$, find x.
6. Simplify:

   a $\left(\sqrt[3]{x}\right)^6$

   b $\dfrac{3y^4\times(10y)^3}{2y^5}$

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[Question part reference style adapted]

7. Express each of the following in the form 3^n:

   a $\dfrac{1}{9}$

   b $\sqrt[3]{3}$

   c $3^{10}\times 9^{135}$

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[Question part reference style adapted]

8. Simplify $\left(x^4+7x^3\times\dfrac{x}{9}\right)^{-\frac{1}{2}}$.
9. Rationalise the denominator of $\dfrac{\sqrt{n}+1}{\sqrt{n}-1}$.
10. **a** Find and simplify an expression for $\left(a+b\sqrt{5}\right)^2$.

    b By considering $(2-\sqrt{5})^4$ show that $\sqrt{5}<\dfrac{161}{72}$.

    c By considering $(2-\sqrt{5})^3$ show that $\sqrt{5}>\dfrac{38}{17}$.

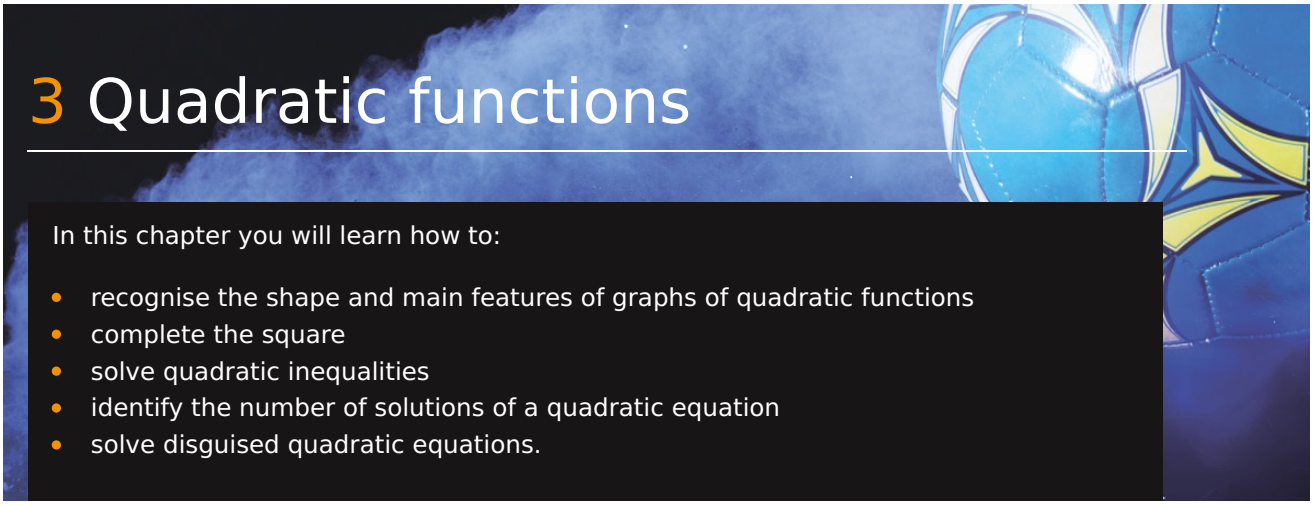
    d **i** Explain why considering $(3-\sqrt{5})^3$ gives a worse upper bound on $\sqrt{5}$ than found in part **b**.

    ii Explain why considering $(4-\sqrt{5})^4$ would not give as good an upper bound on $\sqrt{5}$ as found in part **b**.

Worksheet

For more challenging questions on indices and surds, see Extension sheet 2.

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3 Quadratic functions

In this chapter you will learn how to:

- recognise the shape and main features of graphs of quadratic functions
- complete the square
- solve quadratic inequalities
- identify the number of solutions of a quadratic equation
- solve disguised quadratic equations.

Before you start...

GCSE	You should know how to multiply out brackets.	1 Expand $(3x+1)(2x-3)$.
GCSE	You should know how to solve quadratics by factorising.	2 Solve: a $x^2+x-20=0$ b $2x^2+15x-8=0$ c $5x^2-3x=0$ d $4x^2-9=0$
GCSE	You should know how to solve quadratics using the formula.	3 Solve: a $x^2-4x+2=0$ b $2x^2-10x-5=0$
GCSE	You should know how to solve linear inequalities.	4 Solve $5x-1>2x+5$.

Quadratic phenomena

Many problems in applications of mathematics involve maximising or minimising a certain quantity. They are common in economics and business (minimising costs and maximising profits), biology (finding the maximum possible size of a population) and physics (electrons move to the lowest energy state).

The quadratic function is the simplest function with maximum or minimum points, so it is often used to model such situations. It also arises in many natural phenomena, such as the motion of a projectile or the dependence of power on voltage in an electric circuit.

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Section 1: Review of quadratic equations

A quadratic function is one of the form $f(x) = ax^2 + bx + c$ where a, b and c are constants and $a \neq 0$.

You should be familiar with two methods for solving quadratic equations:

- factorising
- the quadratic formula.

Many calculators have an equation solver that just lets you type in the coefficients a, b and c to generate the solutions, but you need to be able to apply these methods as well, particularly factorising.

Remember that you may first have to rearrange the equation to get it in the form $ax^2 + bx + c = 0$.

Gateway to A Level

For a revision of factorising and using the quadratic formula, see Gateway to A Level sections B and C.

WORKED EXAMPLE 3.1

Find the values of x for which $(2x - 3)^2 = x + 6$.

$(2x - 3)^2 = x + 6$	Expand the brackets. Remember: $(2x - 3)^2 = (2x - 3)(2x - 3)$.
$4x^2 - 12x + 9 = x + 6$	
$4x^2 - 13x + 3 = 0$	Move everything to one side: $f(x) = 0$.
$(4x - 1)(x - 3) = 0$	Factorise and solve.
$x = \frac{1}{4}$ or 3	

WORKED EXAMPLE 3.2

Find the values of x for which $x = 2 + 8x^{-1}$.

$x = 2 + 8x^{-1}$	This might not look like a quadratic at first, but whether or not you spot that it is, it is always a good idea to start by replacing x^{-a} with $\frac{1}{x^a}$.
$x = 2 + \frac{8}{x}$	
$x^2 = 2x + 8$	Now multiply through by x to remove the denominator
$x^2 - 2x - 8 = 0$	Move everything to one side: $f(x) = 0$.
$(x - 4)(x + 2) = 0$	Factorise and solve.
$x = 4$ or -2	

The quadratic formula

Sometimes you can't see how to factorise the equation, for example if the solutions are not whole numbers or simple fractions. In that case you can use the quadratic formula to solve the equation.

Key point 3.1

The solutions of $ax^2 + bx + c = 0$ are given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When using the quadratic formula, you may need to use your knowledge of surds to simplify the answer.

Tip

The quadratic formula is **not** given on the formula sheet.

Fast forward

The other alternative is to solve the equation by completing the square. You will see in Section 3 that this is where the quadratic formula actually comes from.

WORKED EXAMPLE 3.3

Solve the equation $x^2 + 3x = 7x + 3$.

Give your solutions in exact form.

$x^2 + 3x = 7x + 3$ — Move everything to one side: $\mathrm{f}(x) = 0$.

$x^2 - 4x - 3 = 0$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times (-3)}}{2 \times 1}$$ — If you can't see how to factorise it, use the formula:

$a = 1, b = -4, c = -3$

$$= \frac{4 \pm \sqrt{28}}{2}$$

$$= \frac{4 \pm 2\sqrt{7}}{2}$$ — Simplify the surd: $\sqrt{28} = 2\sqrt{7}$

$$= 2 \pm \sqrt{7}$$

Tip

If you are told to give your solutions to a certain number of decimal places or significant figures, or to give exact solutions, it usually means the quadratic won't factorise easily.

Rewind

You learnt to simplify surds in Chapter 2, Section 2.

WORKED EXAMPLE 3.4

Solve the equation $\sqrt{3}x^2 - 6x - 2\sqrt{3} = 0$, giving your answer in simplified surd form.

$$x = \frac{6 \pm \sqrt{36 - 4\sqrt{3}\left(-2\sqrt{3}\right)}}{2\sqrt{3}}$$ — The answers are not going to be whole numbers, so use the quadratic formula.

Use $\sqrt{3} \times \sqrt{3} = 3$.

$$= \frac{6 \pm \sqrt{36+24}}{2\sqrt{3}}$$

$$= \frac{6 \pm \sqrt{60}}{2\sqrt{3}}$$

$$= \frac{6 \pm 2\sqrt{15}}{2\sqrt{3}}$$ Use $\sqrt{60} = \sqrt{4 \times 15} = 2\sqrt{15}$.

$$= \frac{3 \pm \sqrt{15}}{\sqrt{3}}$$

$$= \sqrt{3} \pm \sqrt{5}$$ Use $\frac{3}{\sqrt{3}} = \sqrt{3}$ and $\frac{\sqrt{15}}{\sqrt{3}} = \sqrt{5}$.

EXERCISE 3A

1 By factorising, solve the following equations:

a i $3x^2 + 2x = x^2 + 3x + 6$

ii $2x^2 + 3 = 17x - 7 - x^2$

b i $9x^2 = 24x - 16$

ii $18x^2 = 2x^2 - 40x - 25$

c i $(x - 3)(x + 2) = 14$

ii $(2x + 3)(x - 1) = 12$

d i $2x = 11 + \frac{6}{x}$

ii $3x + \frac{4}{x} = 7$

2 Use the quadratic formula to find the exact solutions of the following equations:

a i $2x^2 + x = x^2 + 4x - 1$

ii $x^2 - 3x + 5 = 6 - 2x$

b i $3x^2 - 4x + 1 = 5x^2 + 2x$

ii $9x - 2 = 5x^2 + 1$

c i $(x + 1)(x + 3) = 5$

ii $(3x + 2)(x - 1) = 2$

d i $2x + \frac{1}{x} = 6$

ii $x = 4 + \frac{3}{x}$

3 Solve the equation $8x - 9 = (3x - 1)(x + 3)$.

4 Solve the equation $6x = 5 + 4x^{-1}$.

5 Find the exact solutions to the equation $x + x^{-1} = 3$.

6 Solve the equation $x^2 + 8k^2 = 6kx$, giving your answer in terms of k.

7 Solve the equation $2\sqrt{5}x^2 + 3x - \sqrt{5} = 0$, giving your answers in simplified surd form.

8 Find the exact solutions of the equation $x^2\sqrt{2} + 2x\sqrt{5} - 3\sqrt{2} = 0$.

9 Rearrange $y = ax^2 + bx + c$ to find x in terms of y.

10 The positive difference between the solutions of the quadratic equation $x^2 + kx + 3 = 0$ is $\sqrt{69}$. Find the possible values of k.

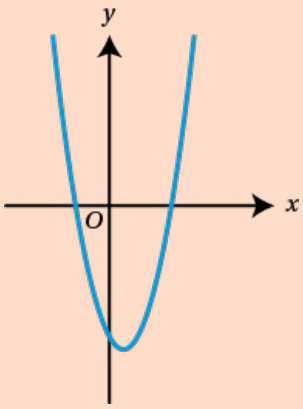
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Section 2: Graphs of quadratic functions

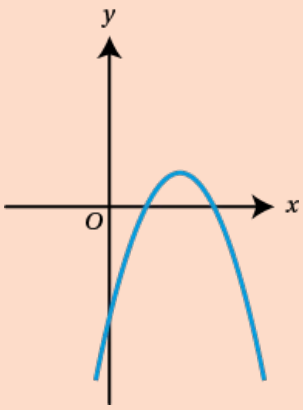
All quadratic graphs are one of two possible shapes. The graph $y = ax^2 + bx + c$ is called a **parabola**. Its shape depends on the coefficient a.

Key point 3.2

- If $a > 0$ the parabola is a **positive** quadratic.



- If $a < 0$ the parabola is a **negative** quadratic.



When sketching a graph, you should show the point(s) where it crosses the coordinate axes.

Key point 3.3

The graph $y = ax^2 + bx + c$ crosses the:

- y-axis at the point $(0, c)$
- x-axis at the root(s) of the equation $ax^2 + bx + c = 0$ (if it crosses the x-axis at all).

Tip

A quadratic curve may not cross the x-axis at all. Try sketching this for yourself to see.

Tip

A root of an equation is a solution.

WORKED EXAMPLE 3.5

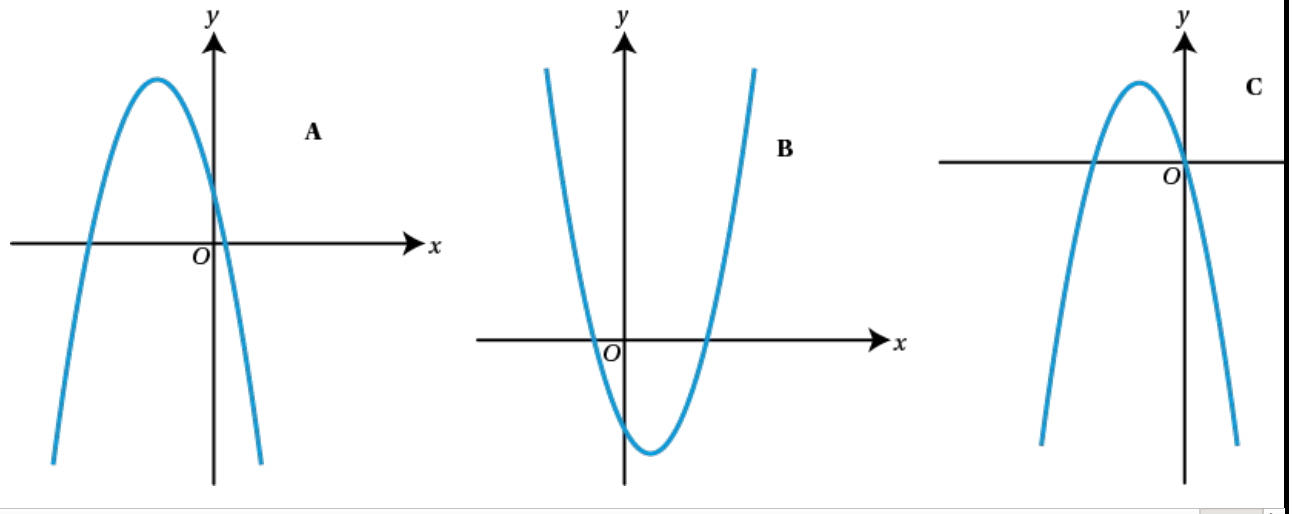
Match each equation to the corresponding graph, explaining your reasons.

a $y = 3x^2 - 4x - 1$

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b $y = -2x^2 - 4x$

c $y = -x^2 - 4x + 2$



Graph B shows a positive quadratic, so graph B corresponds to equation **a**.

Graph B is the only positive quadratic.

Graph A has a positive y-intercept, so graph A corresponds to equation **c**.

Graph C corresponds to equation **b**.

You can distinguish between the other two graphs based on their y -intercepts.

WORKED EXAMPLE 3.6

Sketch the graph of $y = 3x^2 + 2x - 8$.

This is a positive quadratic as $a > 0$.

When $x = 0, y = -8$

Find the x-intercept.

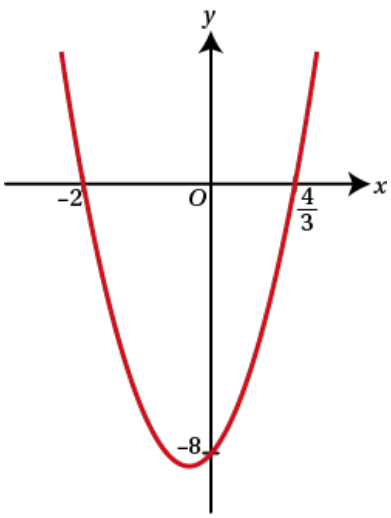
When $y = 0, 3x^2 + 2x - 8 = 0$

Find the x-intercepts. To do this, solve the equation $y = 0$.

$(3x - 4)(x + 2) = 0$

This factorises.

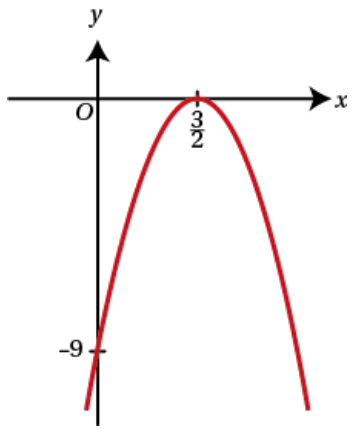
$x = \frac{4}{3}$ *or* -2



Sketch the graph. It does not have to be to scale but it should show all relevant features, and axis intercepts should be labelled.

WORKED EXAMPLE 3.7

Find the equation of this graph, giving your answer in the form $y = ax^2 + bx + c$.



Repeated root at $x = \frac{3}{2} \Rightarrow (2x - 3)^2$ is a factor.

The x-intercepts tell us about the factors.

$y = k(2x - 3)^2$

Write in factorised form. The factor $(2x - 3)^2$ could be multiplied by **any** constant, so we will label this unknown constant k

$-9 = k(0 - 3)^2$

$-9 = 9k$

$k = -1$

To find the value of k, use the fact that when $x = 0, y = -9$.

So the equation is

$y = -(2x - 3)^2$

$= -(4x^2 - 12x + 9)$

$= -4x^2 + 12x - 9$

Expand to give the equation in the form required.

EXERCISE 3B

1 Match the equations and the corresponding graphs.

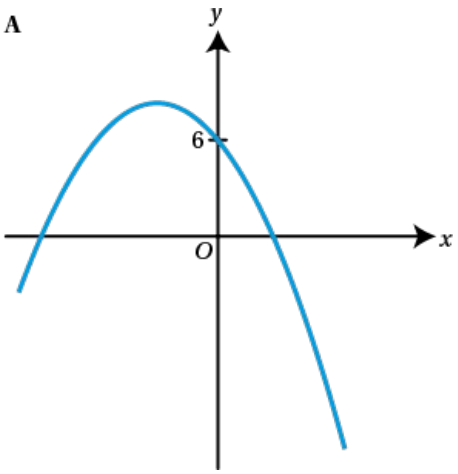
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a i $y = -x^2 - 3x + 6$

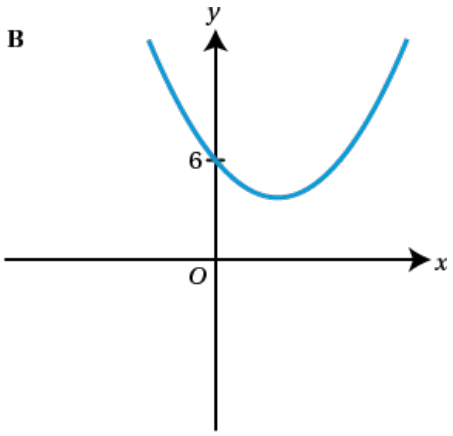
ii $y = 2x^2 - 3x + 3$

iii $y = x^2 - 3x + 6$

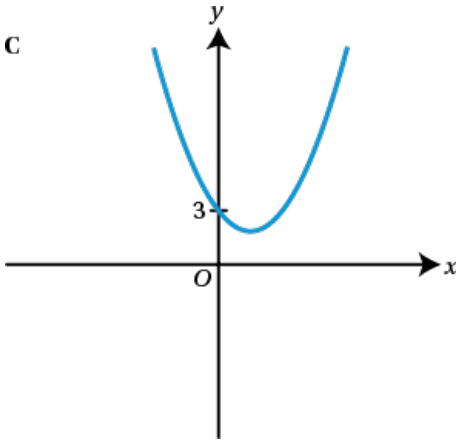
A



B



C



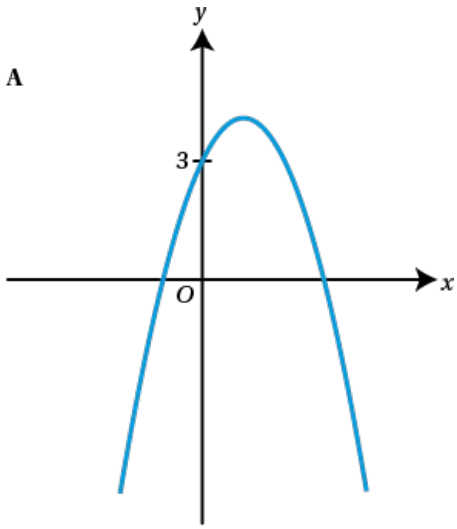
b i $y = -x^2 + 2x - 3$

ii $y = -x^2 + 2x + 3$

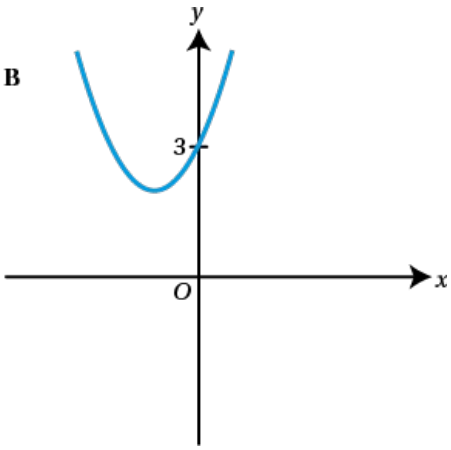
iii $y = x^2 + 2x + 3$

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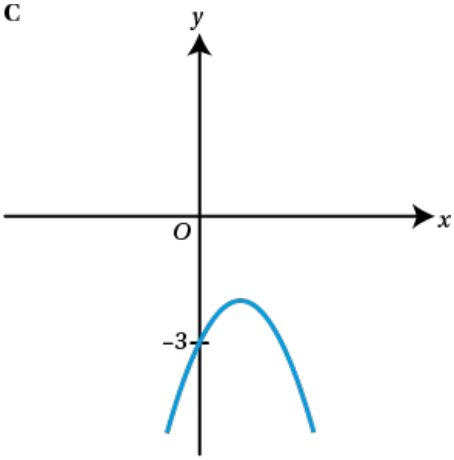
A



B



C



2 Sketch the following graphs, labelling all axis intercepts.

a i $y = x^2 - 3x - 10$

ii $y = 2x^2 + 11x + 12$

b i $y = -3x^2 + 14x - 8$

ii $y = 6 - 5x - x^2$

c i $y = 3x^2 + 6x$

ii $y = 4x - x^2$

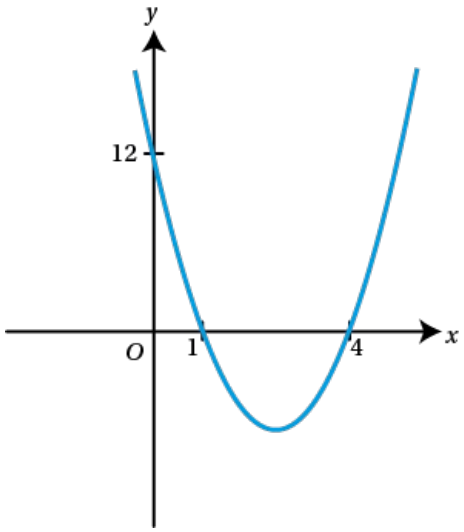
d i $y = -4x^2 - 20x - 25$

ii $y = 4x^2 - 4x + 1$

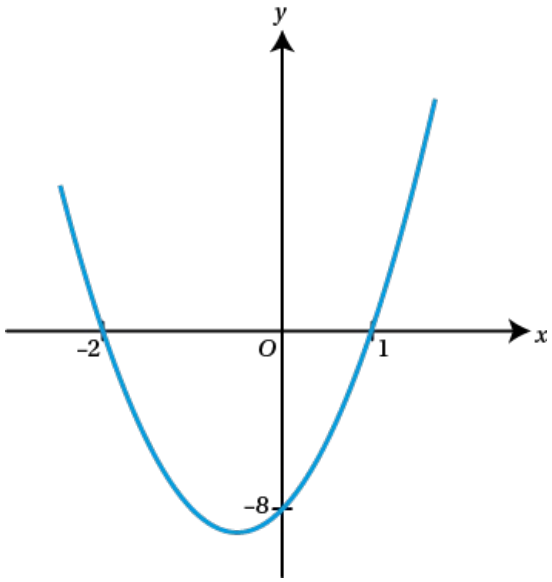
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3 Find the equation of each graph in the form $y = ax^2 + bx + c$.

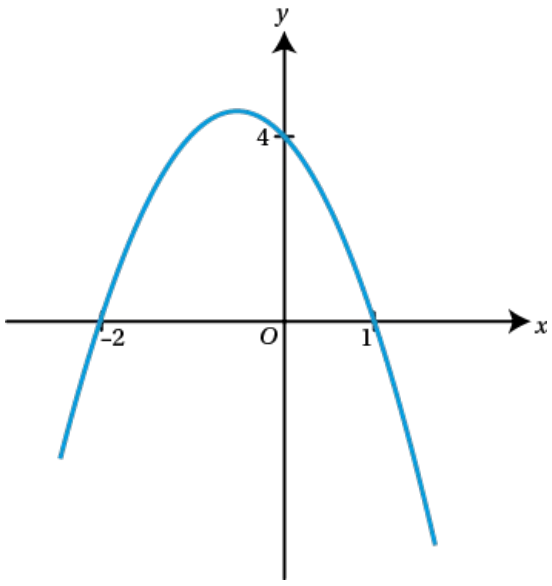
a i



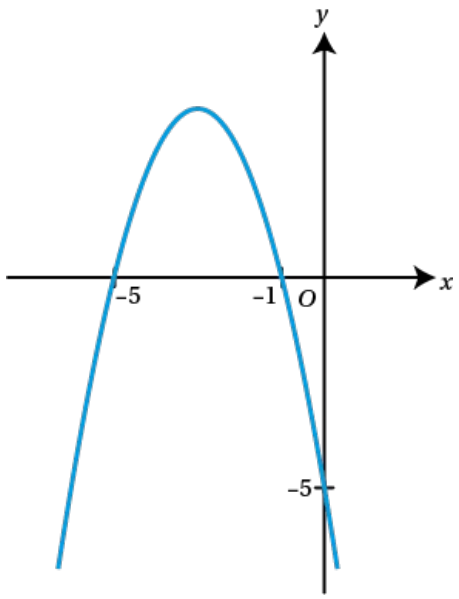
ii



b i



ii



Section 3: Completing the square

It can be useful to rewrite quadratic functions in terms of a bracket squared for two main reasons:

- To solve a quadratic equation
- to find the coordinates of the vertex of a parabola.

Sometimes this just means factorising:

$$x^2 + 10x + 25 = (x + 5)^2$$

But even if this isn't possible, it is only a case of adjusting the constant at the end:

$$x^2 + 10x = (x + 5)^2 - 25$$

The important things to note here are that:

- the number in the bracket is always half the coefficient of x: $\frac{10}{2} = 5$
- the constant to be subtracted at the end is always the number in the bracket squared: $5^2 = 25$

This process, called completing the square, allows you to write any quadratic in the form $a(x + p)^2 + q$ and is illustrated in the next few worked examples.

Rewind

You are probably already familiar with this method from GCSE, but perhaps not in the more tricky cases shown in Worked examples 3.8 and 3.9. If you don't like this method you can use the method of comparing coefficients from Worked example 1.1.

WORKED EXAMPLE 3.8

Express $x^2 - 8x + 3$ in the form $(x + p)^2 + q$, stating the values of p and q.

$x^2 - 8x + 3 = (x - 4)^2 - (-4)^2 + 3$ — Halve the coefficient of x. Subtract $(-4)^2$. The constant term $+3$ is still there.

$= (x - 4)^2 - 16 + 3$ — Simplify.

$= (x - 4)^2 - 13$

$\therefore p = -4, q = -13$

WORKED EXAMPLE 3.9

Express $x^2 + 5x + 7$ in the form $(x + p)^2 + q$, stating the values of p and q.

$x^2 + 5x + 7 = \left(x + \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 7$ — Halve the coefficient of x. Subtract $\left(\frac{5}{2}\right)^2$. The constant term $+7$ is still there.

$= \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + 7$ — Simplify.

$$= \left(x + \frac{5}{2}\right)^2 + \frac{3}{4}$$

$$\therefore p = -\frac{5}{2}, q = \frac{3}{4}$$

If the coefficient of x^2 isn't 1, you will need to factorise the expression before completing the square.

WORKED EXAMPLE 3.10

$2x^2 - 6x + 2 = a(x + p)^2 + q$

a Find the constants a, p and q.

b Hence solve the equation $2x^2 - 6x + 2 = 0$

a $2x^2 - 6x + 2 = 2\{x^2 - 3x\} + 2$ — Factorise 2 from the first two terms.

$= 2\left\{\left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right\} + 2$ — Now complete the square on the terms in the bracket.

$= 2\left\{\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right\} + 2$

$= 2\left(x - \frac{3}{2}\right)^2 - \frac{9}{2} + 2$ — Multiply the 2 back in.

$= 2\left(x - \frac{3}{2}\right)^2 - \frac{5}{2}$ — Simplify.

$\therefore a = 2, p = -\frac{3}{2}, q = -\frac{5}{2}$

b $2x^2 - 6x + 2 = 0$ — Hence' means that we must use the result of part **a**.

$2\left(x - \frac{3}{2}\right)^2 - \frac{5}{2} = 0$

$2\left(x - \frac{3}{2}\right)^2 = \frac{5}{2}$ — Now just rearrange to make x the subject.

$\left(x - \frac{3}{2}\right)^2 = \frac{5}{4}$

$x - \frac{3}{2} = \pm\sqrt{\frac{5}{4}}$ — Remember the $\pm$ when square rooting.

$x - \frac{3}{2} = \pm\frac{\sqrt{5}}{2}$

$x = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

$= \frac{3 \pm \sqrt{5}}{2}$

You might think that the answer ends up looking exactly like the sort of answer you get from using the quadratic formula. And you'd be right

You can use exactly the same method for solving the equation in the previous example on the general quadratic equation $ax^2 + bx + c = 0$ to establish the quadratic formula.

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Tip

In part **a** of Worked example 3.10 you could not simply divide both sides by 2, as you would then have a different expression. However, when solving the equation in part **b**, you could have started by dividing **both sides** by 2 first.

PROOF 1

Show that if $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$ax^2 + bx + c = 0$

First divide by a to make the quadratic easier to complete the square.

$x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$

$\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a^2} + \dfrac{c}{a} = 0$

Complete the square: halving $\dfrac{b}{a}$ gives $\dfrac{b}{2a}$.

$\left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2}{4a^2} - \dfrac{c}{a}$

Now rearrange as before to make x the subject.

$= \dfrac{b^2 - 4ac}{4a^2}$

$x + \dfrac{b}{2a} = \pm\sqrt{\dfrac{b^2 - 4ac}{4a^2}}$

Square root both sides, remembering the $\pm$.

$= \pm\dfrac{\sqrt{b^2 - 4ac}}{2a}$

$x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

WORK IT OUT 3.1

Express $-x^2 + 10x - 7$ in the form $a(x + p)^2 + q$.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$-x^2 + 10x - 7 = -(x - 5)^2 - 25 - 7$

$= -(x - 5)^2 - 32$

Solution 2

$-x^2 + 10x - 7 = -[x^2 - 10x] - 7$

$= -[(x - 5)^2 - 25] - 7$

$= -(x - 5)^2 + 25 - 7$

$= -(x - 5)^2 + 18$

Solution 3

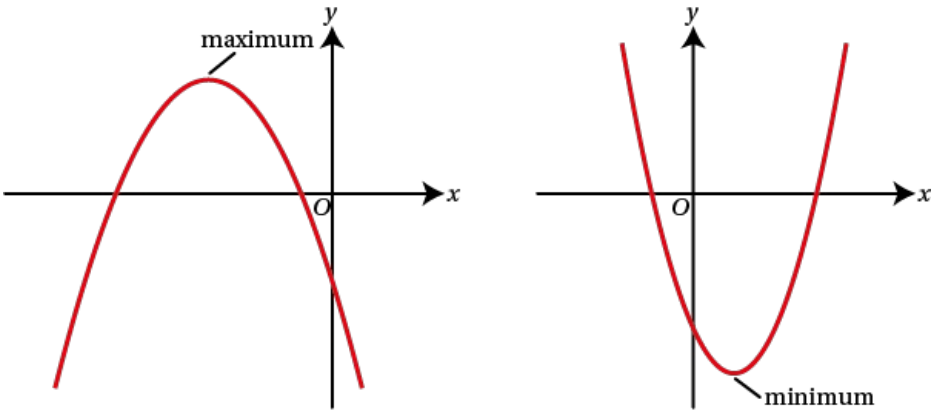
Multiplying by -1:

$x^2 - 10x + 7$

Completing the square:

$(x-5)^2 - 25 + 7$
$= (x-5)^2 - 18$

As well as enabling you to find roots of a quadratic equation, completing the square also gives you a way of finding the coordinates of the maximum or minimum point of a quadratic function:



This may also be referred to as the **vertex** of the quadratic, or as the **turning point** – both are general terms to cover both maximum and minimum points.

From Worked example 3.8 you know that $x^2 - 8x + 3 = (x-4)^2 - 13$.

Since $(x-4)^2 \geqslant 0$ for all $(x-4)^2 - 13 \geqslant -13$ for all x.

In other words, the smallest value the function can ever take is -13.

This will occur when $(x-4)^2 = 0$, i.e. when $x = 4$.

So the minimum point of $y = (x-4)^2 - 13$ is $(4, -13)$.

Quadratic functions have a vertical line of symmetry through their turning point.

Fast forward

You will learn about turning points of other functions in Chapter 14.

Key point 3.4

The quadratic $y = (x+p)^2 + q$ has a turning point at $(-p, q)$ and the line of symmetry $x = -p$.

WORKED EXAMPLE 3.11

State the coordinates of the turning points of the functions in Worked examples 3.9 and 3.10.

a $y = x^2 + 5x + 7$

b $y = 2x^2 - 6x + 2$

a From Worked example 3.9: Use the completed square form.

$x^2 + 5x + 7 = \left(x + \frac{5}{2}\right)^2 + \frac{3}{4}$

$\therefore$ coordinates of turning point: $\left(-\frac{5}{2}, \frac{3}{4}\right)$

$y = (x+p)^2 + q$ has a turning point at $(-p, q)$.

b From Worked example 3.10:

Use the completed square form.

$$2x^2 - 6x + 2 = 2\left(x - \frac{3}{2}\right)^2 - \frac{5}{2}$$

$\therefore$ coordinates of turning point: $\left(\frac{3}{2}, -\frac{5}{2}\right)$

The factor of 2 outside the square bracket does not have any effect.

Tip

When a parabola crosses the x-axis, the x-coordinate of the vertex (and the line of symmetry) is halfway between the roots. However, when the parabola does not cross the x-axis, you need to use completing the square to find the vertex.

You can use the coordinates of the vertex to say whether a quadratic graph crosses the x-axis, that is, whether the corresponding quadratic equation has solutions.

Fast forward

In Section 5, you will meet another way of deciding how many solutions a quadratic equation has.

WORKED EXAMPLE 3.12

a Complete the square for $-5x^2 + 6x - 2$.

b Hence explain why the equation $-5x^2 + 6x - 2$ has no real solutions.

a $-5x^2 + 6x - 2 = -5\left(x^2 - \frac{6}{5}x\right) - 2$

Factorise -5 out of the first two terms.

$$= -5\left\{\left(x - \frac{3}{5}\right)^2 - \left(\frac{3}{5}\right)^2\right\} - 2$$

Complete the square inside the bracket.

$$= -5\left(x - \frac{3}{5}\right)^2 + \frac{9}{5} - 2$$

Multiply the -5 back in.

$$= -5\left(x - \frac{3}{5}\right)^2 - \frac{1}{5}$$

b $\left(x - \frac{3}{5}\right)^2$ is always a non-negative number.

Use the fact that squares cannot be negative.

Hence $-5\left(x - \frac{3}{5}\right)^2 \leqslant 0$

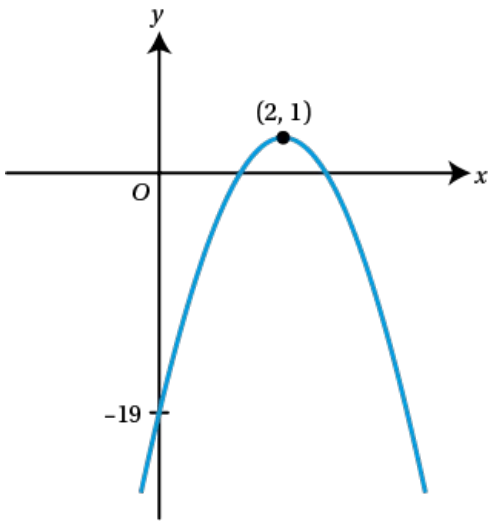
and so $-5\left(x - \frac{3}{5}\right)^2 - \frac{1}{5} \leqslant -\frac{1}{5}$

Therefore $-5x^2 + 6x - 2$ can never equal zero, so the equation has no solutions.

You can also use the completed square form to find the equation of the graph when the coordinates of

WORKED EXAMPLE 3.13

Find the equation of the following quadratic graph:



Turning point is at $(-2, 1)$ so the function must be of the form $y = (x - 2)^2 + 1$

Since you are given the coordinates of the turning point, use the completed square form.

When $x = 0$, $y = -19$:

$-19 = a(0 - 2)^2 + 1$

Use the other given point, $(0, -19)$ to find a.

$-19 = 4a + 1$

$a = -5$

So the equation is $y = -5(x - 2)^2 + 1$

Give the equation. There's no need to express it in the form $y = ax^2 + bx + c$ here.

EXERCISE 3C

1 Write down the coordinates of the vertex of these quadratic functions:

a i $y = (x - 3)^2 + 4$

ii $y = (x - 5)^2 + 1$

b i $y = 2(x - 7)^2 - 1$

ii $y = 3(x - 1)^2 - 5$

c i $y = (x - 1)^2 + 3$

ii $y = (x + 7)^2 - 3$

d i $y = -5(x + 2)^2 - 4$

ii $y = -(x - 1)^2 + 5$

2 Write the following expressions in the form $a(x - k)^2 + h$.

a i $x^2 - 6x + 4$

ii $x^2 + 10x + 21$

b i $x^2 + 4x + 1$

ii $x^2 + 6x - 3$

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c **i** $2x^2 - 12x + 5$

ii $3x^2 + 6x + 10$

d **i** $-x^2 + 2x - 5$

ii $-x^2 - 4x + 1$

e **i** $x^2 + 3x + 1$

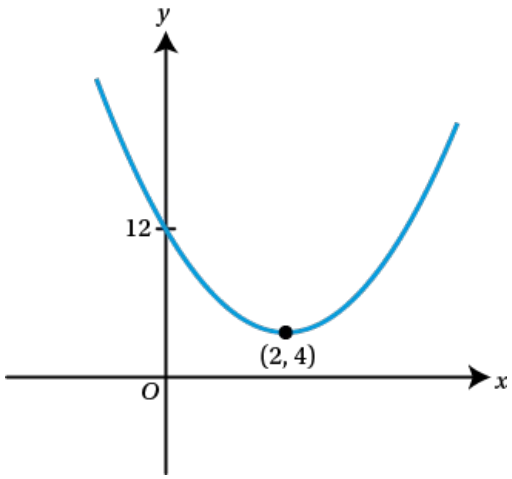
ii $x^2 - 5x + 10$

f **i** $2x^2 + 6x + 15$

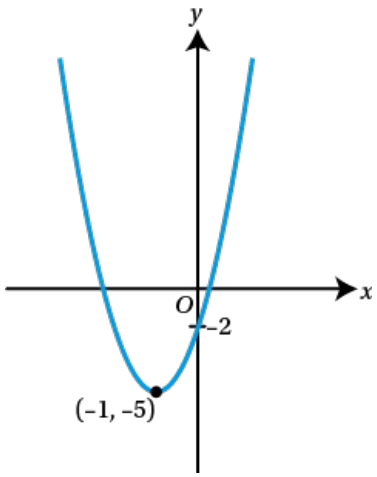
ii $2x^2 - 5x - 1$

3 Find the equation of each graph in the form $y = a(x - k)^2 + h$.

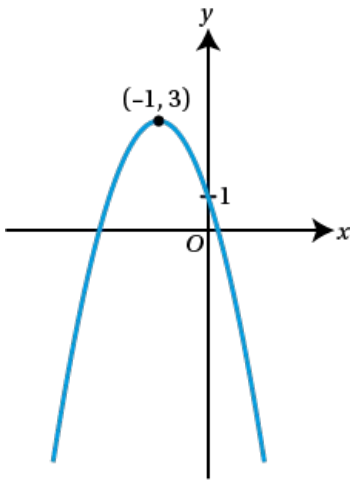
a **i**



ii

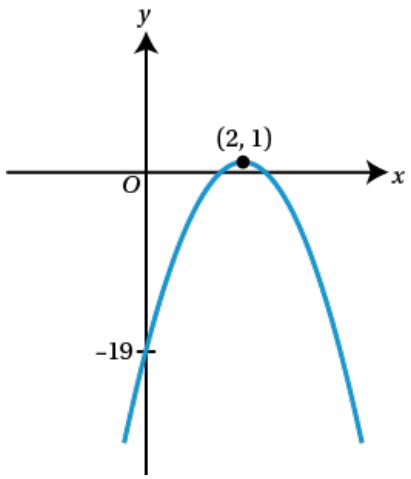


b **i**



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ii



4 $y = x^2 - 6x + 11$

a Write y in the form $(x - a)^2 + b$.

b Find the minimum value of y.

5 The curve $y = a(x + b)^2 + c$ has a minimum point at $(3, 6)$ and passes through the point $(1, 14)$.

a Write down the value of b and c.

b Find the value of a.

6 **a** Express $y = 8x - x^2 - 21$ in the form $a - (x + b)^2$ where a and b are integers to be found.

b Write down the coordinates of the turning point of the graph.

c Hence explain why the equation $8x - x^2 - 21 = 0$ has no real roots.

7 **a** Write $2x^2 + 4x - 1$ in the form $a(x - p)^2 + q$.

b Hence find the exact solutions of the equation $2x^2 + 4x - 1 = 0$.

c Sketch the graph of $y = 2x^2 + 4x - 1$ clearly giving the coordinates of all axis intercepts and of the minimum point.

d Write down the equation of the axis of symmetry of this curve.

8 **a** Write $x^2 + 6x$ in the form $(x + k)^2 + h$.

b Hence find the range of values of p for which $x^2 + 6x = p$ has at least one real solution.

9 Let $\mathrm{f}(x) = 3x^2 + 2x + 1$.

a Complete the square for $\mathrm{f}(x)$.

b Hence explain why the equation $\mathrm{f}(x) = 0$ has no real solutions.

c Write down the equation of the line of symmetry of the graph of $y = \mathrm{f}(x)$.

10 By writing the left-hand side in the form $a(x + p)^2 + q$, show that the equation $-2x^2 + 8x - 13 = 0$ has no real roots.

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Section 4: Quadratic inequalities

As well as quadratic equations, you can also have quadratic inequalities, for example $x^2 < 144$ or $x^2 - 6x - 7 > 0$.

Key point 3.5

To help solve quadratic inequalities always sketch the graph.

Gateway to A Level

For a reminder of linear inequalities, see Gateway to A Level section D.

WORKED EXAMPLE 3.14

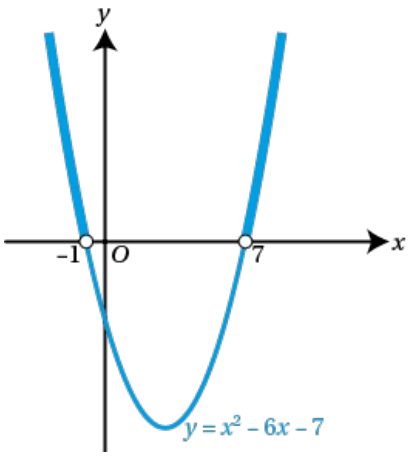
Solve the inequality $x^2 - 6x - 7 > 0$.

$x^2 - 6x - 7 > 0$

$(x - 7)(x + 1 = 0)$

$x = 7$ or -1

To sketch the graph you need the roots of the equation.



Sketch $y = x^2 - 6x - 7$. You want the part where the graph is positive $(y > 0)$

$\therefore x < -1$ or $x > 7$

There are two parts of the graph that give the required values of x, so you need to write two inequalities.

As with quadratic equations, you might need to rearrange the inequality first.

WORKED EXAMPLE 3.15

Solve the inequality $5 + 3x - 2x^2 > 1 - 4x$.

$5 + 3x - 2x^2 > 1 - 4x$

$0 > 2x^2 - 7x - 4$

Rearrange: it is easiest to make the coefficient of x^2 positive.

$2x^2 - 7x - 4 < 0$

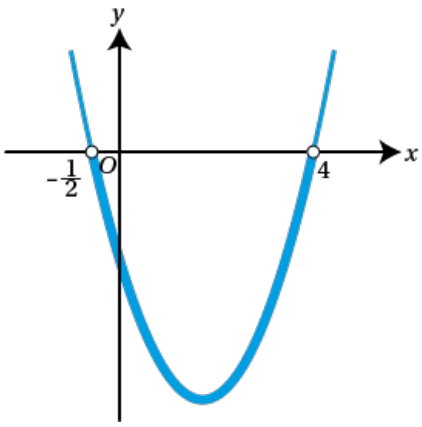
Swap sides so that the expression is on the left.

$2x^2 - 7x - 4 = 0$

$(2x + 1)(x - 4) = 0$

$x = -\frac{1}{2}$ or 4

To find the intercepts of the graph, we need to find where the expression on the left equals zero.



Sketch $y = 2x^2 - 7x - 4$.
You want the part where the graph is negative $(y < 0)$

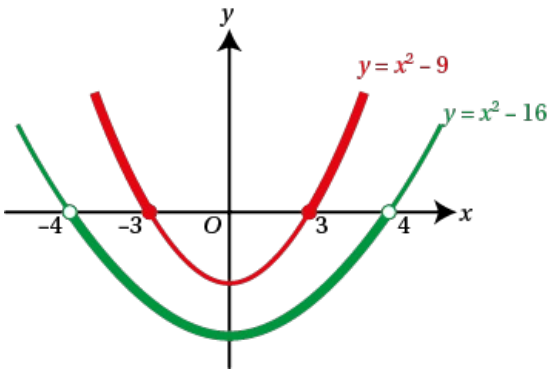
$\therefore -\frac{1}{2} < x < 4$

There is just one part of the graph that gives the required values of x, so write one inequality.

You can illustrate these inequalities on a number line. This is particularly useful if there is more than one inequality.

WORKED EXAMPLE 3.16

Solve simultaneously $x^2 < 16$ and $x^2 \geqslant 9$.



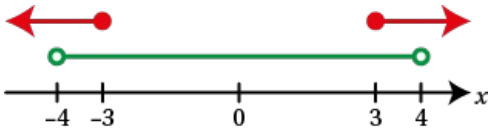
Sketch $y = x^2 - 16$ and $y = x^2 - 9$.

$\therefore -4 < x < 4$

and

$x \leqslant -3$ or $x \geqslant 3$

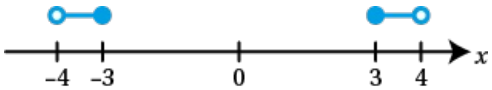
Give the solutions of each.



The solution to the simultaneous inequalities is the region covered by both the red and the green line.

So $x \in (-4, -3] \cup [3, 4)$

Look for the sections of the number line covered by both solutions.



You may need to look back at Chapter 1 for a reminder about interval notation.

EXERCISE 3D

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1 Solve the following quadratic inequalities:

a **i** $x^2 \leqslant 8$

ii $x^2 < 5$

b **i** $x^2 > 6$

ii $x^2 \geqslant 12$

c **i** $(x-4)(x+1) > 0$

ii $(2x-5)(3x+2) < 0$

d **i** $(3-x)(x+1) < 0$

ii $(4-x)(x-2) > 0$

e **i** $(3-x)(12-x) > 0$

ii $(2-x)(-2-x) < 0$

2 Solve the following inequalities. Write your answers using interval notation.

a **i** $x^2 - 5x + 6 < 0$

ii $x^2 + x - 6 < 0$

b **i** $x^2 - 4x - 12 \geqslant 0$

ii $x^2 + 7x + 6 \geqslant 0$

c **i** $2x^2 + x > 6$

ii $3x^2 - x > 10$

d **i** $2x^2 + 3x - 5 \leqslant 0$

ii $5x^2 + 6x + 1 \leqslant 0$

3 Solve the inequality $2x^2 > 6 - x$.

4 Find the set of values of x for which $2x^2 + 3x + 1 \leqslant 11 + 4x - x^2$.

5 A ball is thrown upwards and its height h m, at time t s, is given by $h = 7t - 4.9t^2$. How long does the ball spend more than 1.5 m above ground?

6 **a** Solve the following inequalities:

i $7x - 5 < 3x + 5$

ii $2x^2 - 7x < 4x - 5$

b Hence find the set of value of x for which both $7x - 5 < 3x + 5$ and $2x^2 - 7x < 4x - 5$.

7 Solve simultaneously $x^2 + 6 > 5x$ and $x^2 \geqslant 1$.

8 Find the range of values of x for which both $2x^2 \geqslant 4x$ and $5x^2 - 13x - 6 \leqslant 0$.

9 The cost of producing n items is $£(950 + 63n)$. The items can be sold for $£(280 + 5n)$ **per item**. How many items can be produced and sold in order to make a profit? Give your answer in the form $M \leqslant n \leqslant N$ where M and N are both integers.

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Section 5: The discriminant

If you try to apply the quadratic formula to find solutions of $x^2 - 3x + 3 = 0$, you get the following result:

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times 3}}{2} = \frac{3 \pm \sqrt{-3}}{2}$$

As the square root of a negative number is not a real number, it follows that the expression has no real roots.

This will clearly happen whenever the expression inside the square root, $b^2 - 4ac$, is negative.

Similarly, if $b^2 - 4ac = 0$, then the quadratic formula becomes

$$x = \frac{-b \pm \sqrt{0}}{2a} = \frac{-b}{2a}$$

and there is just the one root at $x = -\frac{b}{2a}$.

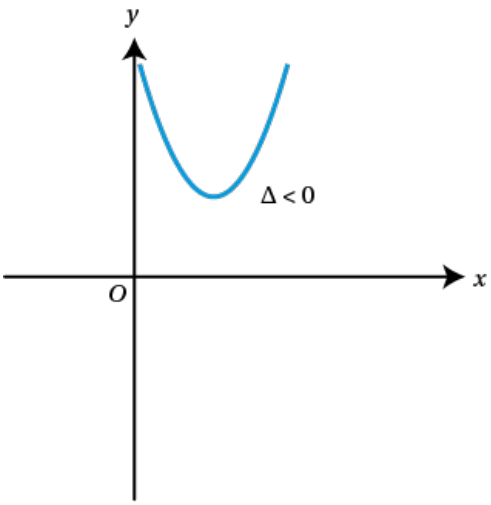
In all other cases there will be two real roots.

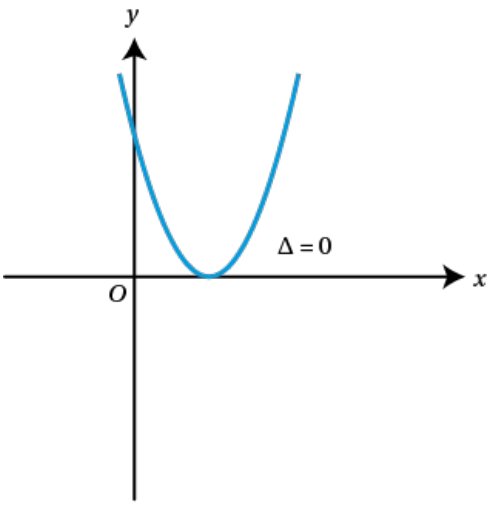
The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic (often symbolised by the Greek letter Δ, which is capital delta).

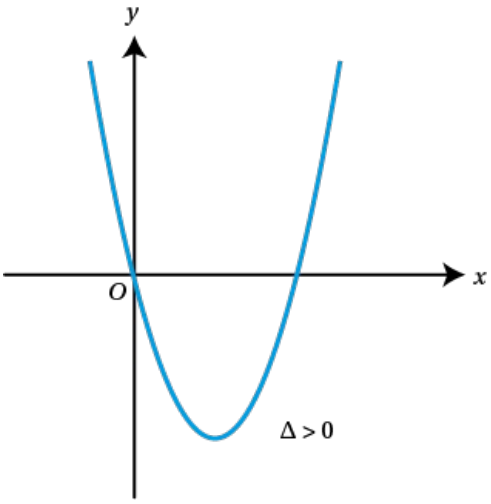
These graphs are examples of the three possible situations.

Fast forward

If you study Further Mathematics, in Pure Core Student Book 1 you will meet a new type of number, called an imaginary number, which makes it possible to find roots of functions like this.







Key point 3.6

For the quadratic equation $ax^2 + bx + c = 0$:

- if $\Delta < 0$ the equation has no real roots
- if $\Delta = 0$ the equation has one (repeated) root
- if $\Delta > 0$ the equation has two distinct real roots

where $\Delta = b^2 - 4ac$ is the discriminant.

Tip

Repeated roots can also be referred to as equal roots.

Tip

Questions of this type often lead to a quadratic equation or inequality for k.

WORKED EXAMPLE 3.17

Find the exact values of k for which the quadratic equation $kx^2 - (k+2)x + 3 = 0$ has a repeated root.

$b^2 - 4ac = 0$ — Repeated root means that $b^2 - 4ac = 0$

$(k+2)^2 - 4(k)(3) = 0$ — $a = k, b = -(k+2), c = 3$

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$k^2 + 4k + 4 - 12k = 0$

This is a quadratic equation in k.

$k^2 - 8k + 4 = 0$

$$k = \frac{8 \pm \sqrt{8^2 - 4 \times 4}}{2}$$
$$= \frac{8 \pm \sqrt{48}}{2}$$
$$= \frac{8 \pm 4\sqrt{3}}{2}$$
$$= 4 \pm 2\sqrt{3}$$

If you can't factorise it use the quadratic formula.

WORKED EXAMPLE 3.18

Find the set of values of k for which the equation $2x^2 - (k+1)x + 5 - k = 0$ has two distinct real solutions.

$b^2 - 4ac > 0$

Two distinct real root $\Rightarrow b^2 - 4ac > 0$.

$(k+1)^2 - 4(2)(5-k) > 0$

$a = 2, b = -(k+1), c = 5 - k$

$k^2 + 2k + 1 - 40 + 8k > 0$

$k^2 + 10k - 39 > 0$

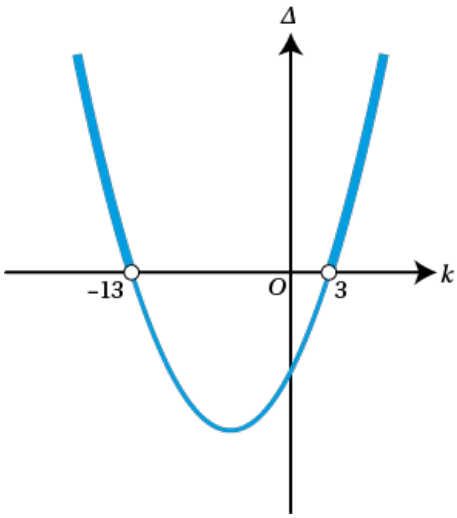
This is a quadratic inequality in k.

$k^2 + 10k - 39 = 0$

$(k+13)(k-3) = 0$

$k = -13, 3$

Solve the equation $k^2 + 10k - 39 = 0$.



You want the region where $\Delta > 0$

$\therefore k < -13$ or $k > 3$

State the range of values of k as required.

Tip

Note that the graph drawn in Worked example 3.18 here isn't the graph of the original quadratic expression (in the variable x) – it's the graph of a quadratic in k. You haven't solved the original equation, you've solved a quadratic inequality for k that ensures there are two distinct solutions for x in the original equation.

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When $\Delta < 0$, the graph does not intersect the x-axis, so it is either entirely above or entirely below it. The two cases are distinguished by the value of a.

Key point 3.7

For a quadratic function with $\Delta < 0$:

- if $a > 0$ then $y > 0$ for all x
- if $a < 0$ then $y < 0$ for all x.

WORKED EXAMPLE 3.19

$y = -3x^2 + kx - 12$

Find the values of k for which $y < 0$ for all x.

No real roots, $\therefore \Delta < 0$

y is a negative quadratic. $y < 0$ means that the graph is entirely below the x-axis. This will happen when $f(x) = 0$ has no real roots.

$b^2 - 4ac < 0$

$k^2 - 4(-3)(-12) < 0$

$a = -3, b = k, c = -12.$

$k^2 - 144 < 0$

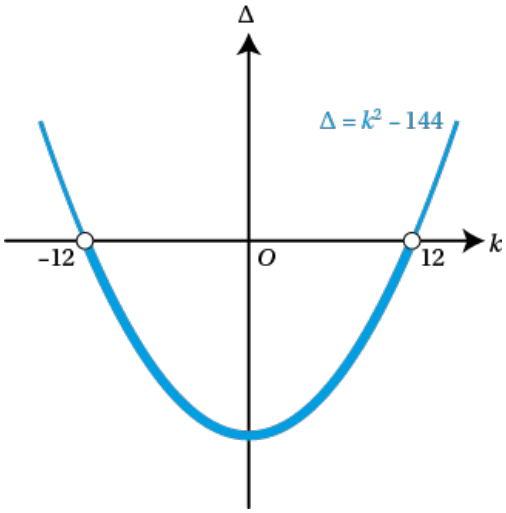
This is a quadratic inequality in k

$k^2 - 144 = 0$

$(k - 12)(k + 12) = 0$

$k = -12, 12$

Solve the equation $k^2 - 144 = 0$.



You want the region where $\Delta > 0$

$\therefore -12 < k < 12$

State the range of values of k as required.

EXERCISE 3E

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1 Evaluate the discriminant of the following quadratic equations:

a i $x^2 + 4x - 5 = 0$

ii $x^2 - 6x - 8 = 0$

b i $2x^2 + x + 6 = 0$

ii $3x^2 - x + 10 = 0$

c i $3x^2 - 6x + 3 = 0$

ii $9x^2 - 6x + 1 = 0$

d i $12 - x - x^2 = 0$

ii $-x^2 - 3x + 10 = 0$

2 State the number of solutions for each equation from question 1.

3 Find the set of values of k for which:

a i the equation $2x^2 - x + 3k = 0$ has two distinct real roots

ii the equation $3x^2 + 5x - k = 0$ has two distinct real roots

b i the equation $5x^2 - 2x + (2k - 1) = 0$ has equal roots

ii the equation $2x^2 + 3x - (3k + 1) = 0$ has equal roots

c i the equation $-x^2 + 3x + (k + 1) = 0$ has real roots

ii the equation $-2x^2 + 3x - (2k + 1) = 0$ has real roots

d i the equation $3kx^2 - 3x + 2 = 0$ has no real solutions

ii the equation $-kx^2 + 5x + 3 = 0$ has no real solutions

e i the quadratic expression $(k - 2)x^2 + 3x + 1$ has a repeated root

ii the quadratic expression $-4x^2 + 5x + (2k - 5)$ has a repeated root

Worksheet

For a further example of this type and more practice questions, see Support sheet 3.

f i the graph of $y = x^2 - 4x + (3k + 1)$ is tangent to the $x - axis$

ii the graph of $y = -2kx^2 + x - 4$ is tangent to the $x - axis$

g i the expression $-3x^2 + 5k$ has no real roots

ii the expression $2kx^2 - 3$ has no real roots

4 Find the values of parameter m for which the quadratic equation $mx^2 - 4x + 2m = 0$ has equal roots.

5 Find the exact values of k such that the equation $-3x^2 + (2k + 1)x - 4k = 0$ has a repeated root.

6 Find the range of values of the parameter c such that $2x^2 - 3x + (2c + 1) \geqslant 0$ for all x.

7 Find the set of values of k for which the equation $x^2 - 2kx + 6k = 0$ has no real solutions.

8 Find the range of values of k for which the quadratic equation $kx^2 - (k + 3)x - 1 = 0$ has no real roots.

9 Find the range of values of m for which the equation $mx^2 + mx - 2 = 0$ has one or two real roots.

10 Find the possible values of m such that $mx^2 + 3x - 4 < 0$ for all x.

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Section 6: Disguised quadratics

You will often meet equations that can be turned into quadratics by making a substitution.

WORKED EXAMPLE 3.20

Solve the equation $x^4 - 3x^2 - 4 = 0$.

Let $y = x^2$	A substitution $y = x^2$ turns this into a quadratic equation since $x^4 = y^2$.
$y^2 - 3y - 4 = 0$	
$(y + 1)(y - 4) = 0$	This is now a standard quadratic equation.
$y = -1$ or $y = 4$	
$x^2 = -1$ (reject)	Use the substitution to find x.
or $x^2 = 4$	Note that some values of y will not lead to a corresponding value of x, since square numbers must be positive.
$x = 2$ or -2	

Other substitutions may not be so clear. In particular, it is quite common to be given an **exponential equation**, an equation with the 'unknown' variable in the power, which needs a substitution. Look out for an a^x and an a^{2x}.

WORKED EXAMPLE 3.21

Solve the equation $4^x - 10 \times 2^x + 16 = 0$.

$4^x - 10 \times 2^x + 16 = 0$	$4^x = (2^2) = 2^{2x}$
$2^{2x} - 10 \times 2^x + 16 = 0$	
Let $y = 2x$	A substitution $y = 2x$ turns this into a quadratic equation since $2^{2x} = (2^x)^2$
$y^2 - 10y + 16 = 0$	This is now a standard quadratic equation.
$(y - 2)(y - 8) = 0$	
$y = 2$ or 8	
$2^x = 2$	Use the substitution to find x
$x = 1$	
or	
$2^x = 8$	
$x = 3$	

Fast forward

You will see how to solve more complicated exponential equations in Chapter 7.

EXERCISE 3F

In this exercise, you must show detailed reasoning.

1 Solve the following equations, giving your answers in an exact form.

a i $a^4 - 10a^2 + 21 = 0$

ii $x^4 - 7x^2 + 12 = 0$

b i $2x^6 + 7x^3 = 15$

ii $a^6 + 7a^3 = 8$

c i $x^2 - 4 = \dfrac{2}{x^2}$

ii $x^2 + \dfrac{36}{x^2} = 12$

d i $x - 6\sqrt{x} + 8 = 0$

ii $x - 10\sqrt{x} + 24 = 0$

e i $3^{2x} - 12 \times 3^x + 27 = 0$

ii $2^{2x} - 17 \times 2^x + 16 = 0$

Tip

'Detailed reasoning' means that your solution must involve algebraic rearrangement and not, for example, the solver feature on your calculator.

2 By letting $y = \sqrt{x}$, solve the equation $x - \sqrt{x} - 6 = 0$.

3 Use an appropriate substitution to solve $x^2 + \dfrac{9}{x^2} = 10$.

4 Use a suitable substitution to solve the equation $\dfrac{1}{(3x+1)^2} + 5 = \dfrac{6}{3x+1}$.

5 Use an appropriate substitution to solve $x^3 - 9x^{1.5} + 8 = 0$.

6 Solve the equation $9(1 + 9^{x-1}) = 10 \times 3^x$.

7 Solve the equation $5^x = 6 - 5^{1-x}$.

8 Solve the equation $\dfrac{3}{(x-1)^2} - \dfrac{5}{x-1} = 2$.

9 **a** Write $x^4 - 4x^3 - 7x^2$ in the form $(x^2 - px)^2 - qx^2$.

b Hence use a substitution of the form $y = x^2 - px$ to solve the equation $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$.

10 Solve the equation $4^{x+0.5} - 17 \times 2^x + 8 = 0$.

11 Solve $x = \sqrt{x} + 12$.

Checklist of learning and understanding

- Quadratic functions have the general form $f(x) = ax^2 + bx + c$. The main features are summarised in this table:

Feature	What to look at	Conclusion
Overall shape	The sign of a.	

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y-intercept	The value of c.	y-intercept $(0, c)$
Turning point	Completed square form: $y = a(x + p)^2 + q$	Turning point $(-p, q)$
Line of symmetry	Completed square form: $a(x + p)^2 + q$ OR x-intercepts $x_{1,}x_2$	Line of symmetry $x = -p = \frac{x_1 + x_2}{2}$
x-intercepts	Factorise $\mathrm{f}(x) = 0$ or use the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Roots x_1 and x_2 x-intercepts $(x_1, 0)$ and $(x_2, 0)$
The number of real roots	Discriminant: $\Delta = b^2 - 4ac$	Two distinct roots: $\Delta > 0$ One root (equal roots, repeated root): $\Delta = 0$ No real roots: $\Delta < 0$

- To solve quadratic inequalities, rearrange to make one side zero and sketch the graph.
- A substitution can transform an equation into a quadratic equation.

Mixed practice 3

In this exercise, you must show detailed reasoning.

1. A quadratic function passes through the points $(k, 0)$ and $(k + 4, 0)$. Find the x-coordinate of the vertex of the graph of the function.

2. Solve algebraically:

   $(2x - 3)(x - 5) = (x - 3)^2$

3. Solve $x^4 - 5x^2 + 4 = 0$.

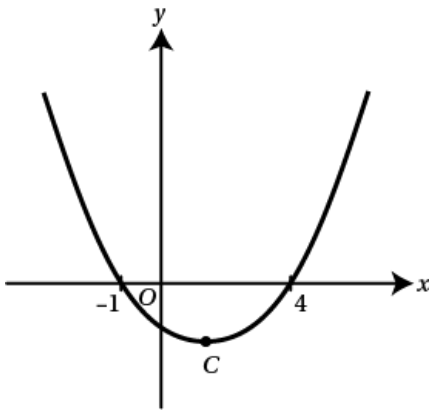
4. The quadratic function $y = (x - a)^2 + b$ has a turning point at $(3, 7)$.

   a State whether this turning point is a maximum or a minimum point.

   b State the values of a and b.

5. The quadratic function $y = a(x - b)^2 + c$ passes through the points $(-2, 0)$ and $(6, 0)$. Its maximum y value is 48. Find the values of a, b and c.

6. The diagram represents the graph of the function $\mathrm{f}(x) = (x + p)(x - q)$.



   a Write down the values of p and q if they are both positive.

   b The function has a minimum value at the point C. Find the C-coordinate of C.

7. **a** Find the discriminant of $kx^2 - 4x + k$ in terms of k.

   b The quadratic equation $kx^2 - 4x + k = 0$ has equal roots. Find the possible values of k.

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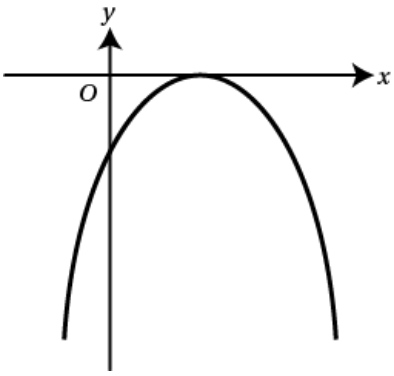
Tip

This means that your solution must involve algebraic rearrangement and not, for example, the solver feature on your calculator.

8. Solve simultaneously $x^2 - 2x > 0$ and $x^2 - 4x + 3 \geqslant 0$.

9. The diagram shows the graph of the function $y = ax^2 + bx + c$.

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Copy and complete this table to show whether each expression is positive, negative or zero.

Expression	Positive	Negative	Zero
a			
c			
$b^2 - 4ac$			
b			

10 **a** Write $x^2 - 10x + 35$ in the form $(x - p)^2 + q$.

b Hence, or otherwise, find the maximum value of $\dfrac{1}{(x^2 - 10x + 35)^3}$.

11 Find the exact values of k for which the equation $2kx^2 + (k+1)x + 1 = 0$ has no real roots.

12 Solve the equation: $x^{\frac{1}{4}} + 2x^{-\frac{1}{4}} = 3$.

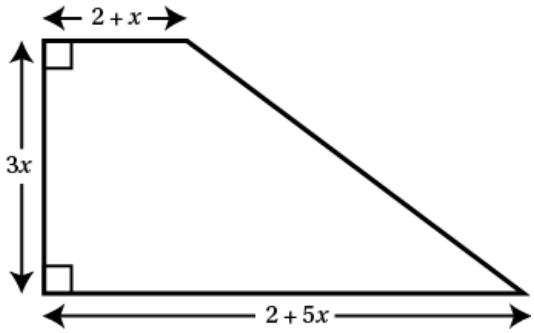
13 Solve the equation $\dfrac{49}{(5x+2)^2} - \dfrac{14}{5x+2} + 1 = 0$.

14 **a** Express $2x^2 - 6x + 9$ in the form $p(x+q)^2 + r$.

b State the coordinates of the vertex of the curve $y = 2x^2 - 6x + 9$.

c State the number of real roots of the equation $y = 2x^2 - 6x + 9 = 0$.

15 A lawn is to be made in the shape shown. The units are metres.



a The perimeter of the lawn is $P\,\text{m}$. Find P in terms of x.

b Show that the area, $A\,\text{m}^2$, of the lawn is given by $A = 9x^2 + 6x$.

The perimeter of the lawn must be at least $39\,\text{m}$ and the area of the lawn must be less than $99\,\text{m}^2$.

c By writing down and solving appropriate inequalities, determine the set of possible values of x.

OCR, GCE Mathematics, Paper 4721, January 2010 [Question part reference style adapted]

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16 Alexia and Michaela were both trying to solve a quadratic equation of the form $x^2 + bx + c = 0$.

Unfortunately Alexia misread the value of b and found that the solutions were 6 and 1.

Michaela misread the value of c and found that the solutions were 4 and 1.

What were the correct solutions?

17 Find the values of k for which the line $y = 2x - k$ is tangent to the curve with equation $x^2 + y^2 = 5$.

18 Let α and β denote the roots of the quadratic equation $x^2 - kx + (k - 1) = 0$.

a Express α and β in terms of the real parameter k.

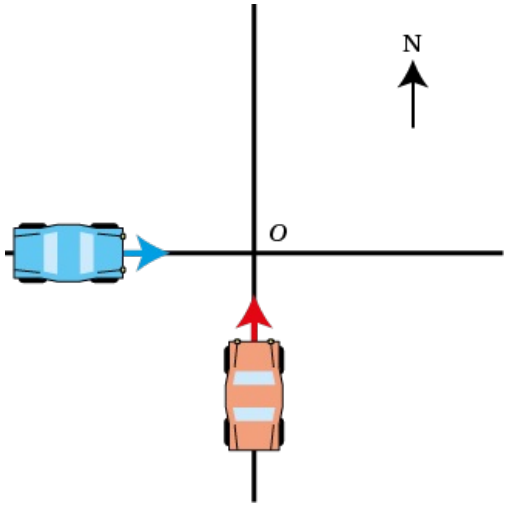
b Given that $\alpha^2 + \beta^2 = 17$, find the possible values of k.

Tip

α and β are Greek letters pronounced alpha, and beta, respectively. These are the lowercase forms of the letters.

19 Let $\mathrm{q}(x) = kx^2 + (k - 2)x - 2$. Show that the equation $\mathrm{q}(x) = 0$ has real roots for all values of k.

20 Two cars are travelling along two straight roads that are perpendicular to each other and meet at the point O, as shown in the diagram. The first car starts 50 km west of O and travels east at the constant speed of 20 km/h. At the same time, the second car starts 30 km south of O and travels north at the constant speed of $15\ \mathrm{km/h}$.



a Show that at time t (hours) the distance d (km) between the two cars satisfies $d2 = 625t^2 - 2900t + 3400$.

b Hence find the closest distance between the two cars.

Worksheet

For a selection of more challenging problems, see Extension sheet 3.

4 Polynomials

In this chapter you will learn how to:

- define a polynomial
- find the product of two polynomials
- find the quotient of two polynomials
- quickly find factors of a polynomial
- sketch polynomials.

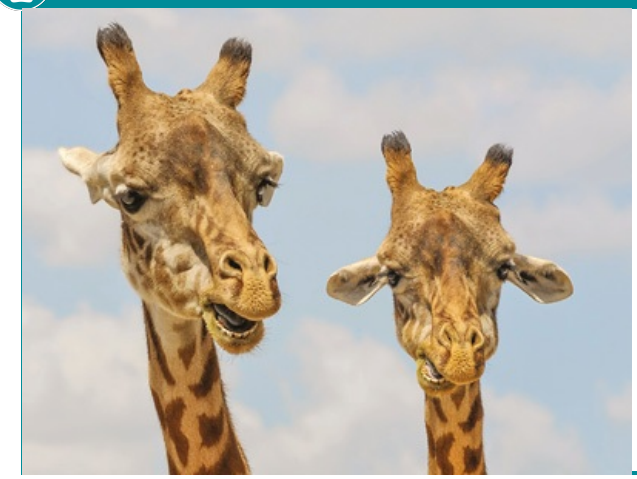
Before you start...

Chapter 2	You should know how to work with indices.	1 Simplify $x^2 \times x^4$.
GCSE	You should know how to multiply out brackets and collect like terms.	2 Expand and simplify $(2x+1)(x-3)$.
GCSE	You should know how to factorise quadratic expressions.	3 Factorise $x^2 - 8x + 15$.
GCSE	You should know how to solve quadratic equations using the quadratic formula.	4 Solve $x^2 + 4x + 2 = 0$.

Why study polynomials?

In Chapter 3 you were introduced to various properties of quadratic functions. As well as their mathematical interest, quadratic functions are used to model many real-world situations, such as the path of a projectile. To include more real-world situations you can extend quadratics to include terms in x^3, x^4 and so on. For example, the relationship between height and mass may be modelled using a cubic equation. This group of functions, called polynomials, turns out to be remarkably powerful.

Did you know?



Polynomial functions can be used to predict the mass of an animal from its length.

Fast forward

A It turns out that many other functions, such as sin x or e^x, can be approximated by (their

graphs can closely match to) particular polynomials, as you will learn if you study Further Mathematics, in Pure Core Student Book 2.

Section 1: Working with polynomials

A polynomial is a function made up of a sum of terms containing non-negative (positive or zero) integer powers of an unknown, such as x. Polynomial functions are classified according to the highest power of the unknown (x) occurring in the function. This is called the **degree** of the polynomial.

Key point 4.1

General form of the function	Degree	Name	Example
$\mathrm{f}(x) = a$	0	constant function	$y = 5$
$\mathrm{f}(x) = ax + b$	1	linear function	$y = x + 7$
$\mathrm{f}(x) = ax^2 + bx + c$	2	quadratic function	$y = 3x^2 + 4x - 1$
$\mathrm{f}(x) = ax^3 + bx^2 + cx + d$	3	cubic function	$y = 2x^3 + 7x$
$\mathrm{f}(x) = ax^4 + bx^3 + cx^2 + dx + e$	4	quartic function	$y = x^4 + x^3 + 2x + \frac{1}{2}$

Did you know?



The Greeks had methods to solve quadratic equations, and the formulae for solving cubic and quartic equations were developed in 16th-century Italy. For over three hundred years, nobody was able to come up with a general solution to the quintic equation until, in 1821, the Norwegian mathematician Niels Abel (pictured) used a branch of mathematics called group theory to prove that there could never be a 'quintic formula'.

The letters a, b, c etc. in Key point 4.1 are called the **coefficients** of the powers of x. The coefficient of the highest power of x in the function (given by a in Key point 4.1) is called the **lead coefficient** and the term containing the highest power of x is the **leading order term**.

Coefficients can take any value, with the restriction that the lead coefficient cannot equal zero (a polynomial of order n with a lead coefficient 0 is in fact a polynomial of order at most $n - 1$).

You should already be familiar with adding and subtracting two polynomials. To multiply two polynomials, you need to expand the brackets and collect like terms. Worked example 4.1 is one suggested way of setting out polynomial multiplication to ensure that you include all of the terms.

Gateway to A Level

See Gateway to A Level section A for a reminder of expanding Brackets.

WORKED EXAMPLE 4.1

Expand $(x^3 + 3x^2 - 2)(x^2 - 5x + 4)$.

$$(x^3 + 3x^2 - 2)(x^2 - 5x + 4)$$
$$= x^3(x^2 - 5x + 4) + 3x^2(x^2 - 5x + 4) - 2(x^2 - 5x + 4)$$
$$= x^5 - 5x^4 + 4x^3$$
$$+3x^4 - 15x^3 + 12x^2$$
$$-2x2 + 10x - 8$$
$$= x^5 - 2x^4 - 11x^3 + 10x^2 + 10x - 8$$

Multiply each term inside the first bracket, in turn, by the whole of the expression in the second bracket.

Then collect like terms.

EXERCISE 4A

1 Decide whether each of the following expressions are polynomials. For those that are polynomials give the degree and the lead coefficient.

a $3x^3 - 3x^2 + 2x$

b $1 - 3x - x^5$

c $5x^2 - x^{-3}$

d $9x^4 - \frac{5}{x}$

e $4e^x + 3e^{2x}$

f $x^4 + 5x^2 \ \ 3\sqrt{x}$

g $4x^5 - 3x^3 + 2x^7 - 4$

h 1

2 Expand and simplify the brackets for the following expressions.

a i $(3x - 2)(2x^2 + 4x - 7)$

ii $(3x + 1)(x^2 + 5x + 6)$

b i $(2x + 1)(x^3 - 8x^2 + 6x - 1)$

ii $(2x + 5)(x^3 - 6x^2 + 3)$

c i $(b^2 + 3b - 1)(b^2 - 2b + 4)$

ii $(r^2 - 3r + 7)(r^2 - 8r + 2)$

d i $(5 - x^2)(x^4 - 2x^3 + 1)$

ii $(x - x^3)(x^3 - x - 1)$

3 In what circumstances might you want to expand brackets? In what circumstances is the factorised form better?

4 **a** Is it always true that the sum of a polynomial of degree n and a polynomial of degree $n - 1$ has degree n?

b Is it always true that the sum of a polynomial of degree n and a polynomial of degree n has degree n?

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Section 2: Polynomial division

From Worked example 4.1 you know that:

$$x^5 - 2x^4 - 11x^3 + 10x^2 + 10x - 8 = (x^2 - 5x + 4)(x^3 + 3x^2 - 2)$$

which can also be written as:

$$\frac{x^5 - 2x^4 - 11x^3 + 10x^2 + 10x - 8}{x^2 - 5x + 4} = x^3 + 3x^2 - 2$$

However, if you had not just done the multiplication it is unlikely that you would have been able to spot this.

Tip

The resultant polynomial, when one polynomial is divided by another, is called the 'quotient'. In this example, $x^3 + 3x^2 - 2$ is the quotient.

One way to do this is by polynomial long division.

1. Divide the leading-order term in the numerator by the leading-order term in the denominator. This is the leading-order term of the answer.
2. Multiply this term by the whole denominator. Subtract the resulting expression from the numerator.
3. Repeat this process until all terms have been accounted for.

Rewind

Another common method is to compare coefficients – this is very similar to the process shown in Chapter 1, Worked example 1.1.

There are several ways to set this process out, but the traditional way is given in Worked example 4.2.

WORKED EXAMPLE 4.2

Given that $(x - 4)$ is a factor of $x^3 - x^2 - 11x - 4$, find the other factor.

Divide the polynomial by $(x - 4)$

The leading term in the denominator is x.

$$\begin{array}{r} x^2 + 3x + 1 \\ (x-4)\overline{)\,x^3 - x^2 - 11x - 4} \\ \underline{x^3 - 4x^2} \\ 3x^2 - 11x - 4 \\ \underline{3x^2 - 12x} \\ x - 4 \\ \underline{x - 4} \\ 0 \end{array}$$

This is the final answer – it gets written down in stages:
To get x^2 we divide the leading term by x.
This is x^2 multiplied by the denominator.
These are the remaining terms. To get $+3x$ we divide $3x^2$ by x.
This is $+3x$ multiplied by the denominator.
These are the remaining terms. To get $+1$ we divide x by x.
This is $+1$ multiplied by the denominator.
There is nothing left over.

Hence the other factor is $x^2 + 3x + 1$, and so:

$$x^3 - x^2 - 11x - 4 = (x - 4)(x^2 + 3x + 1)$$

EXERCISE 4B

1 You are given one factor of each polynoimal. Use polynomial division to find the other factor.

a i $(x+4)$ is a factor of $x^3-4x^2-35x-12$

ii $(x+3)$ is a factor of x^3+x^2-3x+9

b i $(x+1)$ is one factor of x^3+x^2-2x-2

ii $(x+4)$ is one factor of $x^3+4x^2+3x+12$

c i $(2x+5)$ is one factor of $2x^3+x^2-6x+10$

ii $(3x-2)$ is one factor of $3x^3-17x^2+16x-4$

d i $(x-7)$ is one factor of $x^4-12x^3+38x^2-23x+14$

ii $(x-8)$ is one factor of $2x^4-16x^3+3x^2-19x-40$

2 Use polynomial division to simplify the following expressions.

a i $\dfrac{x^3-3x^2-13x-30}{x^2+3x+5}$

ii $\dfrac{x^3+5x^2-5x+63}{x^2-2x+9}$

b i $\dfrac{x^4+7x^3+13x^2+2x-2}{x^2+3x-1}$

ii $\dfrac{x^4+3x^3-32x^2-17x+3}{x^2-4x-3}$

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Section 3: The factor theorem

Algebraic division allows you to factorise a polynomial if you know one factor; but finding the first factor can be difficult. To do this you use something called the **factor theorem**.

Key point 4.2

The factor theorem states that:

$(x - a)$ is a factor of $\mathrm{f}(x)$ if and only if $\mathrm{f}(a) = 0$.

Focus on ...

This is proved in Focus on ... Proof 1.

Notice that the factor theorem says two things:

- If $(x - a)$ is a factor of $\mathrm{f}(x)$ then $\mathrm{f}(a) = 0$.

and

- If $\mathrm{f}(a) = 0$ then $(x - a)$ is a factor of $\mathrm{f}(x)$.

WORKED EXAMPLE 4.3

Show that $(x + 3)$ is a factor of $\mathrm{f}(x) = 2x^3 + x^2 - 9x + 18$.

$$\mathrm{f}(-3) = 2 \times (-3)^3 + (-3)^2 - 9 \times (-3) + 18$$
$$= -54 + 9 + 27 + 18 = 0$$

By the factor theorem, if $\mathrm{f}(-3) = 0$ then $(x + 3)$ is a factor.

Therefore $(x + 3)$ is a factor of $\mathrm{f}(x)$.

A slightly more advanced version of the factor theorem is given in Key point 4.3.

Key point 4.3

$(ax - b)$ is a factor of $\mathrm{f}(x)$ if and only if $\mathrm{f}\left(\frac{b}{a}\right) = 0$.

WORKED EXAMPLE 4.4

Show that $(2x - 3)$ is a factor of $\mathrm{f}(x) = 6x^3 - 9x^2 + 8x - 12$.

$$\mathrm{f}\left(\frac{3}{2}\right) = 6 \times \left(\frac{3}{2}\right)^2 - 9\left(\frac{3}{2}\right)^2 + 8 \times \frac{3}{2} - 12$$
$$= \frac{81}{4} - \frac{81}{4} + 12 - 12 = 0$$

By the factor theorem, if $\mathrm{f}\left(\frac{3}{2}\right) = 0$ then $(2x - 3)$ is a factor.

Therefore $(2x - 3)$ is a factor of $\mathrm{f}(x)$.

WORK IT OUT 4.1

Find the value of a such that $(3x + 5)$ is a factor of $\mathrm{f}(x) = 6x^3 + 19x^2 + ax - 10$.

Only one of these solutions is correct. Identify the errors in the incorrect solutions.

Solution 1

$\mathrm{f}(-5) = 0$:

$-750 + 475 - 5a - 10 = 0$

$5a = 285$

$a = 57$

Solution 2

$\mathrm{f}\left(\frac{5}{3}\right) = 0$:

$\frac{250}{9} + \frac{475}{9} + \frac{5}{3}a - 10 = 0$

$\frac{5a}{3} = \frac{617}{9}$

$a = \frac{617}{15}$

Solution 3

$\mathrm{f}\left(-\frac{5}{3}\right) = 0$:

$-\frac{250}{9} + \frac{475}{9} - \frac{5}{3}a - 10 = 0$

$\frac{5a}{3} = 15$

$a = 9$

Once one factor has been identified, polynomial division can be used to find the remaining factors, and then find the roots of the polynomial as in Worked example 4.5.

WORKED EXAMPLE 4.5

Solve the equation $\mathrm{f}(x) = x^3 + x^2 - 13x + 14 = 0$.

$\mathrm{f}(0) = 14$

$\mathrm{f}(1) = 3$

$\mathrm{f}(-1) = 27$

$\mathrm{f}(2) = 0$

Factorising would help you solve this equation. The factor theorem might help you find a factor.

Therefore $(x - 2)$ is a factor of $\mathrm{f}(x)$.

$$
\begin{array}{r}
x^2 + 3x - 7 \\
(x-2)\overline{)\,x^3 + x^2 - 13x + 14} \\
x^3 - 2x^2 \\
\hline
3x^2 - 13x + 14 \\
3x^2 - 6x \\
\hline
-7x + 14 \\
-7x + 14 \\
\hline
0
\end{array}
$$

Polynomial division will give you the second factor.

So:

$x^3 + x^2 - 13x + 14 = (x - 2)\left(x^2 + 3x - 7\right)$

$(x-2)(x^2+3x-7)=0$
$\Rightarrow x=2$ or $x^2+3x-7=0$

If the product equals zero then one of the factors is zero.

$\Rightarrow x=2$ or $x=\dfrac{3\pm\sqrt{9+28}}{2}$

The second equation is quadratic.

So the solutions are:

$x=2$ or $x=\dfrac{-3+\sqrt{37}}{2}$ or $x=\dfrac{-3-\sqrt{37}}{2}$

Tip

If the expression is going to factorise nicely then you only need to try numbers that are factors of the constant term.

Tip

When questions mention factors it is often tempting to go straight to polynomial division. Always try the factor theorem first.

A very common type of question asks to find unknown coefficients in an expression if factors are given.

WORKED EXAMPLE 4.6

$f(x)=x^3+4x^2+ax+b$ has factors of $(x-1)$ and $(x+1)$. Find the constants a and b.

$f(1)=0$
$1+4+a+b=0$
$a+b=-5$ (1)

Apply the factor theorem with the factor $(x-1)$. If $(x-1)$ is a factor then $f(1)=0$.

$f(-1)=0$
$-1+4-a+b=0$
$-a+b=-3$ (2)

Apply the factor theorem with the factor $(x+1)$.

$2b=-8$
$b=-4$
$a=-1$ (1) + (2)

Solve the simultaneous equations in a and b.

Worksheet

For a further example of this type and more practice questions, see Support sheet 4.

EXERCISE 4C

1 Decide whether each of the following expressions is a factor of $2x^3-3x^2-3x+2$.

a **i** $x-1$
ii $x+1$
b **i** $x-2$

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ii $x+2$

c **i** $x-\frac{1}{2}$

ii $x+\frac{1}{2}$

d **i** $2x-1$

ii $2x+1$

e **i** $3x-1$

ii $3x+2$

2 Fully factorise the following expressions.

a **i** x^3+2x^2-x-2

ii x^3+x^2-4x-4

b **i** $x^3-7x^2+16x-12$

ii $x^3+6x^2+12x+8$

c **i** $x^3-3x^2+12x-10$

ii $x^3-2x^2+2x-15$

d **i** $6x^3-11x^2+6x-1$

ii $12x^3+13x^2-37x-30$

3 Solve the following equations.

a **i** $x^3+12=2x^2+11x$

ii $x^3-x^2-17x=15$

b **i** $x^3-5x^2+7x-2=0$

ii $x^3-6x^2+7x-2=0$

4 Find the roots of the following equations.

a **i** $x^3-6x^2+11x=6$

ii $x^3-2x^2+6=5x$

b **i** $x^3+x^2-x-1=0$

ii $x^3-3x^2-10x+24=0$

5 **a** Show that $(x-2)$ is a factor of $\mathrm{p}(x)=x^3-3x^2-10x+24$

b Hence express $\mathrm{p}(x)$ as the product of three linear factors and solve $\mathrm{p}(x)=0$.

6 **a** Show that $(x-3)$ is a factor of $\mathrm{p}(x)=x^3-x^2-2x-12$.

b Hence show that $\mathrm{p}(x)=0$ only has one real root.

7 x^3+7x^2+cx+d has factors $(x+1)$ and $(x+2)$. Find the values of c and d.

8 $\mathrm{f}(x)=x^3-ax^2-bx+168$. has factor $(x+7)$ and $(x+3)$.

a Find a and b.

b Find the remaining factor of $\mathrm{f}(x)$.

9 The polynomial $x^2+kx-8k$ has a factor $(x-k)$. Find the possible values of k.

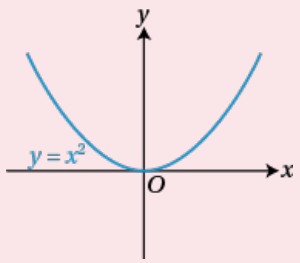
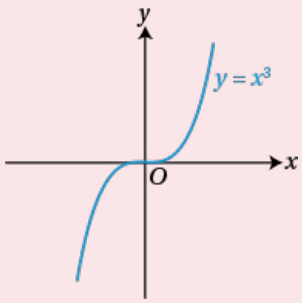
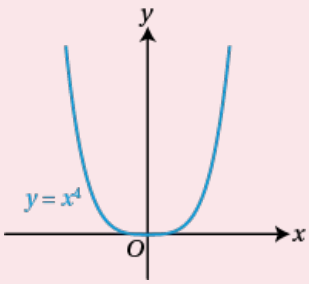
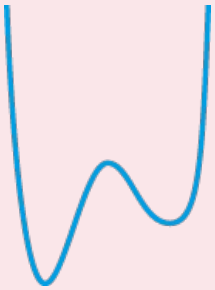
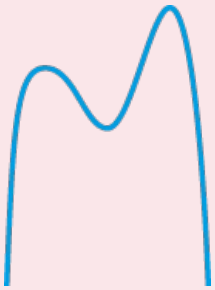
10 The polynomial $x^2-(k+1)x-3$ has a factor $(x-k+1)$. Find k.

11 The polynomial x^2-5x+6 is a factor of $2x^3-15x^2+ax+b$. Find the values of a and b.

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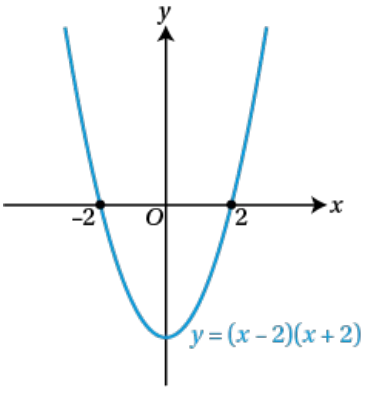
Section 4: Sketching polynomial functions

You need to know the shapes to expect for polynomial functions. The graphs of polynomials of degree zero and one are straight lines. All other polynomial graphs are smooth curves. Just as with quadratic graphs, the shape depends on the sign of the lead coefficient.

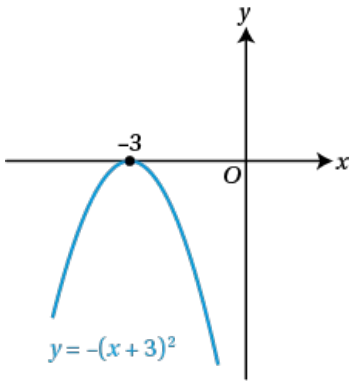
n	$y = x^n$	Positive polynomial	Negative polynomial	x-intercepts	Tur poi
2				0, 1 or 2	1
3				1, 2 or 3	0 or
4				0, 1, 2, 3, or 4	1 or

When a polynomial is given in factorised form, you can easily find the x-intercepts.

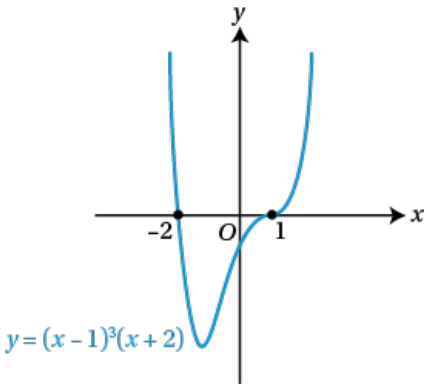
One fact to be aware of is how repeated roots affect the shape of the graph.



If a polynomial has a factor $(x - a)$ then the curve passes through the x-axis at a.



If a polynomial has a double factor $(x-a)^2$ then the curve touches the x-axis at a.



If a polynomial has a triple factor $(x-a)^3$ then the curve passes through the x-axis at a, flattening as it does so.

Fast forward

You do not need to find the coordinates of turning points unless explicitly asked. You will learn how to do this in Chapter 14.

Once you know the x-and y-intercepts you can sketch the graph of a polynomial function. A **sketch** does not need to be to scale, but should show the correct shape and the axis intercepts. If you were asked to **draw** a graph or **plot** some points, you would do so accurately, on graph paper.

Key point 4.4

To sketch the graph of a polynomial function:

1. Decide on the basic shape by considering the order (degree) and the lead coefficient.
2. Set $x = 0$ to find the y-intercept.
3. Write in factorised form if possible.
4. Find x-intercepts.
5. Decide on how the curve meets the x-axis at each intercept.
6. Connect all this information with a smooth curve.

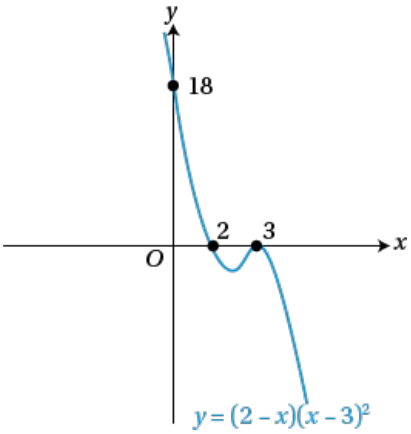
Explore

For higher degree polynomials, there are more options for the shape. Use technology to investigate how many x-intercepts a polynomial of degree n can have.

WORKED EXAMPLE 4.7

Sketch the graph of $y = (2-x)(x-3)^2$.

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This is a negative cubic.	Classify the basic shape. If you were to expand the brackets, the leading order term would be $-x^3$.
When $x = O, y = 2 \times (-3)^2 = 18$	Find the y-intercept.
When $y = 0, x = 2$ or $x = 3$	Find the x-intercepts.
At $x = 2$ curve passes through the x-axis.	Decide whether the curve crosses or touches the x-axis. This depends whether a factor is linear or squared.
At $x = 3$ curve just touches the x-axis.	
	Sketch the curve.

You can sometimes deduce a possible equation of a curve from its graph. If you know that a curve represents a polynomial function, the x-intercepts tell you its factors. However, there could also be a constant factor multiplying the whole equation: for example, $y = 3(x-1)(x-2)$ has the same x-intercepts as $y = (x-1)(x-2)$. You need to know the coordinates of another point on the graph (for example, the y-intercept) in order to find this constant factor.

To find the equation of a polynomial from its graph:

Key point 4.5

To find the equation of a polynomial from its graph:

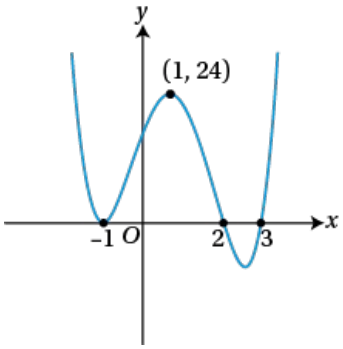
1 Use the shape of the graph and position of the x-intercepts to write down the factors of the polynomial.

2 Use any other point to find the constant factor.

WORKED EXAMPLE 4.8

The diagram shows the graph of a quartic polynomial. Find its equation.

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Single root at $x = 2$ and $x = 3$

Double root at $x = -1$

Describe x-intercepts. They tell you about the factors.

$y = k(x+1)^2(x-2)(x-3)$

Write in factorised form. Don't forget the constant factor outside the brackets.

$24 = k \times (2)^2 \times (-1) \times (-2) = 8k$

$\Rightarrow k = 3$

Use the fact that when $x = 1, y = 24$.

So the equation is $y = 3(x+1)^2(x-2)(x-3)$.

EXERCISE 4D

1 Sketch the following graphs, labelling all axis intercepts.

a i $y = 2(x-2)(x-3)(x-4)$

ii $y = 7(x-5)(x+1)(x-3)$

b i $y = 4(5-x)(x-3)(x-3)$

ii $y = 2(x-1)(2-x)(x-3)$

c i $y = -x(x-4)^2$

ii $y = (x-2)^2(x+2)$

d i $y = x(x^2+4)$

ii $y = (x+1)(x^2-3x+7)$

e i $y = (1-x)^2(1+x)$

ii $y = (2-x)(3-x)^2$

Tip

Remember that a sketch does not need to be to scale. In this exercise, you don't need to find the coordinates of turning points.

2 Sketch the following graphs, labelling all axis intercepts.

a i $y = x(x-1)(x-2)(2x-3)$

ii $y = (x+2)(x+3)(x-2)(x-3)$

b i $y = -4(x-3)(x-2)(x+1)(x+3)$

ii $y = -5x(x+2)(x-3)(x-4)$

c i $y = (x-3)^2(x-2)(x-2)$

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ii $y = x^2(x-1)(x+2)$

d **i** $y = 2(x+1)^3(x-3)$

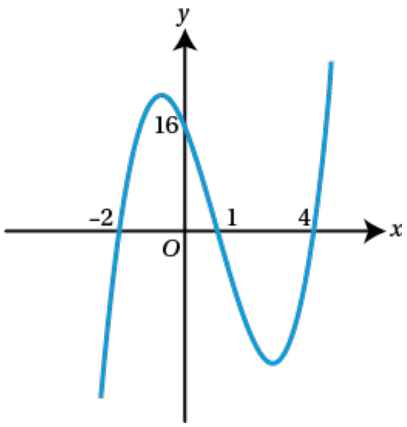
ii $y = -x^3(x-4)$

e **i** $y = \left(x^2+3x+12\right)(x+1)(3x-1)$

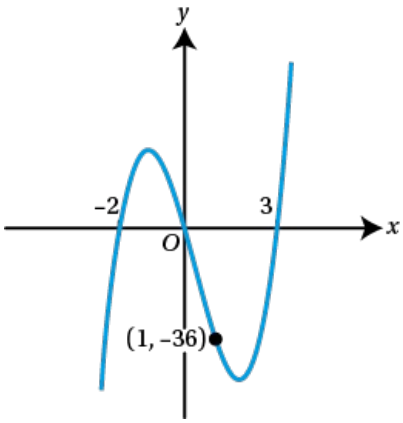
ii $y = (x+2)^2\left(x^2+4\right)$

3 Find the lowest order polynomial equation for each of the following graphs.

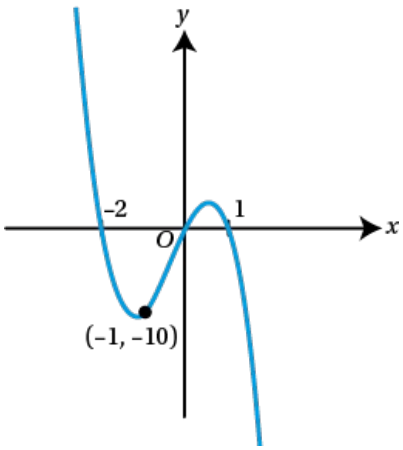
a **i**



ii

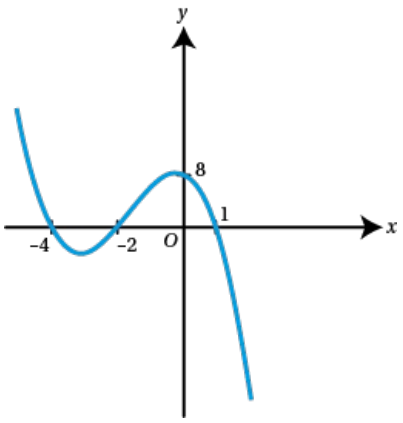


b **i**

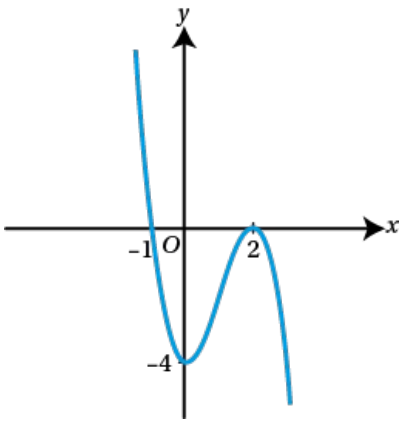


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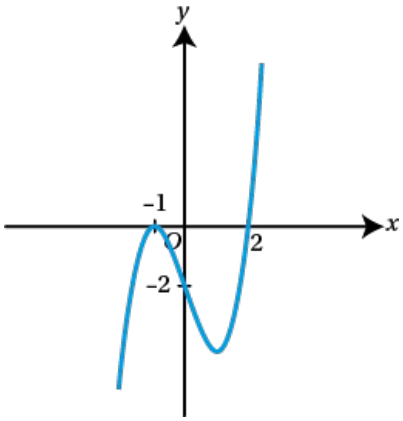
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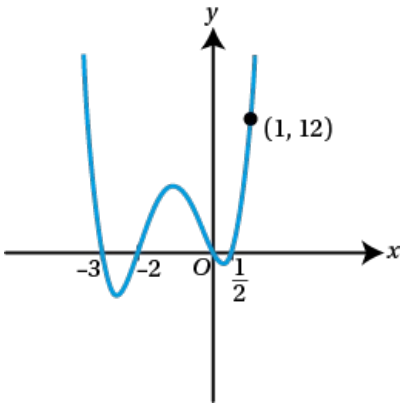
c i



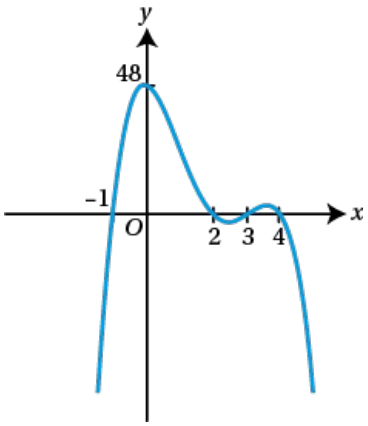
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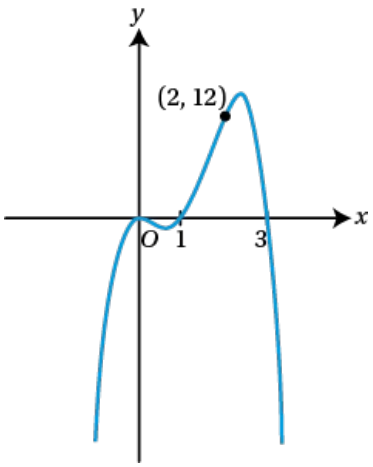
d i



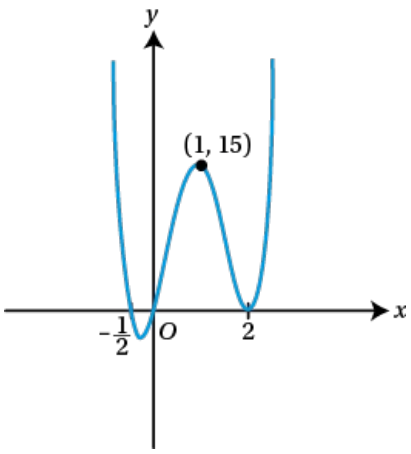
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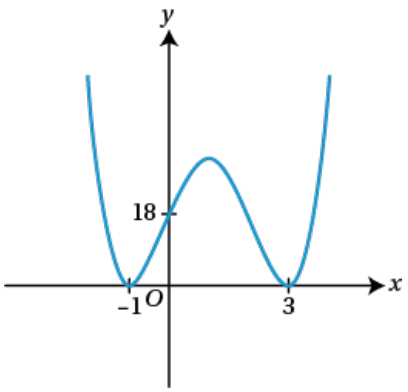
e i



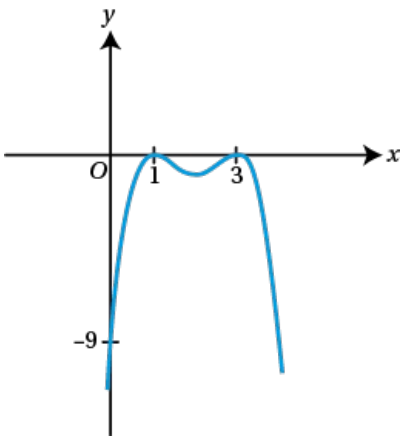
ii



f i



ii



4 **a** Show that $(x-2)$ is a factor of $\mathrm{f}(x) = 2x^3 - 5x^2 + x + 2$.

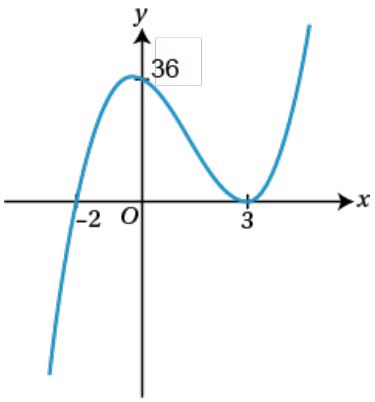
b Factorise $\mathrm{f}(x)$.

c Sketch the graph of $y = \mathrm{f}(x)$.

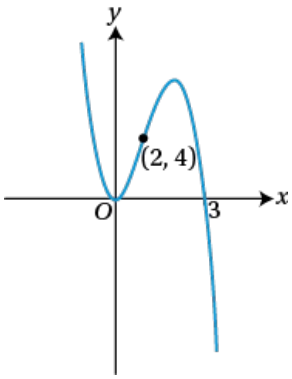
5 Sketch the graph of $y = 2(x+2)^2(3-x)$, labelling any axes intercepts.

6 These two graphs have equations of the form $y = px^3 + qx^2 + rx + s$. Find the values of p, q, r and s for each graph.

a



b



7 **a** Factorise fully $x^4 - q^4$ where q is a positive constant.

b Hence or otherwise sketch the graph of $y = x^4 - q^4$, labelling any points where the graph meets an axis.

8 **a** Sketch the graph of $y = (x-p)^2(x-q)$, where $0 < p < q$.

b How many solutions does the equation $(x-p)^2(x-q) = k$ have when $k > 0$?

Checklist of learning and understanding

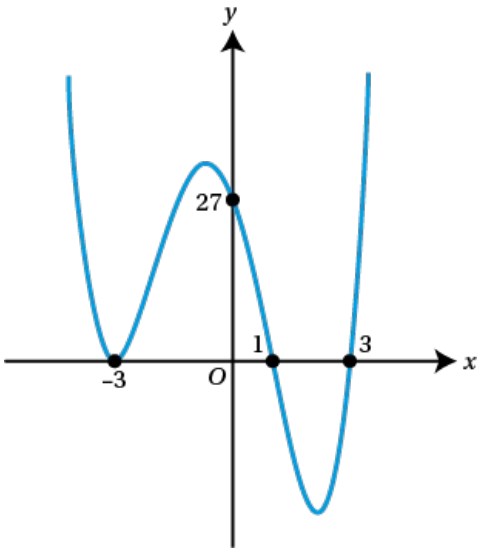
- A polynomial is an expression that is a sum of terms, each of the form ax^k.

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- You can multiply two polynomials by expanding the brackets and collecting like terms.
- You can divide two polynomials by using polynomial division.
- The **factor theorem** tells you when a linear expression is a factor of a polynomial:
  - $(ax + b)$ is a factor of a polynomial $\mathrm{f}(x)$ **if and only if** $\mathrm{f}\left(-\frac{b}{a}\right) = 0$.
- You can sketch the graph of a polynomial function by using its factorised form. The factors tell you the x-intercepts. Repeated factors tell you whether the graph crosses or touches the x-axis.

Mixed practice 4

1 The diagram shows the graph with equation $y = ax^4 + bx^3 + cx^2 + dx + e$. Find the values of a, b, c, d and e.



2 Show that

$$\frac{x^3 + 2x^2 - 3x - 6}{x + 2} = x^2 + bx + c$$

where b and c are integers to be found.

3 **a** Show that $(x - 2)$ is a factor of $\mathrm{f}(x) = x^3 - 4x^2 + x + 6$.

b Factorise $\mathrm{f}(x)$

c Sketch the graph of $y = \mathrm{f}(x)$

4 Two cubic polynomials are defined by $\mathrm{f}(x) = x^3 + (a - 3)x + 2b$ and $\mathrm{g}(x) = 3x^3 + x^2 + 5ax + 4b$, where a and b are constants.

a Given that $\mathrm{f}(x)$ and $\mathrm{g}(x)$ have a common factor of $(x - 2)$, show that $a = -4$ and find the value of b.

b Using these values of a and b, factorise $\mathrm{f}(x)$ fully. Hence show that $\mathrm{f}(x)$ and $\mathrm{g}(x)$ have two common factors.

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[Question part reference style adapted]

5 **a** Given that $(2x - 1)$ and $(x + 2)$ are factors of $2x^3 + ax^2 + 4x + b$, find the values of a and b.

b Hence sketch the graph of $y = 2x^3 + ax^2 + 4x + b$

6 Sketch the graph of $y = (x - a)^2(x - b)(x - c)$ where $b < 0 < a < c$.

7 The cubic polynomial $\mathrm{f}(x)$ is defined by $\mathrm{f}(x) = x^3 + x^2 - 11x + 10$.

a Use the factor theorem to find a factor of $\mathrm{f}(x)$.

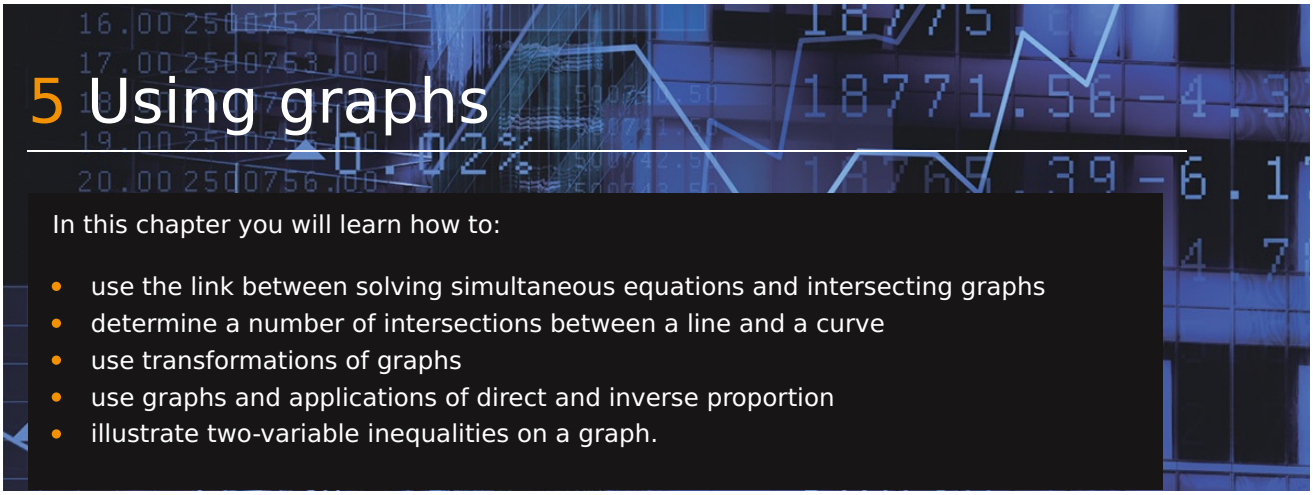
b Hence solve the equation $\mathrm{f}(x) = 0$, giving each root in an exact form.

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[Question part reference style adapted]

8 The polynomial $x^2 - 4x + 3$ is a factor of the polynomial $x^3 + ax^2 + 27x + b$. Find the values of a and b.

Worksheet

For a mixture of more challenging questions, see Extension sheet 4.



5 Using graphs

In this chapter you will learn how to:

- use the link between solving simultaneous equations and intersecting graphs
- determine a number of intersections between a line and a curve
- use transformations of graphs
- use graphs and applications of direct and inverse proportion
- illustrate two-variable inequalities on a graph.

Before you start...

Chapter 3	You should know how to solve quadratic equations.	1 Solve $x^2 + x - 1 = 0$.
Chapter 3	You should know how to use the quadratic discriminant to determine the number of real solutions of a quadratic equation.	2 How many real solutions are there to the equation $x^2 + 4x + 4 = 0$?
GCSE	You should know how to solve simple linear simultaneous equations by elimination.	3 Solve these simultaneous equations. $x + 2y = 5$ $3x + 4y = 11$
Chapter 2	You should know how to solve equations involving indices.	4 Solve $2^x = 8$.
GCSE	You should know how to solve linear inequalities.	5 Solve $3x + 1 > 13$.
GCSE	You should be able to find equations for direct and inverse proportion.	6 V is inversely proportional to the square of r. When $r = 2$, $V = 12$. Find an expression for V in terms of r.

Why use graphs?

Graphs are an alternative way of expressing a relationship between two variables. Understanding the connection between graphs and equations (or inequalities), and being able to switch between the two representations, gives you a much wider variety of tools to solve mathematical problems.

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Section 1: Intersections of graphs

You already know how to solve linear simultaneous equations, and how to use simultaneus equations to find the point of intersection of two straight lines. You can apply similar ideas to find intersections between curves whose equations involve quadratic functions. Whenever you are finding an intersection between two graphs, you are solving simultaneous equations. This means that the values you find for x and y must satisfy both equations.

Gateway to A Level

For revision of linear simultaneous equations, see Gateway to A Level section J.

The intersection of two graphs can always be found using technology (for example, graphing software or a graphical calculator). However, this usually only gives approximate solutions. If you need exact solutions you have to use an algebraic method. In many cases the best method is **substitution**, where you replace every occurrence of one variable in one equation by its expression from the other equation.

WORKED EXAMPLE 5.1

Find the coordinates of the points of intersection of the line $y = 2x - 1$ and the parabola $y = x^2 - 3x + 5$.

$x^2 - 3x + 5 = 2x - 1$

At the intersection points, the y-coordinates for the two curves are equal, so you can replace y in the first equation with the expression for y from the second equation.

$x^2 - 5x + 6 = 0$
$(x - 2)(x - 3) = 0$

This is a quadratic equation. Try to factorise.

$x = 2 \text{ or } 3$
$y = 2x - 1$
$x = 2 : y = 2 \times 2 - 1 = 3$
$x = 3 : y = 2 \times 3 - 1 = 5$

You also need to find the y-coordinates, by substituting back into one of the equations for y (both should give the same answer). Pick the first equation, as it is easier.

The coordinates are $(2, 3)$ and $(3, 5)$.

WORK IT OUT 5.1

Solve $x + y = 3$ and $y^2 + 2x^2 = 9$.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

Squaring $x + y = 3$ gives $x^2 + y^2 = 9$.

Subtracting this from the second equation gives $x^2 = 0$ so $x = 0$. Substituting into the first equation, $y = 3$. Checking this in the second equation gives:

$3^2 + 2 \times 0^2 = 9$

Solution 2

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If $x + y = 3$ then $y = 3 - x$

Substituting into the second equation:

$y^2 + 2x^2 = 9$
$(3 - x)^2 + 2x^2 = 9$
$9 - 6x + x^2 + 2x^2 = 9$
$3x^2 - 6x = 0$

Dividing by $3x$:

$x - 2 = 0$

$x = 2$

Substituting into $y = 3 - x$:

$x = 2, y = 1$

Solution 3

Rearranging the first equation gives $y = 3 - x$. Substituting into the second equation:

$y^2 + 2x^2 = 9$
$(3 - x)^2 + 2x^2 = 9$
$9 - 6x + x^2 + 2x^2 = 9$
$3x^2 - 6x = 0$
$3x\,(x - 2) = 0$
$x = 0 \text{ or } 2$

Substituting into the first equation:

$x = 0, y = 3$
$x = 2, y = 1$

EXERCISE 5A

1 Find the coordinates of intersection of the given curve and the given straight line.

a i $y = x^2 + 2x - 3$ and $y = x - 1$

ii $y = x^2 - 4x + 3$ and $y = 2x - 6$

b i $y = -x^2 + 3x + 9$ and $2x - y = 3$

ii $y = x^2 - 2x + 8$ and $x - y = 6$

2 Solve the following simultaneous equations:

a i $x - 2y = 1, 3xy - y^2 = 8$

ii $x + 2y = 3, y^2 + 2xy + 9 = 0$

b i $xy = 3, x + y = 4$

ii $x + y + 8 = 0, xy = 15$

c i $x + y = 5, y = x^2 - 2x + 3$

ii $x - y = 4, y = x^2 + x - 5$

3 Find the coordinates of the points of intersection of $y = \frac{1}{x}$ and $y = 2x$.

4 Solve simultaneously:

$3^x + 2^x = 13 \quad 3^x - 2^x = 5$

5 Solve simultaneously:

$y = 2^x \quad 4^x + y = 72$

6 The sum of two numbers is 8 and their product is 9.75.

a Show that this information can be written as a quadratic equation.

b What are the two numbers?

7 Solve the equations $xy + x = 0, x^2 + y^2 = 4$.

8 The equations $y = (x - 2)(x - 3)^2$ and $y = k$ have one solution for all $k < m$. Find the largest value of m.

Section 2: The discriminant revisited

Sometimes you only want to know how many intersection points there are, rather than to find their actual coordinates. The discriminant can be used to determine the number of intersections.

WORKED EXAMPLE 5.2

Find the set of values of k for which the line with equation $x + y = k$ intersects the curve with equation $x^2 - 4x + y^2 + 6y = 12$ at two distinct points.

Line equation: $y = k - x$

Substitute into the equation:

Try finding the intersections in terms of k and see if that gives you any ideas.

$$x^2 - 4x + (k - x)^2 + 6(k - x) = 12$$
$$\Rightarrow x^2 - 4x + k^2 - 2kx + x^2 + 6k - 6x = 12$$
$$\Rightarrow 2x^2 - (10 + 2k)x + k^2 + 6k = 12$$

At the intersection points, the y-coordinates for the two curves are equal, so you can replace y in the second equation by the expression for y from the first equation.

$$\Rightarrow 2x^2 - (10 + 2k)x + (k^2 + 6k - 12) = 0$$

This is a quadratic equation, so write it with one side equal to zero.

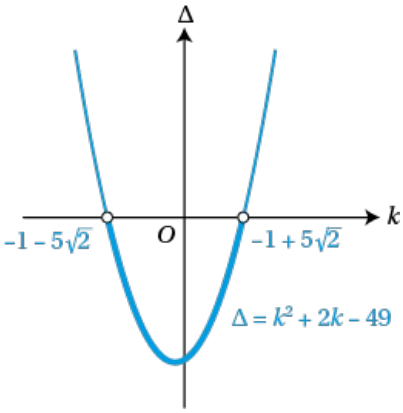
Two solutions $\therefore \Delta > 0$

You know that the discriminant tells you the number of solutions of a quadratic equation.

$$\Delta = (10 + 2k)^2 - 8(k^2 + 6k - 12) > 0$$
$$\Rightarrow 100 + 40k + 4k^2 - 8k^2 - 48k + 96 > 0$$
$$\Rightarrow -4k^2 - 8k + 196 > 0$$

$$\Rightarrow k^2 + 2k - 49 < 0$$

Divide both sides by -4. Remember that this reverses the inequality.



Roots: $k^2 + 2k - 49 = 0$

$$k = \frac{-2 \pm \sqrt{4 + 4 \times 49}}{2}$$
$$= \frac{-2 \pm 2\sqrt{1 + 49}}{2}$$

This is a quadratic inequality. To solve it, find where LHS $= 0$ and sketch the graph.

$$= -1 \pm \sqrt{50} = -1 \pm 5\sqrt{2}$$

The graph shows that the required interval is between the roots.

$$\therefore -1 - 5\sqrt{2} < k < -1 + 5\sqrt{2}$$

Tip

Questions which talk about the number of intersections are often solved using the discriminant.

Fast forward

The equation $x^2 - 4x + y^2 + 6y = 12$ in Worked example 5.2 actually represents a circle. You will study circles in Chapter 6.

EXERCISE 5B

1. Show that the line with equation $x - y = 6$ is a tangent to the curve with equation $x^2 - 6x + y^2 - 2y + 2 = 0$.
2. Find the exact values of m for which the line $y = mx + 3$ is a tangent to the curve with equation $y = 3x^2 - x + 5$.
3. Let C be the curve with equation $4x^2 + 9y^2 = 36$. Find the exact values of k for which the line $2x + 3y = k$ is a tangent to C.
4. Find the values of a for which the curve $y = x^2$ never touches the curve $y = a - (x - a)^2$.
5. Show algebraically that the line $y = kx + 5$ intersects the parabola $y = x^2 + 2$ twice for all values of k.

Tip

A **tangent** touches the curve but does not cross at that point. With quadratic equations this means that there are repeated roots so the discriminant is zero. After studying Chapter 13 you will find another way of finding the tangent to a curve. However, this type of question is still best done using the discriminant.

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Section 3: Transforming graphs

From previous study you may know how changing the function changes the graph as summarised in Key point 5.1.

Key point 5.1

Transformation of $y = \mathrm{f}(x)$	Transformation of graph
$y = \mathrm{f}(x) + c$	Translation c units up.
$y = \mathrm{f}(x + d)$	Translation d units to the left.
$y = p\mathrm{f}(x)$	Vertical stretch, scale factor p
$y = \mathrm{f}(qx)$	Horizontal stretch factor $\frac{1}{q}$
$y = -\mathrm{f}(x)$	Reflection in the x-axis
$y = \mathrm{f}(-x)$	Reflection in the y-axis

Tip

When c is negative the translation is down, and when d is negative it is to the right. When p or q are negative, the stretch is combined with a reflection.

Vertical transformations behave as expected, but the horizontal ones can be counter-intuitive; for example, $\mathrm{f}(x + 3)$ translates the graph to the left.

The following proof shows why that is the case. However, you can use all the results from Key Point 5.1 without proof.

PROOF 2

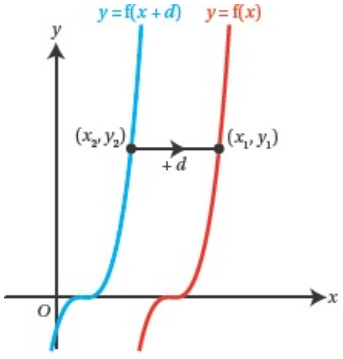
Prove that the graph of $\mathrm{f}(x + d)$ is a translation of the graph of $y = \mathrm{f}(x)$, by d units to the left.

Let (x_1, y_1) be a point on the graph of $y = \mathrm{f}(x)$ and (x_2, y_2) a point on the graph of $y = \mathrm{f}(x + d)$.

Define variables. You have to be very careful and **not** assume that x's are all the same or y's are all the same.

When $x_1 = x_2 + d$ then:
$y_1 = \mathrm{f}(x_1) = \mathrm{f}(x_2 + d) = y_2$

This says that if two points are d units apart horizontally, then they are at the same height.



Hence (x_2, y_2) is d units to the left of (x_1, y_1), and so the graph of $y = \mathrm{f}(x)$ is translated d units to the left to get

Interpret your calculation geometrically and write a conclusion.

the graph of $y = \mathrm{f}(x + d)$.

WORKED EXAMPLE 5.3

The graph of $y = x^2 + 2x$ is translated 5 units to the left. Find the equation of the resulting graph in the form $y = ax^2 + bx + c$.

If $\mathrm{f}(x) = x^2 + 2x$, then the new graph is $y = \mathrm{f}(x + 5)$,	Relate the transformation to function notation.
$y = (x + 5)^2 + 2(x + 5)$ $= x^2 + 12x + 35$	Replace all x by $(x + 5)$ in the equation for the function.

Tip

A translation 5 units to the left could also be described in vector notation as $\begin{pmatrix} -5 \\ 0 \end{pmatrix}$.

WORKED EXAMPLE 5.4

Describe a transformation which transforms the graph of $y = x^2 + 3x$ to the graph of $y = 4x^2 + 6x$.

Let $\mathrm{f}(x) = x^2 + 3x$ Then $4x^2 + 6x = (2x)^2 + 3(2x) = \mathrm{f}(2x)$	Try to relate the two equations by writing the second function in a similar way to the first.
It is a horizontal stretch with scale factor $\frac{1}{2}$.	Relate the function notation to the transformation.

WORKED EXAMPLE 5.5

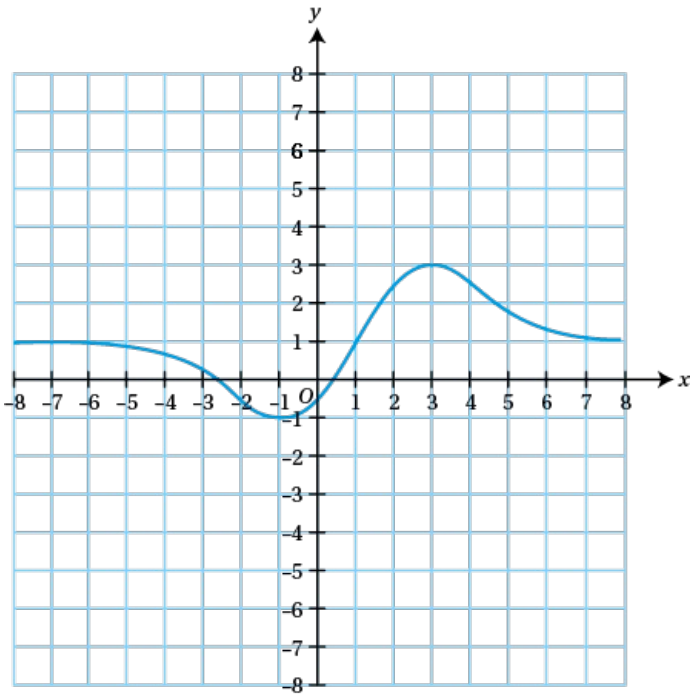
The graph of $y = \mathrm{f}(x)$ has a single maximum point with coordinates $(4, -3)$. Find the coordinates of the maximum point on the graph of $y = \mathrm{f}(-x)$.

The transformation taking $y = \mathrm{f}(x)$ to $y = \mathrm{f}(-x)$ is a reflection in the y-axis.	Relate the function notation to the transformation.
The maximum point is $(-4, -3)$.	Reflection in the y-axis leaves y-coordinates unchanged and changes x to $-x$.

EXERCISE 5C

1 The graph of $y = \mathrm{f}(x)$ is shown. Sketch the graph of the following functions, including the position of the minimum and maximum points.

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a **i** $y = \mathrm{f}(x) + 3$

ii $y = \mathrm{f}(x) + 5$

b **i** $y = \mathrm{f}(x) - 7$

ii $y = \mathrm{f}(x) - 0.5$

c **i** $y = \mathrm{f}(x + 2)$

ii $y = \mathrm{f}(x + 4)$

d **i** $y = \mathrm{f}(x - 1.5)$

ii $y = \mathrm{f}(x - 2)$

e **i** $y = 3\mathrm{f}(x)$

ii $y = 5\mathrm{f}(x)$

f **i** $y = \dfrac{\mathrm{f}(x)}{4}$

ii $y = \dfrac{\mathrm{f}(x)}{2}$

g **i** $y = \mathrm{f}(2x)$

ii $y = \mathrm{f}(6x)$

h **i** $y = \mathrm{f}\left(\dfrac{2x}{3}\right)$

ii $y = \mathrm{f}\left(\dfrac{5x}{6}\right)$

i **i** $y = -\mathrm{f}(x)$

ii $y = \mathrm{f}(-x)$

2 Find the equation of each of these graphs after the given transformation is applied:

a **i** $y = 3x^2$ after a translation of 3 units vertically up

ii $y = 9x^3$ after a translation of 7 units vertically down

b **i** $y = 7x^3 - 3x + 6$ after a translation of 2 units down

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ii $y = 8x^2 - 7x + 1$ after a translation of 5 units up

c **i** $y = 4x^2$ after a translation of 5 units to the right

ii $y = 7x^2$ after a translation of 3 units to the left

d **i** $y = 3x^3 - 5x^2 + 4$ after a translation of 4 units to the left

ii $y = x^3 + 6x + 2$ after a translation of 3 units to the right

Worksheet

To explore how transformations are related to the symmetries of a graph, see Extension sheet 5.

3 Find the required translations:

a **i** transforming the graph $y = x^2 + 3x + 7$ to the graph $y = x^2 + 3x + 2$

ii transforming the graph $y = x^3 - 5x$ to the graph $y = x^3 - 5x - 4$

b **i** transforming the graph $y = x^2 + 2x + 7$ to the graph $y = (x + 1)^2 + 2(x + 1) + 7$

ii transforming the graph $y = x^2 + 5x - 2$ to the graph $y = (x + 5)^2 + 5(x + 5) - 2$

c **i** transforming the graph $y = \sqrt{2x}$ to the graph $y = \sqrt{2x + 6}$

ii transforming the graph $y = \sqrt{2x + 1}$ to the graph $y = \sqrt{2x - 3}$

Worksheet

For more examples like this, see Support sheet 5.

4 Find the equation of the graph after the given transformation is applied.

a **i** $y = 3x^2$ after a vertical stretch factor 7 relative to the x-axis.

ii $y = 9x^3$ after a vertical stretch factor 2 relative to the x-axis.

b **i** $y = 7x^3 - 3x + 6$ after a vertical stretch factor $\frac{1}{3}$ relative to the x-axis.

ii $y = 8x^2 - 7x + 1$ after a vertical stretch factor $\frac{4}{5}$ relative to the x-axis.

c **i** $y = 4x^2$ after a horizontal stretch factor 2 relative to the y-axis.

ii $y = 7x^2$ after a horizontal stretch factor 5 relative to the y-axis.

d **i** $y = 3x^3 - 5x^2 + 4$ after a horizontal stretch factor $\frac{1}{2}$ relative to the y-axis.

ii $y = x^3 + 6x + 2$ after a horizontal stretch factor $\frac{2}{3}$ relative to the y-axis.

5 Describe the following stretches:

a **i** transforming the graph $y = x^2 + 3x + 7$ to the graph $y = 4x^2 + 12x + 28$

ii transforming the graph $y = x^3 - 5x$ to the graph $y = 6x^3 - 30x$

b **i** transforming the graph $y = x^2 + 2x + 7$ to the graph $y = (3x)^2 + 2(3x) + 7$

ii transforming the graph $y = x^2 + 5x - 2$ to the graph $y = (4x)^2 + 5(4x) - 2$

c **i** transforming the graph $y = \sqrt{4x}$ to the graph $y = \sqrt{12x}$

ii transforming the graph $y = \sqrt{2x + 1}$ to the graph $y = \sqrt{x + 1}$

6 Find the equation of the graph after the given transformation is applied.

a **i** $y = 3x^2$ after reflection in the x-axis.

ii $y = 9x^3$ after reflection in the x-axis.

b **i** $y = 7x^3 - 3x + 6$ after reflection in the x-axis.

ii $y = 8x^2 - 7x + 1$ after reflection in the x-axis.

c **i** $y = 4x^2$ after reflection in the y-axis.

ii $y = 7x^3$ after reflection in the y-axis.

d **i** $y = 3x^3 - 5x^2 + 4$ after reflection in the y-axis.

ii $y = x^3 + 6x + 2$ after reflection in the y-axis.

7 Describe the following transformations:

a **i** transforming the graph $y = x^2 + 3x + 7$ to the graph $y = -x^2 - 3x - 7$

ii transforming the graph $y = x^3 - 5x$ to the graph $y = 5x - x^3$

b **i** transforming the graph $y = x^2 + 2x + 7$ to the graph $y = x^2 - 2x + 7$

ii transforming the graph $y = x^2 - 5x - 2$ to the graph $y = x^2 + 5x - 2$

c **i** transforming the graph $y = \sqrt{4x}$ to the graph $y = \sqrt{-4x}$

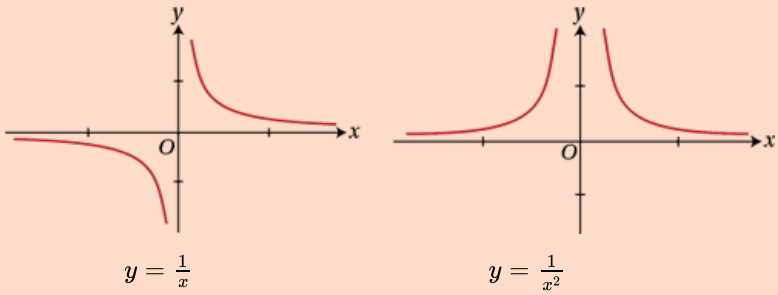
ii transforming the graph $y = \sqrt{2x - 1}$ to the graph $y = \sqrt{-1 - 2x}$

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Section 4: Graphs of $\frac{a}{x}$ and $\frac{a}{x^2}$

You need to be able to sketch the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$.

Key point 5.2



$y = \frac{1}{x}$ $y = \frac{1}{x^2}$

The graphs of $\frac{a}{x}$ and $\frac{a}{x^2}$ are very similar to the graphs in Key point 5.2. They are vertically stretched by a factor of a a.

Both graphs have two **asymptotes**. An asymptote is a line to which the curve gets closer and closer but never meets. These curves have asymptotes at $x = 0$ and $y = 0$.

EXERCISE 5D

1 **a** Write down the transformation that changes the graph of $y = \frac{a}{x}$ into the graph of $y = \frac{a}{x - d}$.

b Hence write down the equations of the asymptotes of the graph $y = \frac{a}{x - d}$.

2 Show that the curve $y = \frac{a}{x^2}$ is a horizontal stretch of the curve $y = \frac{1}{x^2}$ and find the stretch factor.

3 **a** Show that the curves $y = \frac{a}{x}$ and $y = \frac{b}{x^2}$ always intersect at exactly one point, P, and find the coordinates of that point.

b The origin and P are opposite vertices of a rectangle with sides parallel to the coordinate axes. Show that the area of this rectangle is independent of b.

4 Find a condition on m in terms of a and c so that the line $y = mx + c$ is a tangent to the curve $y = \frac{a}{x}$.

5 The function $f(x)$ is a cubic polynomial. Show graphically that the curve $y = \frac{1}{x}$ can intersect this curve in 0, 1, 2, 3 or 4 places.

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Section 5: Direct and inverse proportion

Direct proportion means that the ratio of two quantities is constant. For example, if y is proportional to x^2 you can write $\frac{y}{x^2} = k$ or $y = kx^2$.

Inverse proportion means that the product of two quantities is constant. For example, if y is inversely proportional to x^2 you write $yx^2 = k$ or $y = \frac{k}{x^2}$.

You can use your knowledge of graphs to sketch the graphs of two quantities if you are given information about their proportionality.

Linear functions are closely related to direct proportion: if $y = mx + c$ then $(y - c)$ is directly proportional to x.

Straight-line graphs can be used to represent or model a variety of real-life situations. In some situations, the linear model is only approximate. When making predictions, you should consider its accuracy and limitations.

Gateway to A Level

For a reminder of calculations involving direct and inverse proportion, see Gateway to A Level section K.

Fast forward

A common example where a straight line is used to make predictions is the line of best fit used in statistics. You will learn more about lines of best fit in Chapter 16, Section 4.

WORKED EXAMPLE 5.6

It takes me 12 minutes to drive from my house to the motorway. On the motorway, I drive at an average speed of 65 miles per hour.

a Approximately how long does it take me to drive to Leeds, which is 154 miles away?

b Write down an equation modelling the time, t hours, it takes me to drive to a city d miles away.

c Explain why this model only gives approximate times.

a Time in hours:

$0.2 + \frac{154}{65} = 2.57$ hours

(about $2\ h\ 34$ minutes)

Use time on motorway $= \frac{\text{distance}}{\text{speed}}$.

12 minutes $=0.2$ hours.

b $t = 0.2 + \frac{d}{65}$

c The speed on the motorway is not constant.

It doesn't take into account the time from getting off the motorway in Leeds.

The 154 miles distance is

You are modelling the speed as constant, although in reality this is not the case.

probably not exact; it doesn't specify where in Leeds you are going or exactly where it is measured from.

The speeds and distances quoted are probably only correct to the nearest integer.

All these considerations mean that the model does not give an exact answer, but it is probably good enough to be practical.

EXERCISE 5E

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1 **a** If y is proportional to x^2 and $y = 12$ when $x = 2$, find the value of y when $x = 4$.

b Sketch the graph of y against x.

2 **a** If y is proportional to $x - 4$ and $y = 1$ when $x = 6$, find y when $x = 8$.

b Sketch the graph of y against x.

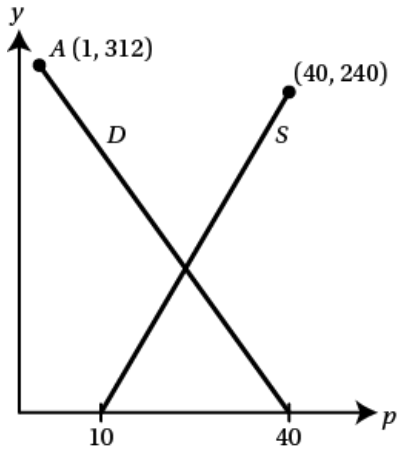
3 **a** If y is inversely proportional to x^2 and $y = 20$ when $x = 1$, find y when $x = 4$.

b Sketch the graph of y against x.

4 **a** If y is inversely proportional to $x + 1$ and $y = 9$ when $x = 3$, find y when $x = 5$.

b Sketch the graph of y against x.

5 Economists use supply and demand curves to model the number of items produced and sold at a particular price. Let £ p be the price of one item. Demand (D) is the number of items that can be sold at this price. Supply (S) is the number of items that the producer will make. The graph shows supply and demand in the simplest model, where both vary linearly with price.



a Show that the equation of line D is $y_D = 320 - 8p$ and find the equation of line S.

b What does the value 320 in the equation of D represent? Suggest why it may not be reasonable to extend the straight line for D beyond point A.

c What is the maximum price that can be charged before there is no more demand?

The market is said to be in equilibrium when supply equals demand.

d Find the equilibrium price of one item.

6 A provider offers two different mobile phone contracts:

A The set-up cost of £65, plus calls at 3 p per minute.

B No set-up costs, calls cost 5 p per minute.

a Write down an equation for the total cost, £C, of making m minutes of calls for each contract.

b Hence find after how many minutes of calls contract A becomes better value.

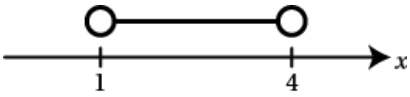
7 y is inversely proportional to x^2 and z is inversely proportional to y. Sketch a graph of z against x.

8 The strength of the Earth's gravitational field is inversely proportional to the square of the distance from the centre of the Earth. If a satellite is put into orbit, the distance to the centre of the Earth is increased by 10%. Find the percentage decrease in the gravitational field strength.

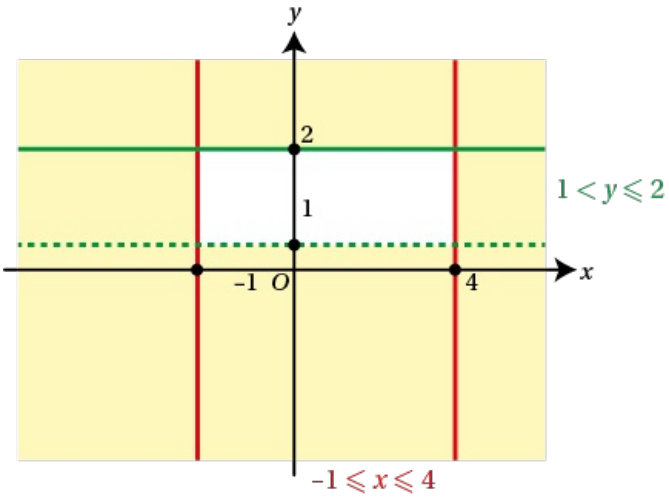
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Section 6: Sketching inequalities in two variables

You can represent inequalities in one variable on a number line. For example, the inequality $1 < x < 4$ can be represented by:

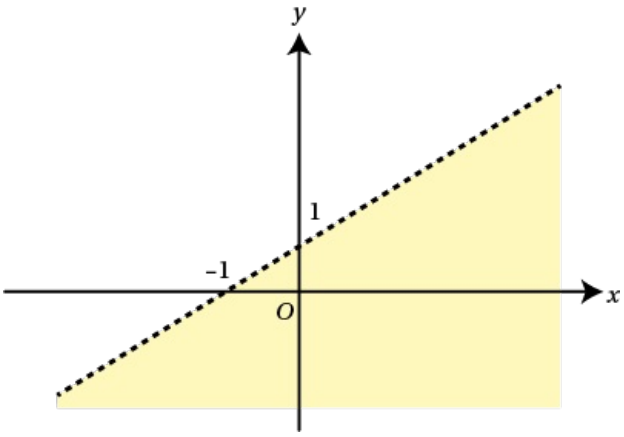


If there is also another variable, you can represent the inequality on a graph:



In this graph you can see the convention that the part that satisfies the inequality is left unshaded. This is so that when you have several inequalities on one graph, the region which satisfies all the inequalities is clear.

If the inequality involves both variables you can still represent the solution by shading. For example, $y > x + 1$ is shown on the following graph. The required region has been left unshaded.



Notice that since the line $y = x + 1$ is not included it is drawn as a dashed line.

Key point 5.3

The general process for illustrating inequalities is:

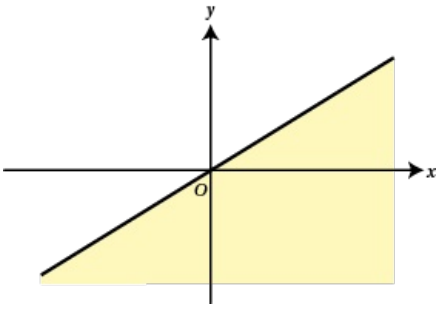
- draw the associated equation on the graph, using a dashed line if the curve is not included
- test a convenient point on one side of the curve
- shade the side that does not satisfy the inequality.

WORKED EXAMPLE 5.7

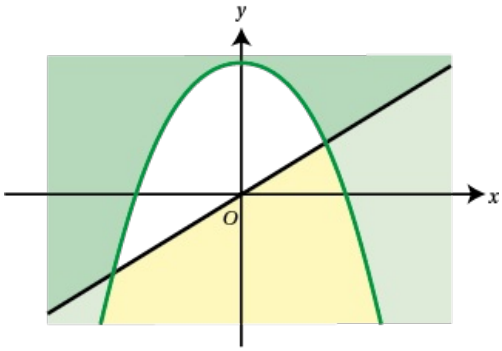
a Represent the inequalities $y \geqslant x$ and $y \leqslant 1 - x^2$ on a graph.

b Find the largest value of x that satisfies these inequalities.

a



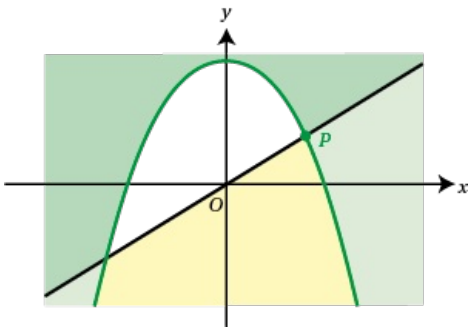
First sketch $y = x$. It is a solid line since the line is included in the inequality. You can try the point $(1, 0)$ and it does not satisfy the inequality, so you shade that side of the line.



Then sketch $y = 1 - x^2$. This is a solid line since the line is included in the inequality. You can try the point $(0, 0)$ and it does satisfy the inequality, so you shade the other side of the curve.

b The largest x value corresponds to the point labelled P.

You need to find the point in the unshaded region with the largest x value.



P occurs where:

To find this point you use simultaneous equations.

$x = 1 - x^2$

So $x \dfrac{-1 \pm \sqrt{5}}{2}$

You can solve this using the quadratic formula.

So the largest value of x is $\dfrac{-1 + \sqrt{5}}{2}$

The larger solution is the one with a plus.

EXERCISE 5F

1 Illustrate the following inequalities on a graph.

a i $y > 1 + 2x$

ii $y < 2 + x$

b i $y + x \geqslant 1$

ii $y + 2x \geqslant 4$

c i $y > x^2$

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ii $y > -x^2$

d **i** $y > x^2 + 3x + 2$

ii $y > x^2 - 7x + 6$

e **i** $y \leqslant x^2 + 2x + 1$

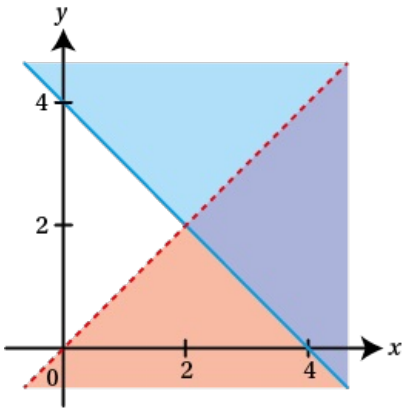
ii $y \leqslant x^2 - 7x + 10$

2 Illustrate the region $x > 0$, $y > 0$, $x + y < 4$ on a graph.

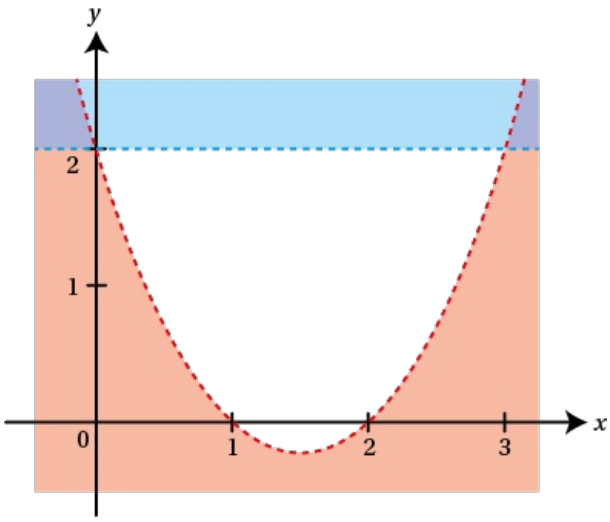
3 Illustrate the region $y \geqslant x^2$, $y < 4$ on a graph.

4 Illustrate the region $y > x^2 - 4x$ and $y < 2x - x^2$ on a graph.

5 Describe using inequalities the unshaded region in this graph.



6 This region is bounded by a parabola and a straight line. Describe using inequalities the unshaded region in this graph.



7 Find the largest integer value of x which satisfies $y < 120x - 2x^2$ and $y > 11x$.

8 Sketch $y > xy$.

Explore

Graphs of inequalities are needed to solve problems about maximising profits or minimising production time. Find out about a technique called **linear programming**.

Checklist of learning and understanding

- You can use substitution to solve simultaneous equations, which allows you to find the

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intersection point of two curves.

- The number of intersections of a quadratic curve and a straight line can be determined using the quadratic discriminant.
- Transforming a function results in a transformation of the graph of the function.

Transformation of $y = \mathrm{f}(x)$	**Transformation of graph**
$y = \mathrm{f}(x) + c$	Translation c units up.
$y = \mathrm{f}(x + d)$	Translation d units to the **left**.
$y = p\mathrm{f}(x)$	Vertical stretch, scale factor P relative to the x-axis
$y = \mathrm{f}(qx)$	Horizontal stretch factor $\frac{1}{q}$ relative to the y-axis
$y = -\mathrm{f}(x)$	Reflection in the x-axis
$y = \mathrm{f}(-x)$	Reflection in the y-axis

- You should be able to sketch the graphs of $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$.
- You should be able to interpret descriptions of the proportionality of two variables and to sketch the associated graph. You should also be able to use a linear model in a variety of contexts and understand that models sometimes only give approximate predictions.
- You can represent inequalities in two variables graphically by shading.

Mixed practice 5

1. Find the intersection of the graphs $x^2 + y^2 = 25$ and $x + y = 7$.

2. **a** Illustrate the region represented by the inequalities $x + y < 3, y \geqslant 0, y < 2x$.

   b Find the upper bound for the values of y that satisfy these inequalities.

3. Find the transformation that transforms the graph of $y = (x - 1)^2$ to the graph of $y = (x + 2)^2$.

4. If z is proportional to x^2 sketch the graph of z against x.

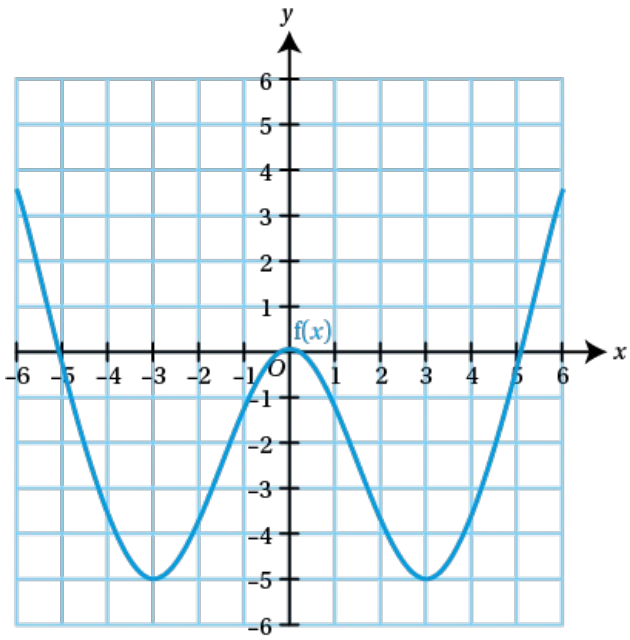
5. Two taxi companies have the following pricing structures:

   Company A charges £1.60 per kilometre.

   Company B charges £1.20 per kilometre plus £1.50 call-out charge.

   Find the length of the journey for which the two companies charge the same amount.

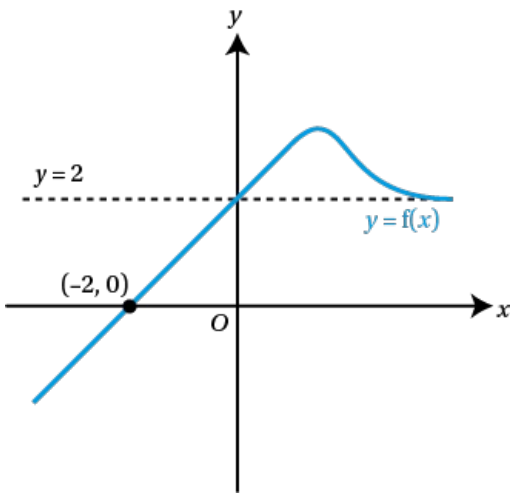
6. The graph of $y = \mathrm{f}(x)$ is shown.



   a Sketch the graph of $y = \mathrm{f}(x - 1) + 2$.

   b State the coordinates of the maximum point of the new graph.

7. The diagram shows a part of the graph of $y = \mathrm{f}(x)$.



   Sketch the graph of $y = \mathrm{f}(3x)$.

8 **a** The curve $y = x^2$ is translated 2 units in the positive x direction. Find the equation of the curve after it has been translated.

b The curve $y = x^3 - 4$ is reflected in the x-axis. Find the equation of the curve after it has been reflected.

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9 A doctor thinks that mass of a baby can be modelled as a linear function of age. A particular baby had a mass of $4.1\ \text{kg}$ aged 2 weeks, and $4.8\ \text{kg}$ aged 5 weeks.

a If M is the mass of the baby aged n weeks, show that the straight line model results in the equation $M = 0.233n + 3.63$, where the coefficients have been rounded to three significant figures.

b Give an interpretation of the values 0.233 and 3.63 in the equation in part **a**.

c The normal mass of a healthy one-year-old baby is approximately between 10 and 12 kg. Is the linear model appropriate for babies as old as one year?

10 **a** Solve the simultaneous equations

$$y = 2x^2 - 3x - 5 \qquad 10x + 2y + 11 = 0$$

b What can you deduce from the answer to part **a** about the curve $y = 2x^2 - 3x - 5$ and the line $10x + 2y + 11 = 0$?

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11 Given that x is inversely proportional to y and z is proportional to x^2 sketch the graph of z against y.

12 **a** By using an appropriate substitution find the exact solutions to the equation $x^4 + 36 = 13x^2$

b Hence solve the inequality $x^4 + 36 \leqslant 13x^2$.

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6 Coordinate geometry

In this chapter you will learn how to:

- find the distance between two points and the midpoint of two points
- find the equation of a straight line in the form $y - y_1 = m(x - x_1)$ and $ax + by + c = 0$
- determine whether two straight lines are parallel or perpendicular
- find the equation of a circle with a given centre and radius
- solve problems involving intersections of lines and circles.

Before you start...

GCSE	You should know how to find the equation of a straight line in the form $y = mx + c$.	1 Find the equation of a straight line a with gradient 3 and y-intercept $(0, -1)$ b with gradient -2 and passing through the point with coordinates $(2, 5)$ c passing through the points with coordinates $(1, 3)$ and $(3, 9)$.
GCSE	You should know how to use the fact that parallel lines have the same gradient.	2 A straight lines passes through the points $(-1, -4)$ and $(4, p)$ and is parallel to the line with equation $y = 3x + 4$. Find the value of p.
GCSE	You should know how to solve two linear simultaneous equations.	3 Solve the simultaneous equations: $3x - 2y = 7$ $x + 3y = 12$
Chapter 3	You should know how to complete the square for an algebraic expression.	4 Write $x^2 - 4x - 3$ in the form $(x - p)^2 + q$.
Chapter 5	You should know how to solve linear and quadratic simultaneous equations, and interpret the solution as the intersection points between a line and a curve.	5 a Solve the simultaneous equations: $y = x^2$ $x + y = 6$ b Show that the line $y = 2x + 1$ is a tangent to the parabola $y = -x^2$.
GCSE	You should know how to use properties of special quadrilaterals.	6 Which of the following properties does each of these quadrilaterals have?

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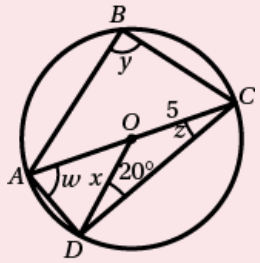
	Parallelogram	Rectangle	Rhombus	Square
opposite sides parallel	☐	☐	☐	☐
opposite sides equal	☐	☐	☐	☐
all four sides equal	☐	☐	☐	☐
sides perpendicular	☐	☐	☐	☐
diagonals equal	☐	☐	☐	☐
diagonals perpendicular	☐	☐	☐	☐
diagonals bisect each other	☐	☐	☐	☐

GCSE

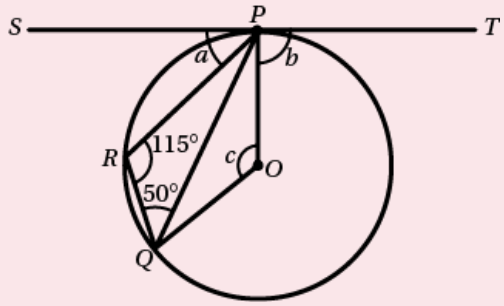
- You should know how to use the properties of tangents and chords of circles:
- the angle in a semi-circle is a right angle
- a tangent to the circle is perpendicular to the radius at the point of contact
- the radius perpendicular to the chord bisects the chord

7 Find the angles and lengths marked with letters, giving reasons for your answers:

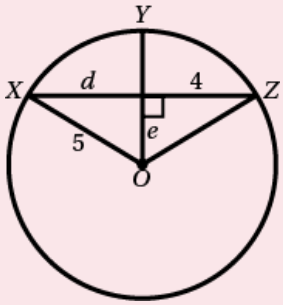
a



b



c



Using equations to represent geometrical shapes

Straight lines and circles are fundamental objects in geometry, and can be used to model many real-life objects. You already know several properties of lines and circles, as well as other geometrical figures made out of them, such as triangles and cones.

In this chapter you will look at using coordinates to represent lines and circles. You will use equations to represent those shapes and to find their intersections.

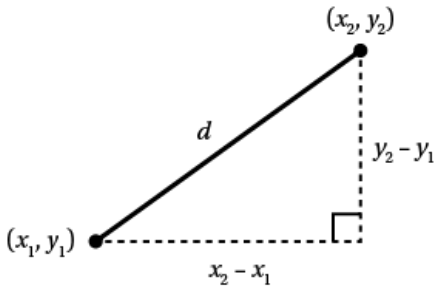
Did you know?

Using equations to represent geometrical shapes is a relatively recent idea in mathematics – it was developed in the 17th century by the French philosopher and mathematician René Descartes. The Cartesian coordinate system is named after him.

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Section 1: Midpoint and distance between two points

You may already have met the idea that if we have two points with coordinates (x_1, y_1) and (x_2, y_2) we can find the distance between these two points using Pythagoras' theorem.



Key point 6.1

The distance between the points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

If the two points are called A and B we use the notation AB to mean the distance between the two points.

We can also find the midpoint of A and B. This is the point on halfway along the line connecting A and B. It can be found by thinking of it as the average of the coordinates of the two points.

Key point 6.2

The midpoint of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Fast forward

You will see how to prove this using vectors in Chapter 17.

WORKED EXAMPLE 6.1

The points A and B have coordinates $(-2, 4)$ and $(5, 2)$. Find:

a the exact distance AB

b the midpoint, M, of A and B.

a The distance is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(5 - (-2))^2 + (2 - 4)^2}$$
$$= \sqrt{7^2 + (-2)^2}$$
$$= \sqrt{53}$$

Use Key point 6.1 with $x_2 = 5, x_1 = -2$ $y_2 = 2, y_1 = 4$.

b The midpoint is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{-2 + 5}{2}, \frac{4 + 2}{2}\right)$$
$$= (1.5, 3)$$

Use Key point 6.2.

Questions can also involve unknown points.

WORKED EXAMPLE 6.2

Find all points of the form (a, a) that are a distance of 5 away from the point $(0, 1)$.

The distance between the points is given by

$$5 = \sqrt{(0-a)^2 + (1-a)^2}$$

Use distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and set this equal to 5.

$$= \sqrt{a^2 + 1 - 2a + a^2}$$
$$= \sqrt{2a^2 - 2a + 1}$$

Simplify the expression under the square root.

So

Square both sides to get rid of the square root.

$$25 = 2a^2 - 2a + 1$$

$$0 = 2a^2 - 2a - 24$$
$$0 = a^2 - a - 12$$
$$0 = (a-4)(a+3)$$
$$a = 4 \text{ or } -3$$

Solve the quadratic equation.

So the points are $(4, 4)$ or $(-3, -3)$.

EXERCISE 6A

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1 Find the exact distance between the points.

a **i** $(0, 0)$ and $(5, 12)$

ii $(3, 4)$ and $(0, 0)$

b **i** $(1, 4)$ and $(2, 6)$

ii $(2, 2)$ and $(3, 5)$

c **i** $(-1, 4)$ and $(3, 2)$

ii $(-1, 3)$ and $(-3, 1)$

d **i** $(-2, -3)$ and $(-3, 0)$

ii $(-1, -5)$ and $(-2, -1)$

2 Find the midpoint for each of the pairs of points in question 1.

Gateway to A Level

For more practice on basic questions involving distances between points and midpoints, see Gateway to A Level section L.

3 Find in terms of a the exact distance between the points $(a, 2a)$ and $(-2a, 8a)$ where $a > 0$.

4 The midpoint of points P and Q is $(1, 1)$. If point P has coordinates (a, b) find the coordinates of Q.

5 **a** The point A has coordinates $(0, 1)$, the point B has coordinates $(4, 4)$ and the point C has coordinates $(7, 8)$. Show that the distance AB equals the distance BC.

b Explain why this does not mean that B is the midpoint of AC.

6 The point $(a, 2a)$ is 3 units away from the point $(3, 1)$. Find the possible values of a.

7 The set of points (x, y) are defined by the property that the distance to the point $(0, 1)$ equals y.

Find the equation connecting x and y.

8 Point A has coordinates (x_1, y_1), point B has coordinates (x_2, y_2) and point M has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Prove that $AM = \frac{1}{2}AB$.

9 The points A and B have coordinates $(-2, 4)$ and $(4a, 2a)$. M is the midpoint of A and B.

a Find and simplify in terms of a:

i the distance AB

ii the midpoint of A and B.

b If O is the origin show that the ratio $AB : OM$ is independent of a.

10 A spider starts at one corner of a cuboidal room with dimensions 5 m by 5 m by 5 m. It can crawl freely across the surface of the wall.

What is the shortest distance it needs to travel to get to the opposite end of the room?

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Section 2: Equation of a straight line

You already know that a non-vertical straight line has an equation of the form $y = mx + c$.

In this equation, m is the **gradient** of the line and c is the **y-intercept**. If you know these two pieces of information you can simply write down the equation.

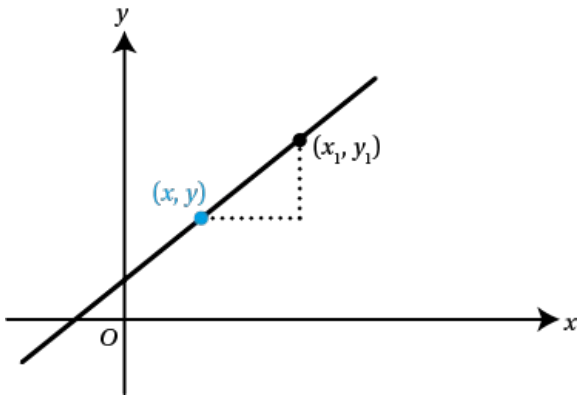
However, this is often not the information you are given. It is more common to know two points the line passes through, or the gradient and one point.

Gateway to A Level

For a reminder and more practice of $y = mx + c$, see Gateway to A Level revision section M.

Equation of the line with a given gradient and one point

This diagram shows a straight line passing through the point with coordinates (x_1, y_1). The gradient of the line is m.



Let (x, y) be any other point on the line. The equation of the line is a rule connecting x and y. You can use the dotted triangle to write an equation for the gradient:

$$m = \frac{y - y_1}{x - x_1}$$

Gateway to A Level

For revision of finding the gradient from two points on the line, see Gateway to A Level section L.

Rearranging this equation gives the form of the equation of the line in Key point 6.3.

Key point 6.3

The line with gradient m through point (x_1, y_1) has equation $y - y_1 = m(x - x_1)$.

If you need to, you can rearrange the equation in Key point 6.3 into the form $y = mx + c$.

WORKED EXAMPLE 6.3

Find the equation of the line with gradient $\frac{2}{3}$ which passes through the point $(-3, 1)$. Give your answer in the form $y = mx + c$.

$y - y_1 = m(x - x_1)$ Use the equation from Key point 6.3.

$y - 1 = \frac{2}{3}(x - (-3))$

$\Leftrightarrow y - 1 = \frac{2}{3}x + 2$ Expand the brackets and rearrange.

$\Leftrightarrow y = \frac{2}{3}x + 3$

Equation of a line through two points

If you know the coordinates of two points the line passes through, you can use their coordinates to find the gradient. You can then use the method used in Worked example 6.3 to find the equation of the line.

WORKED EXAMPLE 6.4

Find the y-intercept of the line containing points $(2, -3)$ and $(-1, 4)$.

Gradient: To find the equation, you need the gradient and one point.

$m = \frac{4 - (-3)}{-1 - 2} = -\frac{7}{3}$

Equation of the line: Now use the equation from Key point 6.3. You can use **either** of the two points.

$y - y_1 = m(x - x_1)$

$y - (-3) = -\frac{7}{3}(x - 2)$

$\Leftrightarrow y + 3 = -\frac{7}{3}(x - 2)$

When $x = 0$: The y-intercept is when $x = 0$.

$y + 3 = -\frac{7}{3}(0 - 2)$

$y = \frac{14}{3} - 3 = \frac{5}{3}$

The y-intercept is $\left(0, \frac{5}{3}\right)$.

You should check that using the point $(-1, 4)$ gives the same equation.

The form $ax + by + c = 0$

Sometimes it is convenient to write the equation of a line in a form other than $y = mx + c$.

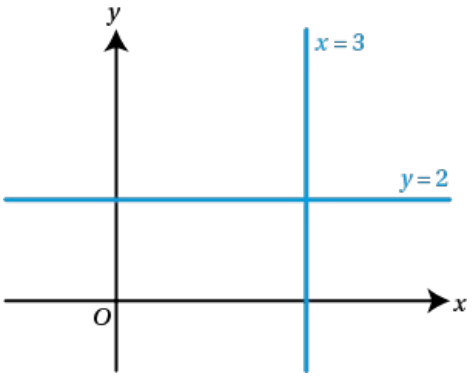
For example, when solving simultaneous equations you may want to write the equations in the form such as $3x + 2y = 7$.

You should also remember that a vertical line has an equation such as $x = 3$ that cannot be written in the form $y = mx + c$.

If you start with the equation of a line in the form $y - y_1 = m(x - x_1)$ it is easy to rearrange it into the form $ax + by + c = 0$.

This form makes it straightforward to find the x- and y-intercepts. If you want to find the gradient of a line given in this form you need to rewrite it as $y = mx + c$.

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WORKED EXAMPLE 6.5

Line l passes through the point $(-2, 2)$ and has the same gradient as the line with equation $3x + 5y = 7$.

a Find the equation of l in the form $ax + by + c = 0$ where a, b and c are integers.

b The line l crosses the coordinate axes at points A and B. Find the exact distance AB.

a Gradient:

$3x + 5y = 7$

$\Leftrightarrow 5y = 7 - 3x$

$\Leftrightarrow y = \frac{7}{5} - \frac{3}{5}x$

So $m = -\frac{3}{5}$

You first need to find the gradient of l. This requires rewriting $3x + 5y = 7$ in the form $y = mx + c$.

The gradient is the coefficient of x.

Equation of l :

$y - 2 = -\frac{3}{5}(x - (-2))$

Now use $y - y_1 = m(x - x_1)$.

$\Leftrightarrow 5y - 10 = -3(x + 2)$

$\Leftrightarrow 5y - 10 = -3x - 6$

$\Leftrightarrow 3x + 5y - 4 = 0$

You need all the coefficients to be integers, so multiply through by 5.

b x-intercept is when $y = 0$:

$3x + 0 - 4 = 0$

$\Rightarrow x = \frac{4}{3}$

You need to find the coordinates of A and B.

y-intercept is when $x = 0$:

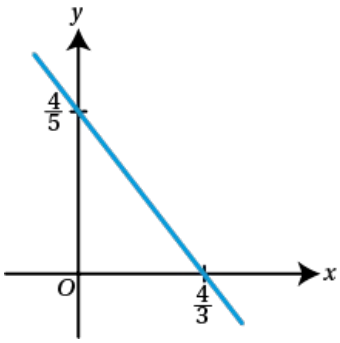
$0 + 5y - 4 = 0$

$\Rightarrow y = \frac{4}{5}$

Distance between $A\left(\frac{4}{3}, 0\right)$ and $B\left(0, \frac{4}{5}\right)$:

The distance between points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ In this case, you can also use the graph to help.

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$$\text{AB} = \sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{16}{9} + \frac{16}{25}}$$
$$= \sqrt{\frac{544}{225}}$$
$$= \frac{4\sqrt{34}}{15}$$

EXERCISE 6B

1 Write down the equation of each line in the form $y - y_1 = m(x - x_1)$.

a i gradient 3, through the point $(4, -1)$

ii gradient 5, through the point $(-3, 2)$

b i gradient $-\frac{1}{2}$, through the point $(-3, 1)$

ii gradient $\frac{4}{3}$, through the point $(1, 3)$

c i through points $(3, 7)$ and $(5, 15)$

ii through points $(4, 1)$ and $(7, 10)$

d i through points $(3, -1)$ and $(-4, 5)$

ii through points $(-1, -7)$ and $(3, 2)$

2 Find the equation of each line in the form $ax + by + c = 0$ where a, b and c are integers.

a i gradient -2, through the point $(-1, 3)$

ii gradient -5, through the point $(3, -2)$

b i gradient $\frac{1}{3}$, through the point $(-1, -7)$

ii gradient $-\frac{3}{2}$, through the point $(2, -5)$

c i through points $(-1, 2)$ and $(2, 7)$

ii through point $(3, 5)$ and $(5, -4)$

d i through points $(2, 1)$ and $(-3, 7)$

ii through points $(-3, 2)$ and $(1, -5)$

3 Find the gradient, the x- and y-intercepts of the lines with the following equations:

a i $y = 3x + 1$

ii $y = -2x + 3$

b i $3x - 2y + 5 = 0$

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ii $4x + 5y - 1 = 0$

c **i** $3y - 3x + 5 = 0$

ii $y - 4x - 6 = 0$

d **i** $4x - 3y = 7$

ii $5x + 2y = 3$

e **i** $y - 3 = 2(x - 2)$

ii $y + 1 = 3(x - 5)$

4 Find the intersections of the following pairs of lines:

a **i** $y = \frac{1}{2}x - 3$ and $y = x + 1$

ii $y = \frac{2}{3}x + 2$ and $y = 2x - 2$

b **i** $2x + 3y = 1$ and $x - 3y + 1 = 7$

ii $3x - y = 3$ and $2x + 5y = 1$

c **i** $5x + 2y + 3 = 1$ and $3x + 4y - 1 = 0$

ii $4x - 2y - 1 = 0$ and $3x - 5y + 5 = 0$

Tip

In part d, do the lines intersect for all values of a and b?

5 **a** Find the equation of the line passing through the points with coordinates $(-4, 3)$ and $(5, -1)$. Give your answer in the form $ax + by + c = 0$ where a, b and c are integers.

b A second line has the same gradient as the line in part **a** and passes through the point $(-4, 2)$. Find the equation of this line.

c Find the value of k such that the point $(3k, k)$ lies on the line in part b.

6 A straight line has gradient $-\frac{2}{3}$ and passes through the point with coordinates $(2, 5)$. It cuts the coordinate axes at points P and Q. Find the area of the triangle OPQ where O is the origin.

7 Line l_1 passes through the points $(-3, -1)$ and $(10, 12)$. Line l_2 passes through the point $(-16, 12)$ and has gradient $-\frac{3}{2}$.

a Find the equations of l_1 and l_2 in the form $ax + by = c$.

b Find the coordinates of the point of intersection, P, of l_1 and l_2.

l_1 intersects the x-axis at Q and l_2 intersects the x-axis at R.

c Find the area of triangle PQR.

8 A line passes through the points $A(2 - 2k, k)$ and $B(k - 1, 2k + 1)$, where k is a constant. The gradient of the line is $\frac{2}{3}$.

a Find the value of k.

b Find the equation of the line.

9 The line with equation $4y - 7x + 14 = 0$ crosses the coordinate axes at points A and B. M is the midpoint of AB. Find the distance of M from the origin, giving your answer in the form $\frac{\sqrt{p}}{q}$ where p and q are integers.

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10 A line passes through the point $\left(\frac{p}{3}, \frac{2}{p}\right)$ and has gradient $-\frac{6}{p^2}$. It crosses the coordinate axes at points A and B. Show that the area of the triangle OAB is independent of p.

Section 3: Parallel and perpendicular lines

It is useful to be able to tell whether two lines are parallel or perpendicular, without having to draw them accurately. You already know how to decide whether two lines are parallel:

Key point 6.4

Two lines are parallel if they have the same gradient.

Tip

The easiest way to identify the gradient is to write the equation of the line in the form $y = mx + c$.

WORKED EXAMPLE 6.6

Find the equation of the line that is parallel to $3x + 5y - 2 = 0$ and passes through the point $(-1, 3)$. Give your answer in the form $ax + by = c$ where a, b and c are integers.

$3x + 5y - 2 = 0$ — Rearrange the equation in order to identify the gradient.

$\Leftrightarrow 5y = -3x + 2$

$\Leftrightarrow y = -\frac{3}{5}x + \frac{2}{5}$

$\therefore m = -\frac{3}{5}$ — Parallel lines have the same gradient.

Line passes through $(-1, 3)$: — Use $y - y_1 = m(x - x_1)$.

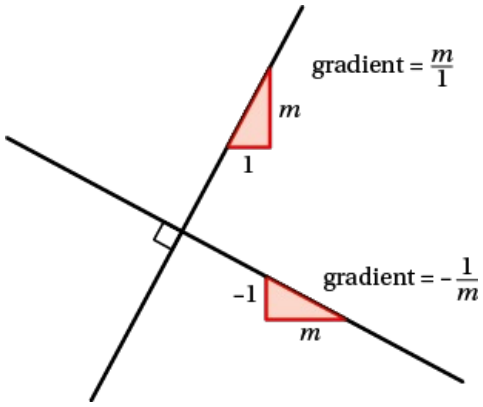
$y - 3 = -\frac{3}{5}(x + 1)$

$\Leftrightarrow 5y - 15 = -3x - 3$ — Rearrange into the required form.

$\Leftrightarrow 3x + 5y = 12$

The diagram shows how the gradients of a pair of perpendicular lines are related.

For a line with gradient m you can draw a right-angled triangle with horizontal side 1 and vertical side m. When the line is rotated through 90° the horizontal and vertical distances are swapped, so that the horizontal side is m and vertical -1.



Key point 6.5

- If a line has gradient m, the gradient of any perpendicular line is $m_1 = \frac{-1}{m}$.
- Two lines with gradients m_1 and m_2 are perpendicular if $m_1 m_2 = -1$.

For example, lines with gradients $-\frac{2}{3}$ and $\frac{3}{2}$ are perpendicular because $-\frac{2}{3} \times \frac{3}{2} = -1$ and lines with gradients -2 and $\frac{1}{2}$ are perpendicular because $-2 \times \frac{1}{2} = -1$.

Tip

Remember that a horizontal line has gradient zero and the gradient of a vertical line is undefined; so this calculation does not apply to those lines.

WORKED EXAMPLE 6.7

Points A, B and C have coordinates $A(3, -2), B(p, 2)$ and $C(-1, 5)$. Find the possible values of p so that ABC is a right angle.

$m_{AB} = \frac{4}{p-3}$, $m_{BC} = \frac{3}{-1-p}$ — The gradients of AB and BC should multiply to -1.

$m_{\mathrm{AB}}\, m_{\mathrm{BC}} = -1$

$\Rightarrow \frac{4}{p-3} \times \frac{3}{-1-p} = -1$

$\Rightarrow 12 = -1(-p - p2 + 3 + 3p)$

$\Rightarrow p^2 - 2p - 15 = 0$

$\Rightarrow (p-5)(p+3) = 0$ — Solve the quadratic equation.

$\Rightarrow p = -3$ or 5

WORK IT OUT 6.1

Find the gradient of a line that is perpendicular to the line with equation $3x + 5y = 12$.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1	Solution 2	Solution 3
$m = \frac{5}{3}$, so $m_1 = -\frac{3}{5}$	$m = -\frac{3}{5}$ so $m_1 = \frac{5}{3}$	$m = -\frac{3}{5}$ so $m_1 = \frac{3}{5}$

One important example of perpendicular lines is the **perpendicular bisector** of a line segment joining two points.

Fast forward

You will meet another important example, the equation of a tangent to a circle, in Section 4.

WORKED EXAMPLE 6.8

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Find the equation of the perpendicular bisector of the line segment joining points $(-3, 4)$ and $(5, 1)$. Give your answer in the form $ax + by = c$.

Gradient of the line segment:

$$m = \frac{1-4}{5-(-3)} = -\frac{3}{8}$$

You need the gradient of the segment to find the perpendicular gradient.

Perpendicular gradient:

$$m_1 = \frac{-1}{\left(-\frac{3}{8}\right)} = \frac{8}{3}$$

Midpoint of the segment joining $(-3, 4)$ to $(5, 1)$:

The perpendicular bisector passes through the midpoint of the line segment.

$$x = \frac{-3+5}{2} = 1$$

$$y = \frac{4+1}{2} = \frac{5}{2}$$

Equation of the line:

$$y - \frac{5}{2} = \frac{8}{3}(x-1)$$

$$\Leftrightarrow 6y - 15 = 16x - 16$$

$$\Leftrightarrow 16x - 6y = -1$$

EXERCISE 6C

1 Find the gradient of a line perpendicular to the given line.

a **i** $3y = 5x - 2$

ii $6y = x - 7$

b **i** $4x = 2y - 3$

ii $5x + 3 = 2y$

c **i** $y = 4 - 2x$

ii $y = 1 - 3x$

d **i** $3x - 7y + 1 = 0$

ii $x - 5y + 3 = 0$

2 Determine whether each pair of lines is parallel, perpendicular or neither.

a **i** $y = 3 - 4x$ and $y = \frac{1}{4}x - 5$

ii $y = 3 - x$ and $y = 5 - x$

b **i** $3x - y + 7 = 0$ and $y - 3x + 5 = 0$

ii $5x - 2y + 3 = 0$ and $2y - 5x + 3 = 0$

c **i** $7x + 2y - 3 = 0$ and $2x - 7y + 4 = 0$

ii $2x + 4y - 4 = 0$ and $4x + 2y + 1 = 0$

d **i** $5x + 3y - 1 = 0$ and $3y - 5x + 2 = 0$

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ii $2y - 7x = 3$ and $7x + 2y = 7$

3 **a** Show that the point $P(1, 4)$ lies on the line l_1 with equation $5x - 2y + 3 = 0$.

b Line l_2 passes through P and is perpendicular to l_1. Find the equation of l_2 in the form $ax + by + c = 0$ where a, b and c are integers.

4 Points A and B have coordinates $A(-2, 3)$ and $B(1, 5)$. O is the origin. Show that the triangle ABO is right-angled, and find its area.

5 **a** Find the coordinates of the midpoint oaf the line segment connecting points $A(5, 2)$ and $B(-1, 7)$.

b Hence find the equation of the perpendicular bisector of AB, giving your answer in the form $y = mx + c$.

6 **a** Find the equation of the line that is parallel to the line with equation $2x + 3y = 6$ and passes through the point $(-4, 1)$.

b The two lines cross the x-axis at points P and Q. Find the distance PQ.

7 Point P has coordinates $(0, 7)$ and point R has coordinates $(12, 4)$. Point Q lies on the x-axis and PQR is a right angle. Find the possible coordinates of Q, giving your answers in surd form.

8 Points P and Q have coordinates $(-4, 3)$ and $(5, 1)$. Find the equation of the perpendicular bisector of PQ in the form $ax + by + c = 0$ where a, b and c are integers.

9 Point M has coordinates $(3, 5)$. Points A and B lie on the coordinate axes and have coordinates $(0, p)$ and $(q, 0)$, so that $A\widehat{M}B$ is a right angle.

a Show that $5p + 3q = 34$.

b Given that $p = 4$, find the value of q and the exact area of the quadrilateral $AOMB$ (where O is the origin).

10 Line l has equation $x - 2y + 3 = 0$ and point P has coordinates $(-1, 6)$.

a Find the equation of the line through P that is perpendicular to l.

b Hence find the shortest distance from P to l.

Gateway to A Level

For a reminder of the properties of the parallelogram and rhombus, see Gateway to A Level section N.

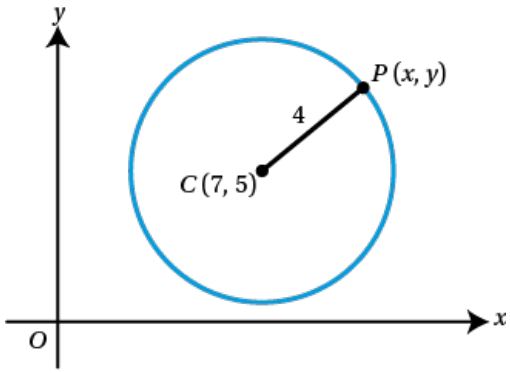
11 Four points have coordinates $A(k, 2), B(k + 1, k + 2), C(k - 3, k + 4)$ and $D(k - 4, 4)$.

a Show that $ABCD$ is a parallelogram for all values of k.

b Find the value of k for which $ABCD$ is a rectangle.

Section 4: Equation of a circle

In this section you will see how to find an equation of a circle with a given centre and radius. As an example, consider the circle with centre at the point $C(7, 5)$ and radius $r = 4$ as shown in the diagram.



An equation of a circle is a rule satisfied by the coordinates of all the points on the circle. Let P be a point on the circle with coordinates (x, y). The distance CP equals the radius of the circle. Using the formula for the distance (or Pythagoras' theorem):

$$(x-7)^2 + (y-5)^2 = 4^2$$

Explore

Find out as much as you can about a curve called an ellipse. What is it? In what ways is it similar to a circle and how is it different?

This equation is satisfied by the coordinates of any point of the circle. This result is generalised in Key point 6.6.

You can use the equation to check whether a point lies on, inside or outside a given circle.

Key point 6.6

The circle with centre (a, b) and radius r has equation $(x-a)^2 + (y-b)^2 = r^2$.

WORKED EXAMPLE 6.9

A circle has radius 5 and the coordinates of its centre are $(-3, 6)$.

a Write down the equation of the circle.

b Determine whether the following points lie on, inside or outside the circle:

i $M(2, 6)$

ii $N(3, 1)$

a $(x-(-3))^2 + (y-6)^2 = 5^2$

$\Leftrightarrow (x+3)^2 + (y-6)^2 = 25$

Take care with negative numbers.

b i Point M:

$(2+3)^2 + (6-6)^2 = 5^2 + 0^2 = 25$

Point M lies on the circle.

Substitute the coordinates into the equation of the circle.

ii Point N:

$61 > 25$, so the point is further than 5 units from the centre of the circle.

$(3+3)^2 + (1-6)^2 = 62 + (-5)^2 = 61$

Point N lies outside the circle.

If you are given an equation of a circle, you can identify the centre and radius. You may need to complete the square first.

Rewind

Completing the square was covered in Chapter 3, Section 3.

WORKED EXAMPLE 6.10

Find the radius and the coordinates of the centre of the circle with equation $x^2 - 3x + y^2 + 4y = 12$.

$\left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + (y+2)^2 - 2^2 = 12$ Complete the square for both x and y.

$\Leftrightarrow \left(x - \frac{3}{2}\right)^2 + (y+2)^2 = 12 + \frac{9}{4} + 4$ Rearrange into the form $(x-a)^2 + (y-b)^2 = r^2$.

$\Leftrightarrow \left(x - \frac{3}{2}\right)^2 + (y+2)^2 = \frac{73}{4}$

The centre is $\left(\frac{3}{2}, -2\right)$ and the radius is $\frac{\sqrt{73}}{2}$. Remember that the number on the right is r^2.

If you are given three points you can draw a circle passing through them (unless the three points lie in a straight line). Finding the centre and radius of the circle involves a long calculation. However there is one special case where you can use a circle theorem to simplify the calculation.

Fast forward

See question 8 in Exercise 6D for how to find a circle determined by three points.

Gateway to A Level

For a reminder and more practice of circle theorems, see Gateway to A Level section O.

WORKED EXAMPLE 6.11

Points $A(3, 9), B(4, 2)$ and $C(11, 3)$ lie on a circle.

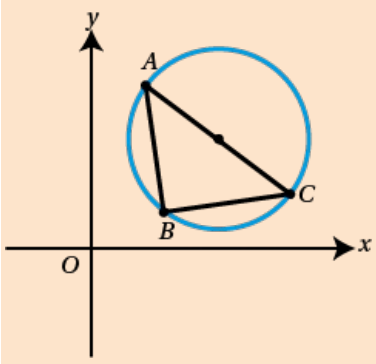
a Show that AC is a diameter of the circle.

b Hence find the equation of the circle.

a $m_{AB} = \frac{2-9}{4-3} = -7$

$m_{BC} = \frac{3-2}{11-4} = \frac{1}{7}$

If AC is a diameter then ABC is a right angle. You can check this by finding the gradients of AB and BC.



$m_{AB}m_{BC} = -1$ so AB is perpendicular to BC.

Since $\text{ABC} = 90^\circ$, AC is a diameter of the circle.

Perpendicular lines have $m_1 m_2 = -1$.

b Find the centre:

$$M = \left(\frac{3+11}{2}, \frac{9+3}{2}\right) = (7, 6)$$

You need to find the centre and the radius. Since AC is a diameter, the centre is the midpoint of AC.

Find the radius:

$$AC^2 = (11-3)^2 + (3-9)^2 = 100$$
$$AC = 10$$
$$\therefore r = 5$$

The radius is half the distance AC.

$$(x-7)^2 + (y-6)^2 = 25$$

Now use the equation of a circle.

EXERCISE 6D

1 Find the equation of the circle with the given centre and radius.

a i centre $(3, 7)$, radius 4

ii centre $(5, 1)$, radius 6

b i centre $(3, -1)$, radius $\sqrt{7}$

ii centre $-4, 2$, radius $\sqrt{5}$

2 Write down the centre and radius of the following circles:

a i $(x-2)^2 + (y+3)^2 = \frac{9}{4}$

ii $(x+1)^2 + (y+5)^2 = \frac{4}{25}$

b i $(x-3)^2 + \left(y - \frac{1}{2}\right)^2 = 6$

ii $\left(x + \frac{3}{4}\right)^2 + \left(y - \frac{1}{5}\right)^2 = 3$

3 Find the centre and radius of the following circles:

a i $x^2 + 4x + y^2 - 6y + 4 = 0$

ii $x^2 - 8x + y^2 + 2y + 8 = 0$

b i $x^2 - 2x + y^2 + 6y + 1 = 0$

ii $x^2 - 10x + y^2 + 4y - 1 = 0$

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c **i** $x^2 + 5x + y^2 - y + 2 = 0$

ii $x^2 - 3x + y^2 + 7y - 3 = 0$

d **i** $x^2 + y^2 - 5y = 12$

ii $x^2 + y^2 + 3x = 10$

4 Determine whether each point lies on, inside or outside the given circle.

a **i** point $(1, 7)$, circle centre $(-2, 3)$, radius 5

ii point $(2, -1)$, circle centre $(-3, 3)$, radius $\sqrt{41}$

b **i** point $(-1, 1)$, circle centre $(3, 6)$, radius 5

ii point $(2, 1)$, circle centre $(5, -1)$, radius 7

5 **a** Write down the equation of the circle with centre $(-6, 3)$ and radius $\sqrt{117}$.

b Find the coordinates of the points where the circle cuts the y-axis.

6 **a** Find the centre and the radius of the circle with equation $x^2 - 5x + y^2 + y = 3$.

b Determine whether the point $A(-1, 3)$ lies inside or outside the circle.

7 A circle with centre $(3, -5)$ and radius 7 crosses the x-axis at points P and Q. Find the exact distance PQ.

8 Points A, B and C have coordinates $A(-7, 3), B(3, 9)$ and $C(12, -6)$.

a Show that ABC is a right angle.

b Find the distance AC.

c Hence find the equation of the circle passing through the points A, B and C.

9 The circle with equation $(x - p)^2 + (y + 3)^2 = 26$ where p is a positive constant passes through the origin.

a Find the value of p.

b Determine whether the point $(3, 2)$ lies inside or outside the circle.

10 A diameter of a circle has endpoints $\mathrm{P}(a, b)$ and $Q(c, d)$. Let $Z(x, y)$ be any other point on the circle.

a Write down the size of the angle PZQ.

b Hence prove that the equation of the circle can be written as $(x - a)(x - c) + (y - b)(y - d) = 0$.

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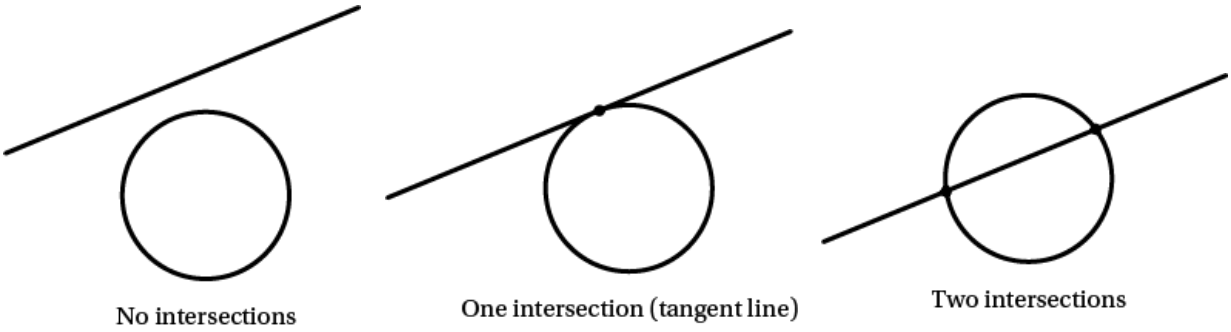
Section 5: Solving problems with lines and circles

In this section you will solve a variety of problems involving lines and circles. You start by looking at intersections, which involves solving simultaneous equations.

Rewind

Before starting this section, you may want to review the circle theorems listed at the start of this chapter.

For the intersection of a line and a circle, there are three possibilities, shown in these diagrams:



No intersections | One intersection (tangent line) | Two intersections

Because the equation of the circle is quadratic, you can use the discriminant to determine whether there are two, one or no intersections.

Rewind

See Chapter 5, Sections 1 and 2, for a reminder of quadratic simultaneous equations.

WORKED EXAMPLE 6.12

A circle has centre $(3, 2)$. Find the radius of the circle so that the circle is tangent to the line with equation $x + 5y = 20$.

Equation of the circle:
$(x-3)^2 + (y-2)^2 = r^2$

Start by writing both equations and trying to find the intersection.

Equation of the line:
$x + 5y = 20 \Rightarrow x = 20 - 5y$

Intersection: $(17-5y)^2 + (y-2)^2 = r^2$

Substitute x from the equation of the line into the equation of the circle.

$\Leftrightarrow (289 - 170y + 25y^2) + (y^2 - 4y + 4) = r^2$

$26y^2 - 174y + (293 - r^2) = 0$

Write in standard quadratic form in order to find the discriminant.

Tangent means one solution so the discriminant is zero:

$174^2 - 4(26)(293 - r^2) = 0$

$\Rightarrow 104r^2 - 196 = 0$

$\Rightarrow r^2 = \frac{49}{26}$

$\Rightarrow r = 1.37$

The radius of the circle is a positive number.

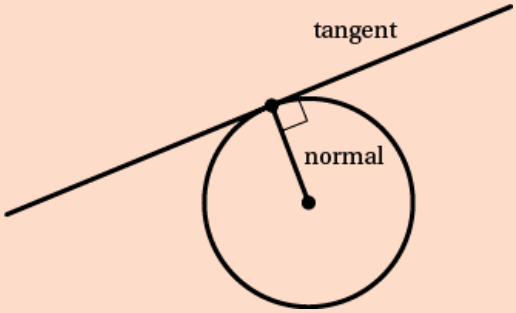
To find the equation of the tangent to a circle at a given point you can use one of the circle theorems: that the tangent is perpendicular to the radius at the point of contact. You can therefore find the gradient of the tangent by using $m_1 m_2 = -1$ for perpendicular lines.

Focus on ...

Focus on ... Problem solving 1 explores alternative methods of solving the problem from the previous worked example.

Key point 6.7

- The tangent to the circle is perpendicular to the radius at the point of contact.
- The normal is the line containing the point of contact and the centre of the circle.



Fast forward

The line that is perpendicular to the tangent at the point of contact is called a **normal** to the curve. In the case of the circle, the normal is the same line as the radius of the circle.

You will learn about tangents and normals to other curves in Chapter 13.

WORKED EXAMPLE 6.13

Find the equation of the tangent and normal to the circle $(x-3)^2 + (y+5)^2 = 5$ at the point $(2, -7)$.

Coordinates of the centre: $(3, -5)$

Gradient of the radius:

$$m_1 = \frac{-7-(-5)}{2-3} = -\frac{-2}{-1} = 2$$

Gradient of the tangent:

$$m_1 m_2 = -1 \Rightarrow m_2 = -\frac{1}{2}$$

You can tell that $(2, -7)$ does indeed lie on the circle because $(-1)^2 + (-2)^2 = 5$. The tangent is perpendicular to the radius, which is the line connecting $(2, -7)$ to the centre.

Tangent passes through $(2, -7)$:

Now use the equation of a straight line.

$$y-(-7) = -\frac{1}{2}(x-2)$$

$$\Rightarrow y = -\frac{1}{2}x - 6$$

Normal: $y-(-7) = 2(x-2)$

$$\Rightarrow y = 2x - 11$$

The normal is the line connecting $(2, -7)$ to the centre. You already know that its gradient is 2.

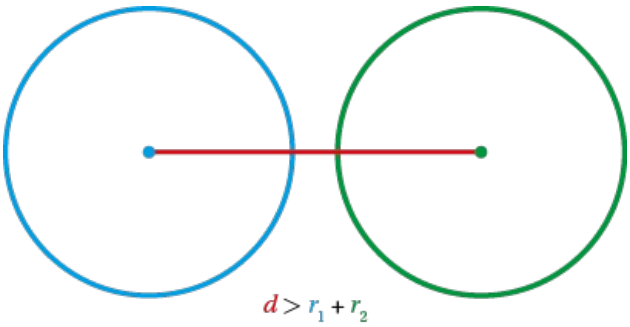
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Worksheet

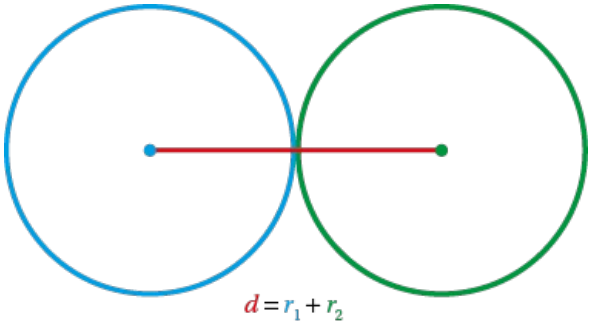
For another example of finding a tangent to a circle, see Support sheet 6.

Intersection of two circles

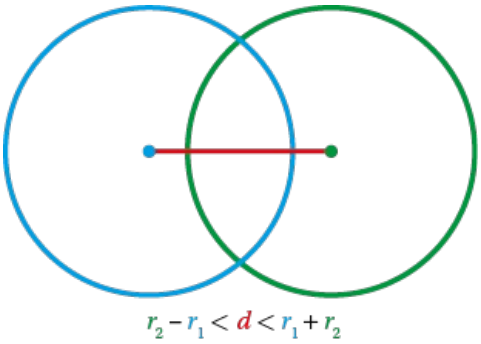
There are five possibilities for the relative position of two distinct circles. A clever way to distinguish between them involves comparing the distance between their centres to the radii of the circles.



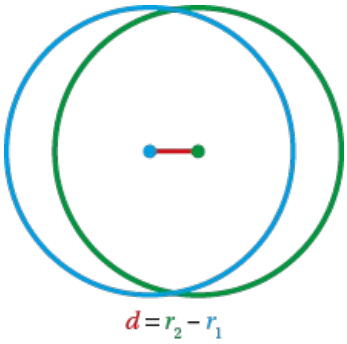
$d > r_1 + r_2$



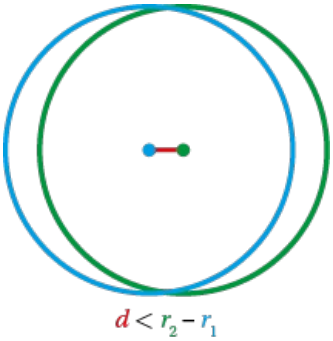
$d = r_1 + r_2$



$r_2 - r_1 < d < r_1 + r_2$



$d = r_2 - r_1$



$d < r_2 - r_1$

WORKED EXAMPLE 6.14

Two circles have equations $x^2 - 6x + y^2 - 20y + 45 = 0$ and $(x - 15)^2 + (y - 5)^2 = r^2$.

a In the case $r = 7$ show that the two circles intersect at two different points.

b Given that the two circles are externally tangent to each other, find the value of r.

a $x^2 - 6x + y^2 - 20y + 45 = 0$

$\Leftrightarrow (x - 3)^2 + (y - 10)^2 - 9 - 100 + 45 = 0$

$\Leftrightarrow (x - 3)^2 + (y - 10)^2 = 64$

You need to compare the distance between the centres to the radii of the circles. So first you need to identify the centre and radius of both circles.

The first circle has centre $(3, 10)$ and radius 8.

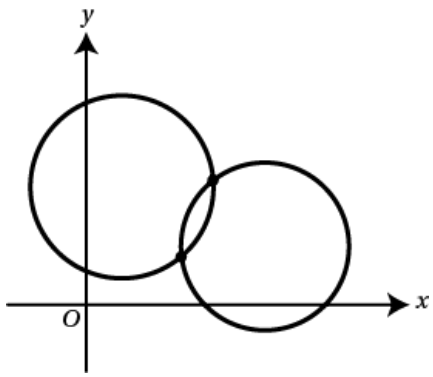
The second circle has centre $(15, 5)$ and

The distance between the centres is less than the sum of the radii but more than their difference.

radius 7.

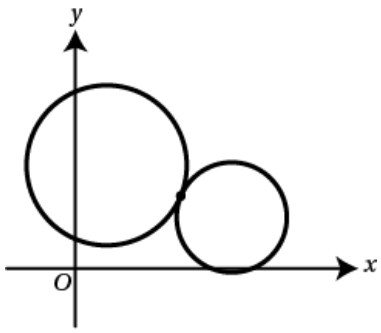
The distance between the centres is

$d = \sqrt{(3-15)^2 + (10-5)^2} = 13$



So the two circles intersect.

b



'Externally tangent' means that they touch on the outside. The distance between the centres is equal to the sum of the radii.

$8 + r = 13$

$\Rightarrow r = 5$

Finding points of intersection of two circles can be difficult, as it involves solving two simultaneous quadratic equations. However, there are some special cases where it is possible to use the substitution method as you did when intersecting a circle with a line.

WORKED EXAMPLE 6.15

Find the coordinates of the points of intersection of the circles $(x-2)^2 + y^2 = 36$ and $(x-2)^2 + (y-10)^2 = 64$.

From the first equation:

$(x-2)^2 = 36 - y^2$

Substitute into the second:

$36 - y^2 + (y-10)^2 = 64$

$\Rightarrow 36 - y^2 + y^2 - 20y + 100 = 64$

$\Rightarrow 20y = 72$

$\Rightarrow y = 3.6$

You need to solve the two simultaneous equations. The term $(x-2)^2$ is common to both equations so you can substitute it from one equation into the other.

$(x-2)^2 + 3.6^2 = 36$

$\Rightarrow (x-2)^2 = 23.04$

Now substitute back to find the x-coordinates.

$\Rightarrow x - 2 = \pm 4.8$

$\therefore x = 6.8$ or -2.8

The coordinates are $(6.8, 3.6)$ and $(-2.8, 3.6)$.

EXERCISE 6E

1 Find the equations of the following:

a i tangent to the circle $x^2 - 2x + y^2 = 15$ at the point $(1, 4)$

ii tangent to the circle $x^2 + y^2 + 6y = 25$ at the point $(-3, 2)$

b i normal to the circle $x^2 + 4x + y^2 - 6y = 0$ at the point $(1, 1)$

ii normal to the circle $x^2 - 4x + y^2 = 9$ at the point $(5, 2)$

c i tangent to the circle with centre $(1, 2)$ and radius $\sqrt{5}$ at the point $(3, 3)$

ii tangent to the circle with centre $(-3, 1)$ and radius $\sqrt{32}$ at the point $(1, 5)$

2 Determine whether the line and the circle intersect. Where they do, find the coordinates of the point(s) of intersection.

a i $x^2 + 3x + y^2 = 24$ and $y = 2x + 1$

ii $x^2 - 5x + y^2 + y = 17$ and $y = 2x - 1$

b i $x^2 - 12x + y^2 - 10y + 41 = 0$ and $2x - y + 3 = 0$

ii $x^2 + y^2 = 32$ and $x + y = 8$

c i $x^2 + y^2 - 10y + 9 = 0$ and $x - y = 7$

ii $x^2 - 8x + y^2 - 6y = 25$ and $x + y = 20$

3 Determine whether the two circles intersect, are disjointed or tangent to each other, or whether one circle is completely inside the other one.

a i $(x - 3)^2 + (y - 5)^2 = 16$ and $(x + 1)^2 + (y - 2)^2 = 25$

ii $(x + 2)^2 + (y - 1)^2 = 64$ and $(x - 1)^2 + (y + 2)^2 = 64$

b i $(x - 4)^2 + (y + 1)^2 = 20$ and $(x + 3)^2 + (y - 5)^2 = 17$

ii $x^2 + (y - 2)^2 = 10$ and $(x - 3)^2 + (y + 3)^2 = 12$

c i $x^2 + y^2 = 30$ and $(x + 1)^2 + (y - 1)^2 = 6$

ii $(x + 3)^2 + (y - 2)^2 = 9$ and $(x + 7)^2 + (y + 1)^2 = 4$

4 Line l_1 has equation $5x - 2y = 7$ and line l_2 is perpendicular to l_1 and passes through the point $(-3, 1)$. Find the coordinates of the point of intersection of the two lines.

5 **a** Show that the point $P(-3, 2)$ lies on the circle with equation $(x - 1)^2 + (y + 2)^2 = 32$.

b Write down the coordinates of the centre of the circle.

c Find the equation of the tangent to the circle at P.

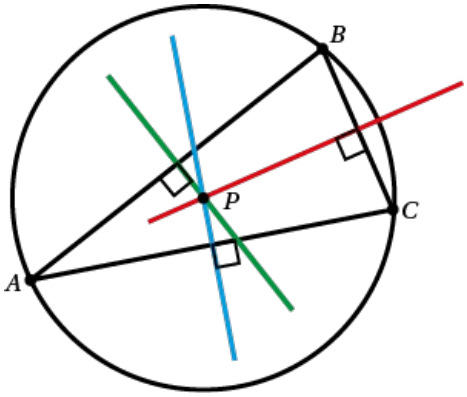
6 Line l_1 has equation $2x + y - 10 = 0$ and line l_2 is perpendicular to l_1 and crosses the x-axis at the point $A(-2, 0)$.

a Find the equation of l_2.

b Find the coordinates of M, the point of intersection of l_1 and l_2.

c Line l_1 crosses the x-axis at B. Find the exact area of the triangle AMB.

7 The circumcircle of a triangle is the circle containing all three vertices.



Its centre is the point of intersection of the perpendicular bisectors of the sides.

A triangle has vertices $A(1, 3), B(5, 7)$ and $C(2, 9)$.

a Find the equations of the perpendicular bisectors of AB and AC.

b Find the coordinates of their intersection, P. This is the centre of the circumcircle.

c Find the equation of the perpendicular bisector of BC and verify that it also passes through P.

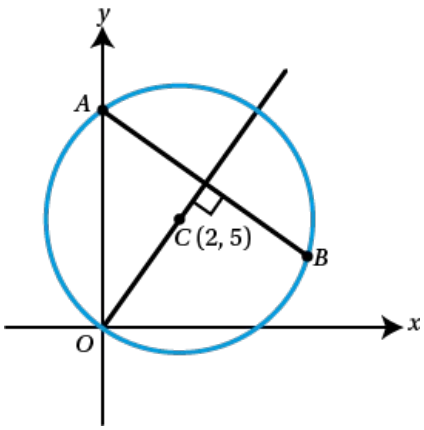
d Find the exact value of the radius of the circumcircle of triangle ABC.

8 A circle with centre $C(2, 5)$ passes through the origin.

a Find the equation of the circle.

b Show that the point $A\ (0, 10)$ lies on the circle.

B is another point of the circle such that the chord AB is perpendicular to the radius OC (extended).



c Find the length of AB correct to three significant figures.

9 Circle C_1 has centre $(-2, 5)$ and radius 7. Circle C_2 has centre $(12, 5)$.

a Given that the two circles are tangent to each other, find the two possible values for the radius of C_2.

b Given instead that the radius of C_2 is 16, find the coordinates of the intersection points of C_1 and C_2.

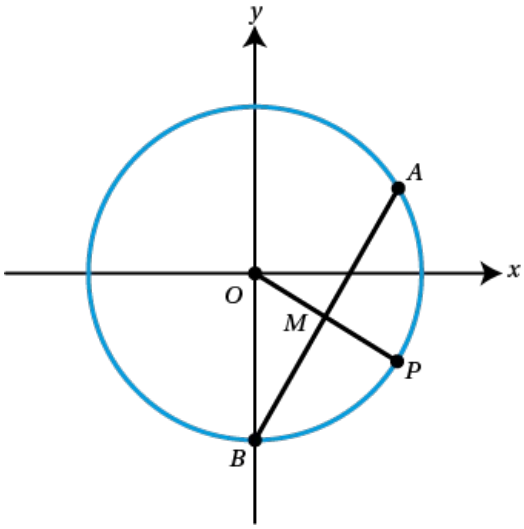
10 A circle with centre at the origin passes through the point $(2, 6)$. The tangent to the circle at $(2, 6)$ cuts the coordinate axes at points P and Q. Find the area of the triangle OPQ.

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11 Find the values of k for which the line $y = kx$ is tangent to the circle with centre $(3, 6)$ and radius 2.

12 The line $3x - y = 3$ is tangent to the circle with centre $(5, -1)$ and radius r. Find the value of r.

13 The circle with centre at the origin and radius 5 cuts the negative y-axis at point B. Point $A(4, 3)$ lies on the circle. Let M be the midpoint of the chord AB. The line through O and M cuts the circle at the point P, as shown in the diagram.



a Find the coordinates of M.

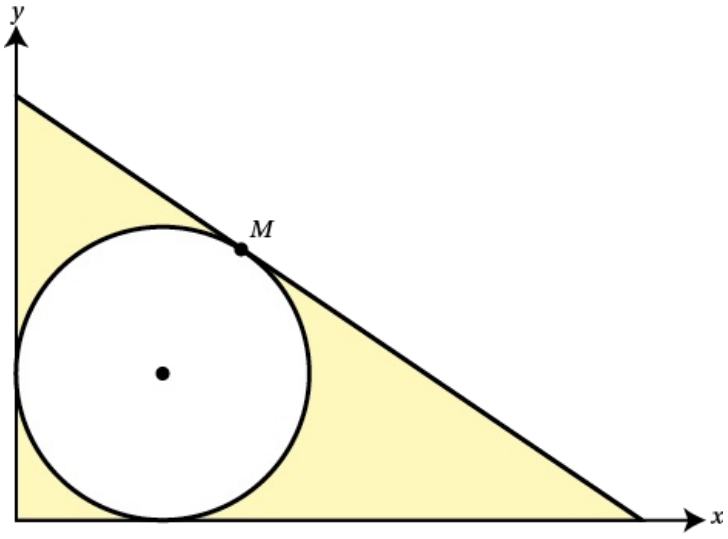
b Show that the quadrilateral $OAPB$ is not a rhombus.

14 A circle has equation $x^2 + y^2 - 10x - 10y + 25 = 0$.

a Show that the circle is tangent to both coordinate axes.

b Show that the point $M(8, 9)$ lies on the circle.

The diagram shows the circle and the tangent at M.



c Find the exact value of the shaded area.

15 Find a condition on a and b so that the curve $x^2 + y^2 = 1$ touches the curve $(x - a)^2 + (y - b)^2 = r^2$ at exactly one point.

Checklist of learning and understanding

- Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$:
  - the distance $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

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    - the midpoint of AB is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
- The equation of the straight line through the points (x_1, y_1) and (x_2, y_2) is $y - y_1 = m(x - x_1)$:
    - the gradient is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$
    - the equation of a line is often written in the form $y = mx + c$ or $ax + by + c = 0$.
- Parallel lines have the same gradient.
- The gradients of perpendicular lines satisfy $m_1 m_1 = -1$.
- A circle with centre (a, b) and radius r has equation $(x - a)^2 + (y - b)^2 = r^2$.
    - You may need to complete the square in order to find the centre and radius.
- You can find the intersection of two lines, a line and a circle, or (sometimes) two circles by solving simultaneous equations.
    - You can tell whether two circles intersect, are disjoint or tangent to each other by comparing the sum and the difference of the radii of the circles to the distance between their centres.
- A tangent to a circle is perpendicular to the radius at the point of contact. The normal is in the direction of the radius.
- A radius that is perpendicular to a chord bisects that chord.
- An angle in a semi-circle is a right angle. It follows that, if A, B and C are three points on a circle and ABC is a right angle, then AC is a diameter of the circle.

Mixed practice 6

1 Find the radius of the circle $x^2 - 8x + y^2 + 6y = 144$.

2 Line l_1 has equation $3x - 2y + 7 = 0$.

a Point $A(2k, 2k + 1)$ lies on l_1. Find the value of k.

b Point B has coordinates $(-2, p)$. Find the value of p so that AB is perpendicular to l_1.

c Line l_2 is parallel to l_1 and passes through B. Find the equation of l_2 in the form $ax + by + c = 0$ where a, b and c are integers.

d l_2 crosses the x-axis at the point C. Find the coordinates of C.

3 Circle C has equation $x^2 - 2x + y^2 - 10y - 19 = 0$.

a Find the coordinates of the centre, P, of the circle.

b Show that point $A(7, 2)$ lies on the circle.

Point M has coordinates $(1, -1)$. Line l is perpendicular to PA and passes through M. It cuts PA at the point S.

c Find the coordinates of S.

4 A circle has equation $x^2 + y^2 + 6x - 4y - 4 = 0$.

a Find the centre and radius of the circle.

b Find the coordinates of the points where the circle meets the line with equation $y = 3x + 4$.

OCR, GCE Mathematics, Paper 4721, January 2010
[Question part reference style adapted]

5 $y = -3x + 5$ is tangent to the circle C at the point $(4, -7)$. The centre of C is at the point $(k - 4, k + 3)$. Find the value of k.

6 Consider the points $A(4, 3), B(3, -2)$ and $C(9, 2)$.

a Show that BAC is a right angle.

b Hence find the equation of the circle through A, B and C.

c Find the equation of the tangent to the circle at B. Give your answer in the form $ax + by + c = 0$ where a, b and c are integers.

7 A circle has centre $(3, 0)$ and radius 5. The line $y = 2x + k$ intersects the circle in two points. Find the set of possible values of k, giving your answers in surd form.

8 A circle has centre $C(7, 12)$ and passes through the point $D(4, 10)$. The tangent to the circle at D cuts the coordinate axes at points A and B. Find the area of these triangles:

a AOB

b ABC

9 The points $A\ (-3, 7)$ and $B(5, -1)$ are endpoints of the diameter of a circle. Find the equation of the circle in the form $x^2 + ax + y^2 + by + c = 0$.

10 Find the exact values of k for which the line $y = kx + 3$ is tangent to the circle with centre $(6, 3)$ and radius 2.

11 **a** Find the equation of the circle with radius 10 and centre $(2, 1)$, giving your answer in the form $x^2 + y^2 + ax + by + c = 0$.

b The circle passes through the point $(5, k)$ where $k > 0$. Find the value of k in the form $p + \sqrt{q}$.

c Determine, showing all working, whether the point $(-3, 9)$ lies inside or outside the circle.

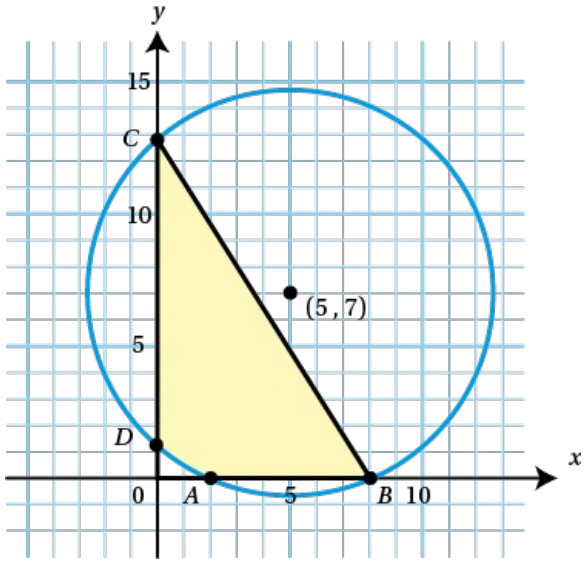
d Find an equation of the tangent to the circle at the point $(8, 9)$.

OCR, GCE Mathematics, Paper 4721, June 2008
[Question part reference style adapted]

12 Find the shortest distance from the point $(-3, 2)$ to the line with equation $3x + 2y = 19$. Give your answer in exact form.

13 Show that each of the circles $x^2 - 6x + y^2 + 10y + 18 = 0$ and $x^2 - 14x + y^2 - 6y + 49 = 0$ lies entirely outside the other one.

14 A circle has centre $(5, 7)$. It crosses the x-axis at points $A(2, 0)$ and $B(p, 0)$, where $p > 2$.



a Find the value of p and write down the equation of the circle.

b The circle crosses the y-axis at points C and D. Find the area of the quadrilateral $ABCD$.

Worksheet

For a selection of more challenging problems, see Extension sheet 6.

7 Logarithms

In this chapter you will learn how to:

- undo exponential functions using an operation called a logarithm
- use laws of logarithms
- use logarithms to find exact solutions of some exponential equations
- use a special number called e.

Before you start...

GCSE	You should know how to work with expressions involving exponents.	1 Answer 'true' or 'false': a $2 \times 3^2 = 36$ b when $x = 25$, $4x^{\frac{1}{2}} = 10$ c $(2 \times 3)^7 = 2^7 \times 3^7$ d $\frac{1}{2x^3} = 2^{-3}x^{-3}$
GCSE	You should know how to evaluate fractional and negative powers.	2 Evaluate the following without a calculator: a $27^{\frac{4}{3}}$ b $9^{-\frac{3}{2}}$
Chapter 2	You should know how to use laws of indices.	3 Write the following in the form x^p: a $x^2 \times \sqrt{x}$ b $\frac{x\sqrt{x}}{x^2}$
GCSE	You should know how to solve equations involving fractions.	4 Solve this equation. $\frac{x+1}{2x-3} = 2$
Chapter 3	You should know how to solve quadratic equations.	5 Solve these equations. a $(x-1)(x+3) = 5$ b $2x^2 - \frac{1}{x^2} = 1$

Discovering logarithms

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If you were asked to solve $x^2 = 3$ for $x \geqslant 0$ then you could either find a decimal approximation (for example by using a calculator or trial and improvement) or take the square root:

$$x = \sqrt{3}$$

This statement just says that 'x is the positive value that, when squared, gives 3'.

Similarly, to solve $10^x = 50$ you could use trial and improvement to seek a decimal value:

$$10^1 = 10$$

$$10^2 = 100$$

So x is between 1 and 2.

$$10^{1.5} \approx 31.6$$

$$10^{1.6} \approx 39.8$$

$$10^{1.7} \approx 50.1$$

So the answer is approximately 1.7.

Just as you can take the square root to answer the question: 'What is the number, which when squared, gives 3?', there is also a function to answer the question: 'What is the number, which when put as the exponent of 10, gives 50?' This power is called a base-10 **logarithm**.

In this chapter you will learn about laws of logarithms and how to use them to solve some problems involving **exponential functions**.

Section 1: Introducing logarithms

In the example in the introduction, the solution to the equation $10^x = 50$ can be written as $x = \log_{10} 50$. More generally, the equation $y = 10^x$ can be re-expressed as $x = \log_{10} y$. In fact, the base does not need to be 10, but could be any positive value other than 1.

Tip

Remember that the symbol $\Leftrightarrow$ means that the statements are equivalent and you can switch between them.

Key point 7.1

Converting between index and logarithmic forms:

$a = b^c \Leftrightarrow c = \log_b a$

WORKED EXAMPLE 7.1

Evaluate without a calculator:

a $\log_2 8$

b $\log_{10} 0.01$

c $\log_{25} 5$

a $\log_2 8 = 3$ Ask the question: '2 raised to what power equals 8'?

b $\log_{10} 0.01 = -2$ Because $10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$.

c $\log_{25} 5 = \frac{1}{2}$ Because $25^{\frac{1}{2}} = \sqrt{25} = 5$.

Whenever you raise a positive number to an exponent, whether it is a positive or negative exponent, the answer is always positive. So there is **no** answer to a question such as 'to what power do you raise 10 to get -3?'

Key point 7.2

The logarithm of a negative number or zero is **not** a real number.

Two common bases have abbreviations for their logarithms. Since a decimal system of counting is commonly used, base 10 is the default base for a logarithm, so that $\log_{10} x$ is usually written simply as $\log x$.

There is also a special number, called e, which is important when studying rates of change. $\mathrm{e}(\approx 2.71828)$ is an irrational number. The logarithm in base e is called the **natural logarithm**, denoted by $\ln x$. Although it looks different, $\ln x$ follows all the same rules as other logarithms.

Fast forward

It is possible to take logarithms of negative numbers if you use complex numbers. This is a new type of number you will meet if you study Further Mathematics in Pure Core Student Book 1, Chapter 3.

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Fast forward

You will look at the number e in more detail in Chapter 8.

Key point 7.3

- $\log_{10} x$ is written as $\log x$.
- $\log_e x$ is written as $\ln x$.

Since taking a logarithm **reverses** the process of raising to a power, the facts listed in Key point 7.4 follow:

Tip

Your calculator has special buttons for log and ln. Some calculators can also evaluate logs in other bases.

Key point 7.4

- $\log_a(a^x) = x$
- $a^{\log_a x} = x$

Did you know?

Logarithms were introduced by the Scottish mathematician John Napier (1550–1617). He originally studied logarithms in base $\frac{1}{e}$.

These are referred to as the **cancellation principles**. This sort of 'cancellation', similar to squaring a square root, is frequently useful when simplifying logarithm expressions, but you can only apply it when the base of the logarithm and the base of the exponential match and are immediately adjacent in the expression.

WORKED EXAMPLE 7.2

Simplify:

a $\ln(e^5)$

b $10^{2+\log 3}$

a $\ln(e^5) = 5$

Apply the first cancellation principle (with base e) from Key point 7.4.

b $10^{2+\log 3} = 10^2 \times 10^{\log 3}$

Use the laws of indices.

$= 100 \times 3$

$= 300$

Apply the second cancellation principle (with base 10) from Key point 7.4.

You can also use the fact that logarithms reverse raising to a power to solve equations and rearrange formulae.

WORKED EXAMPLE 7.3

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Make x the subject of each equation.

a $\log_2 x = y + 3$

b $\log_x(y+1) = 2$

a $\log_2 x = y + 3$

Use $\log_b a = c \Leftrightarrow b^c = a$ (Key point 7.1).

$x = 2^{y+3}$

b $\log_x(y+1) = 2$

Use $\log_b a = c \Leftrightarrow b^c = a$ (Key point 7.1).

$x^2 = y + 1$

$x = \sqrt{y+1}$

The base of a logarithm has to be positive, so you do not need to use $\pm$.

WORKED EXAMPLE 7.4

Find the exact solution to the equation $3\ln(2+x) = 6$.

$\ln(2+x) = 2$

Divide both sides by 3 to get the logarithm by itself.

$e^{\ln(2+x)} = e^2$

$2 + x = e^2$

Turn each side into a power with e as the base; then e and ln will cancel.

$x = e^2 - 2$

Now just solve for x. You are asked for an exact answer so this is the form to leave it in. You should not try to write it as a decimal.

Remember that log x represents a single value, so it can be treated like any algebraic expression, In particular, $\log x \times \log x = (\log x)^2$.

Tip

$(\log x)^2$ is **not** the same as $\log(x^2)$.

WORKED EXAMPLE 7.5

Expand and simplify:

a $(\log x + 1)(\log x - 3)$

b $\dfrac{(\ln x)^2 - 4}{\ln x + 2}$

a $(\log x + 1)(\log x - 3) = (\log x)^2 + \log x - 3\log x - 3$

Expand the brackets as normal.

$= (\log x)^2 - 2\log x - 3$

Group like terms.

b $\dfrac{(\ln x)^2 - 4}{\ln x + 2} = \dfrac{(\ln x - 2)(\ln x + 2)}{\ln x + 2}$

Factorise the numerator (difference of two squares).

$= \ln x - 2$

Cancel common factors in the numerator and denominator.

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EXERCISE 7A

1 Evaluate the following without using a calculator:

a **i** $\log_3 27$

ii $\log_4 16$

b **i** $\log_5 5$

ii $\log_3 3$

c **i** $\log_{12} 1$

ii $\log_{15} 1$

d **i** $\log_3 \frac{1}{3}$

ii $\log_4 \frac{1}{64}$

e **i** $\log_4 2$

ii $\log_{27} 3$

f **i** $\log_8 \sqrt{8}$

ii $\log_2 \sqrt{2}$

g **i** $\log_8 4$

ii $\log_{81} 27$

h **i** $\log_{25} 125$

ii $\log_{16} 32$

i **i** $\log_4 2\sqrt{2}$

ii $\log_9 81\sqrt{3}$

j **i** $\log_{25} 0.2$

ii $\log_4 0.5$



Tip

Although you can evaluate logarithms on a calculator, this exercise will help you develop your understanding of this new concept.

2 Use a calculator to evaluate each of the following, giving your answer correct to 3 s.f.

a **i** $\log 50$

ii $\log\left(\frac{1}{4}\right)$

b **i** $\ln 0.1$

ii $\ln 10$

c **i** $\log_3 8.5$

ii $\log_5 0.6$

d **i** $\log_{0.2} 3$

ii $\log_{0.8} 0.6$

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3 Find an equivalent form for each of the following expressions:

a i $7\log x - 2\log x$

ii $2\log x + 3\log x$

b i $(\log x - 1)(\log y + 3)$

ii $(\log x + 2)^2$

c i $\dfrac{\log a + \log b}{\log a \log b}$

ii $\dfrac{(\log a)^2 - 1}{\log a - 1}$

Tip

There are often different ways of expressing the answer but usually only one 'sensible' one.

4 Make x the subject of the following:

a i $\log_3 x = y$

ii $\log_4 x = 2y$

b i $\log_a x = 1 + y$

ii $\log_a x = y^2$

c i $\log_x 3y = 3$

ii $\log_x y = 2$

d i $y = 2 + \ln x$

ii $\ln y = \ln x - 2$

5 Find the value of x in each of the following:

a i $\log_2 x = 5$

ii $\log_2 x = 4$

b i $\log_5 25 = 5x$

ii $\log_{49} 7 = 2x$

c i $\log_x 36 = 2$

ii $\log_x 10 = \dfrac{1}{2}$

In questions 6 to 11, you must show detailed reasoning. This means that you need to use algebraic rearrangement rather than, for example, the equation solver on your calculator.

6 Solve the equation $\log_{10}(9x + 1) = 3$.

7 Solve the equation $\log_8 \sqrt{1 - x} = \dfrac{1}{3}$.

8 Find the exact solution to the equation $\ln(3x - 1) = 2$.

9 Solve the equation $3(1 + \log x) = 6 + \log x$.

10 Find all values of x that satisfy $(\log_3 x)^2 = 4$.

11 Solve the simultaneous equations:

$\log_3 x + \log_5 y = 6 \qquad \log_3 x - \log_5 y = 2$

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12 Evaluate $\sqrt[6]{(\pi^4+\pi^5)}$. What do you notice about this result?

Section 2: Laws of logarithms

There is a series of laws that hold when performing arithmetic with exponents. There are corresponding laws that apply to logarithms. To investigate this it is useful to look at some examples:

1	$\log_2 4 = 2$	$\log_2 8 = 3$	$\log_2 32 = 5$
2	$\log_{10} 1000 = 3$	$\log_{10} 10 = 1$	$\log_{10} 10000 = 4$
3	$\log_3 27 = 3$	$\log_3 \left(\frac{1}{9}\right) = -2$	$\log_3(3) = 1$

In each case it seems that when you add two logs with the same base you get the log of the product. You can put this into an algebraic rule.

$$\log_a(xy) = \log_a x + \log_a y$$

This can be derived from the laws of indices.

PROOF 3

Prove that $\log_a(xy) = \log_a x + \log_a y$.

Let $P = \log_a x$ and $Q = \log_a y$	Often in proofs it is useful to define certain parts that you are interested in, and then manipulate them using known laws.
So $x = a^P$ and $y = a^Q$	You don't have many algebraic properties of logarithms to work with. You need to use the defining feature of logarithms (Key point 7.1) to turn it into something you are more familiar with: index form.
So $xy = a^P \times a^Q = a^{(P+Q)}$	You can now try to apply the laws of indices to xy.
Taking logs of both sides: $\log_a(xy) = \log_a a^{P+Q}$	You want $\log_a(xy)$ so you need to take logs of both sides
$= P + Q$	Using the cancellation principle (Key point 7.4).
$= \log_a x + \log_a y$	Substitute back in to write the answer in terms of x and y.

Other laws can be derived in a similar way.

Key point 7.5

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a x^k = k \log_a x$$

Explore

The initial purpose of logarithms was as a calculation aid. Because they turn multiplication into addition and division into subtraction, they were used to multiply and divide large numbers. Find out how this was done using 'log tables'.

It is useful to remember these special values:

Key point 7.6

For any base $a > 0, \log_a a = 1$.

The logarithm of 1 is always 0, irrespective of the base.

$\log_a 1 = 0$

Tip

A useful particular case of the third law is when $k = -1$:

$$\log_a \frac{1}{x} = -\log_a x$$

These laws can be used to manipulate expressions involving logarithms.

Tip

Note that, since $\ln x$ is just a logarithm in base e, $\ln \mathrm{e} = 1$ and $\ln 1 = 0$.

WORKED EXAMPLE 7.6

If $x = \log_{10} a$ and $y = \log_{10} b$, express $\log_{10} \frac{100a^2}{b}$ in terms of x and y.

$\log_{10} \frac{100a^2}{b} = \log_{10} 100a^2 - \log_{10} b$ Use laws of logs to isolate $\log_{10} a$ and $\log_{10} b$. First, using the law relating to the log of a fraction...

$= \log_{10} 100 + \log_{10} a^2 - \log_{10} b$ then the log of a product...

$= \log_{10} 100 + 2\log_{10} a - \log_{10} b$ then the log of an exponent.

$= 2 + 2\log_{10} a - \log_{10} b$ $\log_{10} 100 = 2$.

$= 2 + 2x - y$

WORK IT OUT 7.1

If $e^{2y} = x + 2$ write y in terms of x.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$2y = \ln(x + 2)$

$= \ln x + \ln 2$

So $y = \frac{\ln x + \ln 2}{2}$

Solution 2

$2y = \ln(x + 2)$

$y = \frac{1}{2}\ln(x + 2)$

$y = \ln \sqrt{x + 2}$

Solution 3

$2y = \ln(x+2) = \ln x \times \ln 2$

So $y = \frac{1}{2}\ln x \times \frac{1}{2}\ln 2$

The laws of logarithms can also be used to solve equations. The usual tactic is to combine all the logarithms into one.

Tip

It is important to know what you **cannot** do with logarithms and not make up your own rules!

WORKED EXAMPLE 7.7

Solve the equation $\log_2 x + \log_2(x+4) = 5$.

$\log_2 x + \log_2(x+4) = 5$	Rewrite one side as a single logarithm.
$\log_2(x(x+4)) = 5$	
$2^{\log_2(x(x+4))} = 2^5$	Undo the logarithm by raising 2 to the power of both sides.
$x^2 + 4x = 32$	Use the cancellation principle.
$x^2 + 4x - 32 = 0$	Solve the quadratic equation.
$(x+8)(x-4) = 0$	
$x = -8$ or $x = 4$	
When $x = -8$:	Check your solution in the original equation.
LHS: $\log_2(-8) + \log_2(-4)$ is not real so this solution does not work.	State the final solution.
When $x = 4$:	
LHS: $\log_2 4 + \log_2 8 = 2 + 3 = 5 = \text{RHS}$	
So $x = 4$ is the only solution.	

Tip

Checking your solutions is about more than looking for an arithmetic error. It is possible to introduce false solutions through algebraic manipulation.

EXERCISE 7B

In this exercise, you must show detailed reasoning.

1 Given $b > 0$, simplify each of the following:

a i $\log_b b^4$

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ii $\log_b \sqrt{b}$

b **i** $\log_{\sqrt{b}} b^3$

ii $\log_b b^2 - \log_b \sqrt{b}$

2 If $x = \log a, y = \log b$ and $z = \log c$, express the following in terms of x, y and z:

a **i** $\log b^7$

ii $\log a^2 b$

b **i** $\log\left(\frac{ab^2}{c}\right)$

ii $\log\left(\frac{a^2}{bc^3}\right)$

c **i** $\log\left(\frac{100}{bc^5}\right)$

ii $\log(5b) + \log(2c^2)$

d **i** $\log a^3 - 2\log ab^2$

ii $\log(4b) + 2\log(5ac)$

3 Solve the following for x:

a **i** $\log_3\left(\frac{2+x}{2-x}\right) = 1$

ii $\log_2(7x + 4) = 5$

b **i** $\log_3 x - \log_3(x - 6) = 1$

ii $\log_8 x - 2\log_8\left(\frac{1}{x}\right) = 1$

c **i** $\log_3(x - 7) + \log_3(x + 1) = 2$

ii $2\log(x - 2) - \log(x) = 0$

4 Find the value of x for which $3\log_b x = \log_b 64$.

5 Solve the equation $\ln x = 2\ln 9 - \ln 3$.

6 Solve the equation $\log(3x + 6) = \log(3) + 1$.

7 If $x = \log a, y = \log b$ and $z = \log c$ express the following in terms of x, y and z:

a $\log(a^2 b)$

b $\log\left(\frac{100a}{\sqrt{c}}\right)$

8 If $a = \ln 2$ and $b = \ln 5$ find in terms of a and b:

a $\ln 50$

b $\ln 0.16$

9 Find the exact solution of the equation $2\ln x + \ln 9 = 3$ giving your answer in the form Ae^B where A and B are rational numbers.

Worksheet

For a further example and more practice of this type of question, see Support sheet 7.

10 Solve the equation $\log(x + 5) - 1 = \log(x - 1)$.

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11 Solve the equation $\log_2(x+2) = 3 - \log_2 x$.

12 Solve the equation $\log\left(x^2+1\right) = 1 + 2\log\left(x^2\right)$.

Section 3: Solving exponential equations

One of the main purposes of logarithms is to make it possible to solve equations with the unknown in the exponent. By taking logarithms the unknown becomes a factor, which is easier to deal with.

Key point 7.7

To solve an equation with the unknown in the power, take logarithms (to any base) of both sides.

WORKED EXAMPLE 7.8

Find the exact solution of the equation $3^{x-2} = 5$.

$3^{x-2} = 5$	
$\log\left(3^{x-2}\right) = \log 5$	Take logarithms of both sides. You can use any base. Here base 10 is used.
$(x-2)\log 3 = \log 5$	Use the rule $\log a^p = p \log a$.
$x\log 3 - 2\log 3 = \log 5$	Multiply out the brackets to isolate the term with x.
$x\log 3 = \log 5 + 2\log 3$	Collect terms without x and divide by the coefficient of x.
$x = \dfrac{\log 5 + 2\log 3}{\log 3}$	

There is often more than one correct way to write an answer in terms of logarithms. See if you can work out which one of the answers in Work it out 7.2 is an alternative way of solving the equation in Worked example 7.8.

WORK IT OUT 7.2

Check each of these solutions to the equation $3^{x-2} = 5$.

Which is the corarect solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$x - 2 = \log_3 5$

$x = \log_3 5 + 2$

Solution 2

$\ln\left(3^{x-2}\right) = \ln 5$

$(x-2)\ln 3 = \ln 5$

$x - 2 = \ln\left(\dfrac{5}{3}\right)$

$x = \ln\left(\dfrac{5}{3}\right) + 2$

Solution 3

$(x-2)\log 3 = \log 5$

$x\log 3 = 2\log 3 + \log 5 = \log 45$

So, $x = \dfrac{\log 45}{\log 3} = \log 42$

The strategy of taking logs of both sides works particularly well when the unknown appears in the power on both sides of the equation. Again, you can choose any base (**unless** a particular base is specified in the question).

WORKED EXAMPLE 7.9

Solve the equation $2^{x+3} = 5^{x-1}$ giving your answer in the form $\dfrac{\ln p}{\ln q}$.

$(x+3)\ln 2 = (x-1)\ln 5$	Take logs of both sides. The question says to use ln.
$x\ln 2 + 3\ln 2 = x\ln 5 - \ln 5$	Expand the brackets.
$3\ln 2 + \ln 5 = x\ln 5 - x\ln 2$	Group terms with x on one side and terms without x on the other.
$3\ln 2 + \ln 5 = x(\ln 5 - \ln 2)$	You can factorise the expression on the right so that x occurs only once in the equation.
$\ln\left(2^3 \times 5\right) = x\ln\left(\dfrac{5}{2}\right)$	se rules of logarithms to combine into a single log on each side.
$x = \dfrac{\ln\ 40}{\ln\ 2.5}$	You can now divide by the coefficient of x to get the answer in the required form.

WORK IT OUT 7.3

Solve the equation $3 \times 2^x = 7^{x-2}$.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$\log 3 \times \log\left(2^x\right) = \log\left(7^{x-2}\right)$

$\log 3 \times x\log 2 = (x-2)\log 7$

$x\log 3 \times \log 2 = x\log 7 - 2\log 7$

$x\log 6 - x\log 7 = -2\log 7$

$x\log\left(\dfrac{7}{6}\right) = 2\log 7$

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$$x = \frac{\log(49)}{\log\left(\frac{7}{6}\right)}$$

Solution 2

$\ln 3 + x \ln 2 = (x-2)\ln 7$

$\ln 3 + 2\ln 7 = x\ln 7 - x\ln 2$

$$x = \frac{\ln 147}{\ln 3.5}$$

Solution 3

$6^x = 7^{x-2}$

$x = (x-2)\log_6(7)$

$x = x\log_6 7 - \log_6 49$

$\log_6 49 = x(\log_6 7 - 1)$

$$x = \frac{\log_6 49}{\log_6 7 - 1}$$

EXERCISE 7C

In this exercise, you must show detailed reasoning.

1 Solve for x, giving your answers correct to 3 s.f.

a i $3 \times 4^x = 90$

ii $1000 \times 1.02^x = 10000$

b i $6 \times 7^{x+1} = 1.2$

ii $5 \times 2^{2x-5} = 94$

c i $3^{2x} = 4^{x-1}$

ii $5^x = 6^{x-1}$

d i $3 \times 2^{3x} = 7 \times 3^{3x-2}$

ii $4 \times 8^{x-1} = 3 \times 5^{2x+1}$

2 Solve the following equations, giving your answers in terms of natural logarithms.

a i $2 \times 3^x = 5$

ii $5 \times 7^x = 3$

b i $4e^x = 1$

ii $3e^x = 1$

c i $2^{2x} = 5$

ii $10^{2x} = 7$

d i $2^{3x+1} = 10$

ii $5^{2x-3} = 4$

e i $2^{x+1} = 5^x$

ii $5^{x-2} = 3^x$

f i $2^{3x} = 3e^x$

ii $e^{2x} = 5 \times 2^x$

g i $5 \times 3^x = 8^{x+2}$

ii $3^{2x+1} = 7 \times 2^{x-2}$

h i $3 \times 2^{3x} = 7 \times 3^{3x-2}$

ii $4 \times 8^{x-1} = 3 \times 5^{2x+1}$

3 Solve the equation $5^{4x+3} = 28$, giving your answer correct to 3 s.f.

4 Find the exact solution of the equation $4 \times 3^{x-5} = 1$.

5 Find, in terms of the natural logarithm, the exact solution of the equation $e^{3-2x} = 10$.

6 Find the exact solution of the equation $10^x = 5 \times 2^{3x}$, giving your answer in the form $x = \dfrac{\log p}{q + \log r}$, where p, q and r are integers.

7 Solve the equation $2^{3x-1} = 5^{2-x}$ giving your answer in the form $x = \dfrac{\ln a}{\ln b}$ where a and b are integers.

8 If $3^x \times 4^{2x+1} = 6^{x+2}$, show that $x = \dfrac{\ln 9}{\ln 8}$.

Section 4: Disguised quadratics

It is usually impossible to simplify expressions such as $3^x + 5^x$ which contain powers with different bases. This means that you cannot (algebraically) solve the equation $3^x + 5^x = 17$. However, there is a special type of equation that looks very similar to this but which can be solved exactly, by writing it as a quadratic equation.

Rewind

You already met these 'disguised quadratics' in Chapter 3, Section 6.

Fast forward

You will meet disguised quadratics again in Chapter 10, Section 7 on trigonometry.

WORKED EXAMPLE 7.10

Find the exact solution of the equation $4^x + 2^x = 12$.

$4^x + 2^x = 12$

$(2^x)^2 + 2^x = 12$

$4^x = (2^2)^x = (2^x)^2$

Let $y = 2^x$.

Then $y^2 + y = 12$

Make a substitution to turn the equation into a quadratic.

$y^2 + y - 12 = 0$

$(y + 4)(y - 3) = 0$

$y = -4$ or 3

Solve the quadratic.

$2^x = -4$ is impossible.

No real power of a positive number can be negative.

When $2^x = 3$:

$x \log 2 = \log 3$

$x = \dfrac{\log 3}{\log 2}$

The other solution for y **does** give a valid solution for x.

EXERCISE 7D

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1 Find the exact solution(s) of each equation.

a i $4^x - 5 \times 2^x + 6 = 0$

ii $9^x - 6 \times 3^x + 8 = 0$

b i $9^x - 8 \times 3^x = 9$

ii $25^x - 5^x = 6$

c i $e^{2x} + 16e^x = 80$

ii $e^{2x} - 9e^x + 20 = 0$

d i $25^x - 15 \times 5^x + 50 = 0$

ii $4^x - 7 \times 2^x + 12 = 0$

e i $\log_4 x = (\log_4 x^2)^2$

ii $(\log_3 x)^2 - 3\log_3 x + 2 = 0$

2 **a** By letting $y = 3^x$, show that $3^{2x} - 3^{x+2} + 20 = 0$ can be written in the form $y^2 - 9y + 20 = 0$.

b Hence solve $3^{2x} - 3^{x+2} + 20 = 0$, giving your answer correct to 3 s.f.

3 Find exact solutions of the equation $4^x - 10 \times 2^x + 16 = 0$.

4 Solve the equation $2e^{2x} - 9e^x + 4 = 0$ giving your answers in the form k ln 2.

5 Solve the equation $5^{2x+1} - 14 \times 5^x - 3 = 0$.

6 Solve, in exact form, the equation $e^x = 8 - 15e^{-x}$.

Checklist of learning and understanding

- Logarithms answer the question: 'To what power do I need to raise b to get a?'
  - You can convert between logarithmic and index forms: $c = \log_b a \Leftrightarrow b^c = a$.
- You can only take a logarithm of a positive number.
- Cancellation rules can be used to simplify some expressions:
  - $\log_a a^x = x = a^{\log_a x}$
- Logarithms in base 10 are written as log x.
- Logarithms in base e (natural logarithms) are written as ln x.
- Logarithms obey the following laws:
  - $\log_a(xy) = \log_a x + \log_a y$
  - $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
  - $\log_a\left(\frac{1}{x}\right) = -\log_a x$
  - $\log_a x^k = k\log_a x$
  - $\log_a 1 = 0$ and $\log_a a = 1$ for any base $a > 0$.
- Many exponential equations can be solved by taking a logarithm of both sides and using log rules.
- Some equations can be turned into quadratic equations by using a substitution of the form $y = a^x$.

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Mixed practice 7

In this exercise, you must show detailed reasoning.

1. Solve $\log_5\left(\sqrt{x^2+49}\right)=2$.

2. If $8^x=16$, then x equals:

   a $\log_{16}8$

   b $\sqrt[8]{16}$

   c $\frac{16}{8}$

   d $\frac{4}{3}$

3. Given that $a=\log x, b=\log y$ and $c=\log z$ (with all logs being to the base 10), express the following in terms of a, b, c and integers:

   a $\log\frac{x^2\sqrt{y}}{z}$

   b $\log\sqrt{0.1x}$

   c $\log_{100}\left(\frac{y}{z}\right)$

4. Solve the equation $3e^{2x+1}=17$, giving your answer to three 3 s.f.

5. Solve the equation $4\log_a x=\log_a 81$.

6. Given that $\log_x 4=9$, find the value of x correct to 3 s.f.

7. If $\log_a y+\log_a 7=4$, express y in terms of a.

8. The curve $y=3^{2x-1}$ intersects the line $y=4$ at the point P. Find the exact value of the x-coordinate of P.

9. Solve the simultaneous equations:

   $\ln x+\ln y^2=8$

   $\ln x^2+\ln y=6$

10. Given that $4\log_b x-\log_b 9=2$, express b in terms of x.

11. Given that $\log_5 y=8$, find the value of $\log_5(125y)$.

12. If $y=\ln x-\ln(x+2)+\ln(4-x^2)$, express x in terms of y.

13. Solve, correct to 3 s.f., $3^{2x}-3^{x+1}-10=0$.

14. Solve the equation $\log(x^2+1)=1+2\log x$.

15. Solve the equation $3^{2x+1}=5^{200}$ giving your answer to three s.f.

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[Question and question part reference style adapted]

16. **a** Given that $\log_a x=p$ and $\log_a y=q$ express the following in terms of p and q.

    i $\log_a(xy)$

    ii $\log_a\left(\frac{a^2x^3}{y}\right)$

    b **i** Express $\log_{10}(x^2-10)-\log_{10}x$ as a single logarithm.

    ii Hence solve the equation $\log_{10}(x^2-10)-\log_{10}x=2\log_{10}3$.

17 Find the exact value of x satisfying the equation $2^{3x-2} \times 3^{2x-3} = 36^{x-1}$ giving your answer in simplified form $\frac{\ln p}{\ln q}$ where $p, q \in \mathbb{Z}$.

18 Solve the equation $5 \times 4^{x-1} = \frac{1}{3^{2x}}$ giving your answer in the form $x = \frac{\ln p}{\ln q}$ where p and q are rational numbers.

19 Find the exact solutions to $e^x + e^{-x} = 4$.

20 Find the value of x for which $\left(\log_3 x\right)^2 = \log_3 x^3 - 2$.

Worksheet

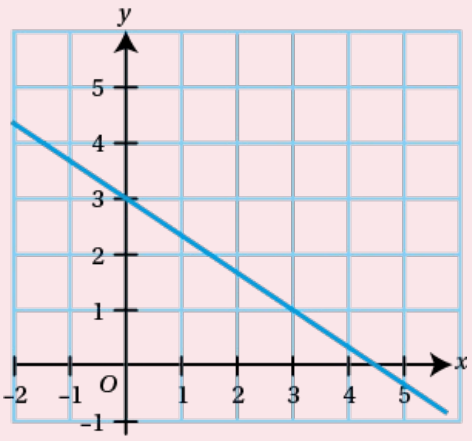
For a selection of more challenging questions, see Extension sheet 7.

8 Exponential models

In this chapter you will learn:

- about graphs of exponential functions
- why exponential functions are often used in modelling
- how to use logarithms to transform curved graphs into straight lines.

Before you start...

Chapter 7	You should be able to use the number e and natural logarithms.	1 Simplify each expression. a $\ln\left(5e^2\right)$ b $e^{l+\ln(5)}$
Chapter 7	You should be able to use the laws of logarithms.	2 If $y = 100x^3$ write $\log y$ in the form $n + k\log x$.
Chapter 5	You should be able to transform graphs.	3 Describe the effect of changing $y = \mathrm{f}(x)$ into $y = \mathrm{f}(2x)$.
Chapter 6	You should be able to work with equations of straight lines.	4 What is the gradient of $3y + 2x = 5$? 5 Find, in the form $y = mx + c$, the equation of this line: 

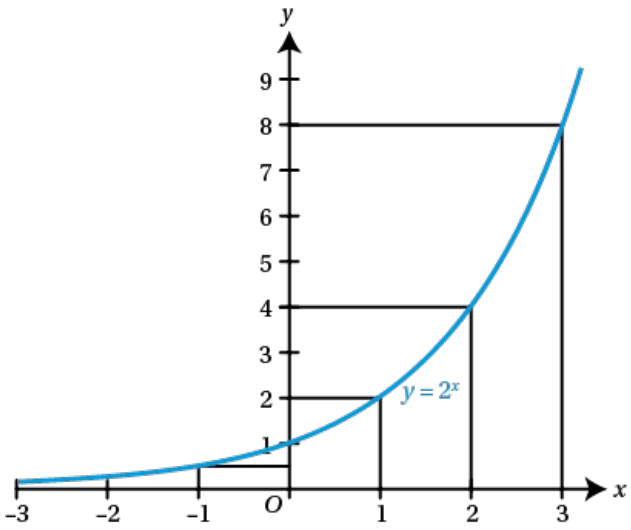
Why use exponential models?

In many situations in the real world the rate of growth of a quantity is approximately proportional to the amount that is there. For example, the more people there are in a country, the more babies will be born. It turns out that the only functions that have this property are exponential functions of the form $y = Ca^x$.

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Section 1: Graphs of exponential functions

This is the graph of $y = 2^x$:

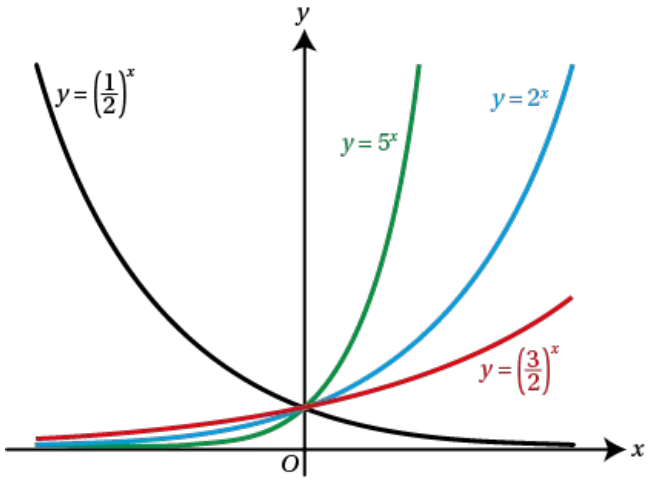


For very large positive values of x, the y value approaches infinity, and for very large negative values of x the y value approaches (but never reaches) 0. In this case, you would say that the x-axis is an **asymptote** to the graph.

If you look at the graphs of exponential functions with different bases you can start to make some generalisations.

Rewind

You may notice that the black line is a reflection of the blue line. This is because $\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$. You know from Chapter 5, Section 3 that replacing x by $-x$ results in the graph being reflected in the y-axis.



Key point 8.1

For all the graphs $y = a^x$:

- the y-intercept is always $(0, 1)$ because $a^0 = 1$
- the graph of the function lies entirely above the x-axis
- the x-axis is an asymptote.

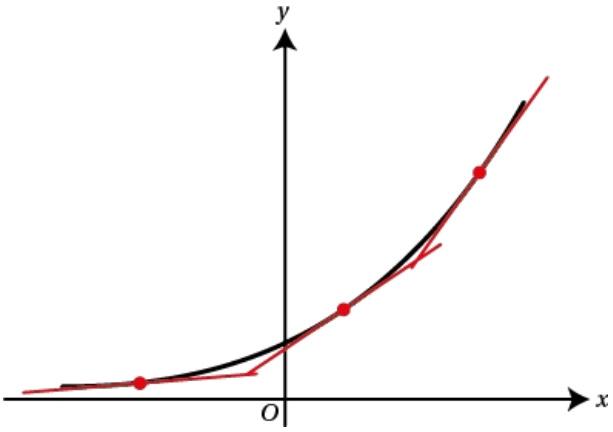
If $a > 1$, then as x increases so does y. This is called **exponential growth**.

If $0 < a < 1$, then as x increases, y decreases. This is called **exponential decay**.

Gradient of an exponential graph

For an exponential growth graph the gradient also increases with x. In fact, the gradient is exactly proportional to the y value at every point on the graph.

You can find the gradient of a curved graph by drawing a tangent and calculating its gradient. The diagram shows some tangents to the graph of $y = 2^x$.



x	-1	0	1	2
y	0.5	1.0	2.0	4.0
Gradient	0.35	0.7	1.4	2.8

The table shows the y values and the gradient for each of the points. You can check that gradient $\approx 0.69y$.

You can do a similar calculation for the graph of $y = 3.5^x$ and find that the gradient $\approx 1.25y$.

These examples suggest that the constant of proportionality depends on the base of the exponential. There is one special value of the base where this constant is 1 so that the gradient at any point is exactly equal to the y value. This value is the number e that you met in Chapter 7.

Tip

Remember that $\mathrm{e} \approx 2.7$ Since $\mathrm{e} > 1, e^x$ is a type of exponential growth, while $e^{-x} = \left(\frac{1}{e}\right)^x$ represents exponential decay.

Key point 8.2

The gradient of e^x equals e^x.

You can extend this result to exponential functions of the form e^{kx} for any constant k.

Key point 8.3

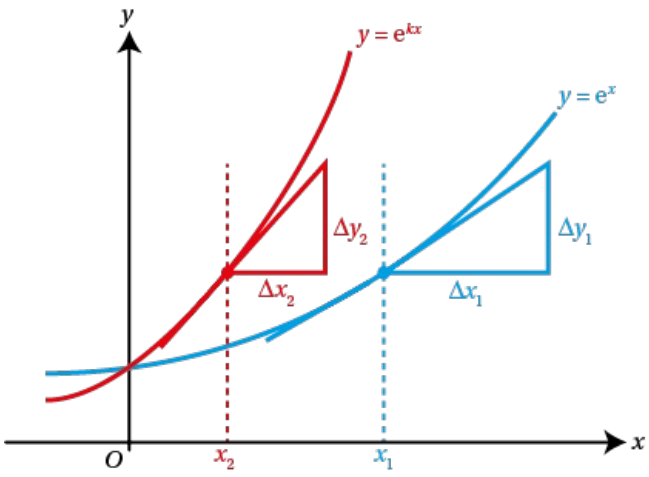
The gradient of e^{kx} equals ke^{kx}.

PROOF 4

Prove that the gradient of e^{kx} is ke^{kx}.

Consider the graph of $y = e^x$ and $y = e^{kx}$.

You can compare the two graphs using the fact that they are related by a horizontal stretch.



A triangle is drawn from the tangent on the blue graph. The gradient is $\frac{\Delta y_1}{\Delta x_1} = e^{x_1}$.

The red graph is a horizontal stretch, factor $\frac{1}{k}$, of everything on the blue graph, including the triangle.

This means that $\Delta y_2 = \Delta y_1$ but $\Delta x_2 = \frac{\Delta x_1}{k}$.

So the gradient of the red line at this point is $\frac{\Delta y_2}{\Delta x_2} = k\frac{\Delta y_1}{\Delta x_1} = ke^{x_1}$.

You already know that $\frac{\Delta y_1}{\Delta x_1} = e^{x_1}$.

But we also have $x_2 = \frac{x_1}{k}$ so $x_1 = kx_2$.

You now need to write the gradient in terms of the x-coordinate of the point on the red graph.

So the gradient of the red line when $x = x_2$ is ke^{kx_2}.

In general the gradient of e^{kx} is ke^{kx}.

WORKED EXAMPLE 8.1

a Find the gradient of $e^{1.2x}$ when $x = 2.6$.

b Find the value of x when the gradient of $e^{1.2x}$ equals 15.

c Find the gradient of the graph of $y = e^{1.2x}$ at the point where $y = 6.5$.

a gradient $= 1.2e^{1.2x}$

The gradient equals 1.2 times the value of the function.

When $x = 2.6$:

$$\text{the gradient} = 1.2e^{1.2\times 2.6}$$
$$= 27.2 \text{ (3s. f.)}$$

b $1.2e^{1.2x} = 15$

You found the expression for the gradient in part **a**.

$e^{1.2x} = 12.5$

$1.2x = \ln(12.5)$

$x = 2.1 (2\text{s.f.})$

This is an exponential equation. Isolate the exponential term and then use logarithms.

c $\text{gradient} = 1.2y$

$= 1.2 \times 6.5$

$= 7.8$

The gradient is proportional to the *y*-value. You have found that the gradient $= 1.2e^{1.2x}$ and $y = e^{1.2x}$ so the gradient $= 1.2y$.

Changing the base of an exponential

The gradient of other exponential functions is more difficult to find. Luckily, any exponential function can be converted into an exponential with base e.

WORKED EXAMPLE 8.2

Given that 3^x can be written in the form e^{kx}, find the value of k.

$3^x = e^{kx}$

$\Rightarrow \ln(3^x) = \ln(e^{kx})$

This is an exponential equation, so take a logarithm of each side. Since you want to find k, choose base e.

$\Rightarrow x \ln 3 = kx$

So $k = \ln 3$

We can compare coefficients of x.

Key point 8.4

Any exponential function a^x can be written in the form e^{kx}.

- If $a > 1, k$ is positive and e^{kx} represents exponential **growth**.
- If $a < 1, k$ is negative and e^{kx} represents exponential **decay**.

WORKED EXAMPLE 8.3

a Write $\left(\frac{3}{4}\right)^x$ in the form e^{kx} giving the value of k to three significant figures.

b Hence find the gradient of $\left(\frac{3}{4}\right)^x$ when $x = 2.5$.

a $\left(\frac{3}{4}\right)^x = e^{kx}$

$\Rightarrow \ln\left(\frac{3}{4}\right)^x = kx$

Take ln of both sides to solve the equation for k.

$\Rightarrow x \ln\left(\frac{3}{4}\right) = kx$

We can compare the coefficients of x.

$\Rightarrow k = \ln\left(\frac{3}{4}\right) \approx -0.288$

b The gradient of $e^{-0.288x}$ is $-0.288e^{-0.288x}$

The gradient of e^{kx} is ke^{kx}.

When $x = 2.5$:

$$\text{gradient} = -0.288e^{-0.288\times 2.5}$$
$$= -0.14 (2\text{s. f.})$$

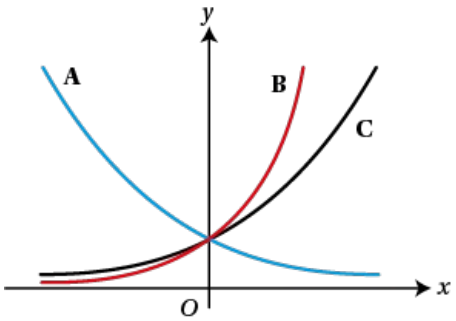
EXERCISE 8A

1 Match each exponential graph with its equation.

a **i** $y = 1.5^x$

ii $y = e^x$

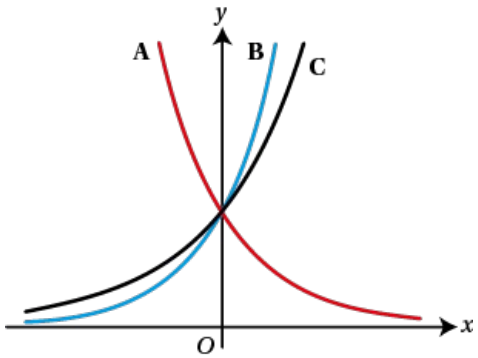
iii $y = 0.7^x$



b **i** $y = e^{1.5x}$

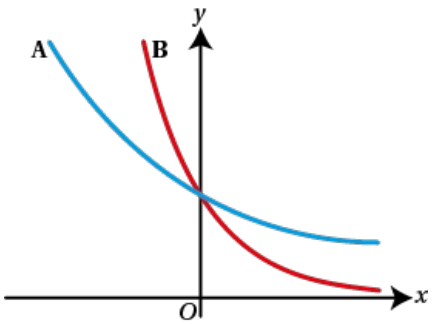
ii $y = e^{2.3x}$

iii $y = e^{-2x}$



c **i** $y = e^{-0.5x}$

ii $y = e^{-1.7x}$



2 Find an expression for the gradient of each function.

a **i** $e^{3.2x}$

ii $e^{0.6x}$

b **i** $e^{-1.3x}$

ii e^{-x}

3 Find the gradient of each graph at the given value of x.

a $y = e^{3.5x}$ when $x = 0.8$

b $y = e^{2.9x}$ when $x = -1.2$

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c $y = e^{-1.2x}$ when $x = 3.6$

d $y = e^{-0.5x}$ when $x = -0.9$

4 Find the gradient of each graph at the given value of y.

a i $y = e^{1.5x}$ when $y = 17$

ii $y = e^{4x}$ when $y = 0.6$

b i $y = e^{-0.6x}$ when $y = 3.5$

ii $y = e^{-x}$ when $y = 0.5$

5 Find the gradient of the graph of $y = e^{1.5x}$ when:

a $x = -2.1$

b $y = 12$.

6 For the graph of $e^{-2.3x}$:

a find the gradient when $y = 0.5$

b find the value of x where the gradient is -2.5.

7 The graph of $y = e^{kx}$ has gradient 26 at the point where $y = 8$.

a Find the value of k.

b Find the gradient of the graph when $x = -1$.

8 The gradient of the graph of $y = e^{ax}$ at the point where $y = 4.6$ equals -1.2. Find the value of x at the point where the gradient is -5.

9 **a** Find the value of k so that $8^x = e^{kx}$.

b Hence find the gradient of the graph of $y = 8^x$ at the point where $x = -0.5$.

10 **a** Find the value of p such that $0.3^x = e^{px}$.

b Hence find the gradient of the curve $y = 0.3^x$ at the point where $y = 0.065$.

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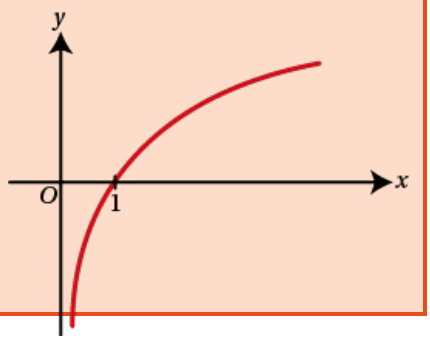
Section 2: Graphs of logarithms

You need to know the graph of the natural logarithm function.

Key point 8.5

The graph of $y = \ln(x)$:

- passes through the point $(0, 1)$
- has the y-axis as a vertical asymptote.



You can combine these facts with your knowledge of other graphs and graph transformations to solve a variety of problems.

EXERCISE 8B

1 **a** On the same diagram sketch the graphs of $y = \ln(x)$ and $y = \frac{2}{x}$.

b Hence state the number of solutions of the equation $x \ln(x) = 2$.

2 Sketch the graphs of $y = \ln(x)$ and $y = \ln(x - 2)$ on the same diagram. Label all intercepts with the coordinate axes.

3 Let k be a positive constant. Use a graphical method to prove that the equation $kx + \ln(x) = 0$ has exactly one solution.

4 The graph of $y = \ln(x)$ can be transformed into the graph of $y = \ln(3x)$ either by a translation or by a stretch. Find the translation vector and the scale factor of the stretch.

5 **a** Sketch the graphs of $y = 2\ln(x)$ and $y = \ln(x + 3)$ on the same graph.

b Find the exact solution of the equation $2\ln(x) = \ln(x + 3)$.

6 Given that $\log_{10}(x) = k\ln(x)$:

a by raising 10 to the power of both sides, show that $k = \frac{1}{\ln 10}$

b describe fully the transformation that transforms the graph of $y = \ln(x)$ to the graph of $y = \log_{10}(x)$.

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Section 3: Exponential functions and mathematical modelling

You know that the gradient of an exponential function $y = e^{kx}$ is proportional to the y value. This means that if a quantity grows (or decays) exponentially, then its **rate of change** (the rate at which one variable changes in relation to another variable) is proportional to the quantity itself.

Explore

The exponential function is the only function where the gradient is proportional to the y-value. Can you find a proof? It relies on the fact that exponential is the only function for which $\mathrm{f}(x + y) = \mathrm{f}(x)\mathrm{f}(y)$.

There are many situations where the rate of change of a quantity is proportional to its size. Here are some examples:

- A population increases at a rate proportional to its size.
- The rate of a chemical reaction is proportional to the amount of the reactant.
- The rate of radioactive decay is proportional to the amount of the substance remaining.
- The value of an investment that is subject to compound interest increases exponentially.

Exponential population growth is an example of a **mathematical model** where you try to capture the important part of a real-world situation using equations. Mathematical models are rarely perfect, so you should always be aware that they might not always work in predicting the real world. For example, there may be other factors that affect the rate of population growth (such as environmental conditions) that are not included in the exponential model.

Focus on ...

You can use technology to investigate how different factors affect population growth. See Focus on ... Modelling 1.

In Section 1 of this chapter, you only considered exponential functions of the form $y = a^x$ or $y = e^{kx}$.

All of these functions have the value 1 when $x = 0$. If you want an exponential model in which the initial value is different from 1 (such as in Worked example 8.4), you need to modify the equation by including another constant.

In many exponential models the quantity varies with time, so in this section you will use t to denote the independent variable.

WORKED EXAMPLE 8.4

The number of bacteria in a culture medium is modelled by the equation $N = 1000 \times 2^{4t}$ where t is the number of hours elapsed since $8\ \mathrm{a.m.}$

a What was the size of the population at $8\ \mathrm{a.m.}$?

b At what time will the population first reach one million?

c What does this model predict about the size of the population in the long term? Explain why this is not a realistic prediction.

a $N = 1000 \times 2^0 = 1000$ This is when $t = 0$.

b $1000 \times 2^{4t} = 1000000$ You are solving the equation $N = 1000000$.

Isolate the term containing t.

$$2^{4t} = 1000$$

$\log\left(2^{4t}\right) = \log 1000$ — Take logarithms of both sides.

$4t \log 2 = 3$ — Use the rule $\log_a x^p = p \log_a x$.

$t = \dfrac{3}{4 \log 2} = 2.49$ — Divide both sides by $4 \log 2$.

The population will first reach 1 million at 10:29 on the same day. — $0.49 \times 60 = 29$ so this is 2 hours 29 minutes after 8 a.m.

c The model predicts that the population will increase faster and faster.

This is not realistic, as the growth will eventually be limited (for example, by lack of food or space).

The exponential function grows with an increasing gradient. This is not realistic in the long term – there are lots of different reasons why the growth might slow down, so there are many possible answers here.

Key point 8.6

For a function of the form $y = Ae^{kt}$:

- the initial value (when $t = 0$) is A
- the rate of growth is $ky = kAe^{kt}$.

WORKED EXAMPLE 8.5

The mass (m grams) of one of the substances in a chemical reaction is modelled by the equation $m = Ae^{-1.2t}$ where t seconds is the time since the start of the reaction. The initial mass of the substance was 72 g.

a State the value of A.

b Find the rate at which the mass is decreasing 5 seconds after the start of the reaction.

a $A = 72$ — A is the value of m when $t = 0$.

b When $t = 5$: — The rate of change is -1.2 m, so you need to find m first.

$m = 72e^{-1.2 \times 5} = 0.178$

The rate of change is

$-1.2m = -1.2 \times 0.178$

$= -0.214$

The mass is decreasing at the rate of 0.214 grams per second. — The negative rate means that the amount is **decreasing**.

WORK IT OUT 8.1

A population grows according to the exponential model $N = 250e^{0.02t}$ where t is measured in months. Find the rate at which the population is increasing after 7 months.

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Which of the following solutions is correct? Identify the mistakes in the other two.

Solution 1

The rate of change of e^{kx} is ke^{kx} so the rate of growth is:

$$\begin{aligned} 0.02e^{0.02t} &= 0.02e^{0.14} \\ &= 0.023 \end{aligned}$$

Solution 2

The rate of change of e^{kx}iske^{kx} so the rate of growth is:

$$\begin{aligned} 250 \times 0.02e^{0.02t} &= 5e^{0.14} \\ &= 5.75 \end{aligned}$$

Solution 3

Initial population is 250.

Population after 7 months is $250e^{0.02\times 7} = 288$

So the rate of growth is $\dfrac{288-250}{7} = 5.43$

Sometimes you need to use experimental data to find the parameters in the model.

WORKED EXAMPLE 8.6

A simple model of a population of bacteria states that the number of bacteria (N thousand) grows exponentially, so that $N = Ae^{kt}$ where t is time in minutes since the start of the experiment.

Initially, there were 2000 bacteria in the dish and after 5 minutes this number has grown to 7000.

a Find the values of constants A and k.

b According to this model, how many bacteria will there be in the dish after another 5 minutes?

c Give two reasons why this model will not provide a good prediction for the number of bacteria in the dish 12 hours later.

a When $t = 0$:

$$2 = Ae^0 \Rightarrow A = 2$$

When $t = 5$:

$$7 = 2e^{5k}$$

$$e^{5k} = 3.5$$

$$k = \frac{1}{5} ln\, 3.5 (= 0.251)$$

Use the equation for $t = 0$ and $t = 5$. Remember that N is in thousands.

Use logarithms to solve the equation for k.

b $N = 2e^{10k}$

$$= 24.5$$

The model predicts that there will be 24 500 bacteria.

In another 5 minutes, $t = 10$.

Notice that, instead of using the value of k *found in part* **a**, you could use the fact that $e^{10k} = \left(e^{5k}\right)^2 = 3.5^2$.

c The model predicts that the

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bacteria population will continue growing indefinitely, but in reality it will eventually slow down as food and space become limiting factors.

The information given in the model is only approximate so in 12 hours errors in this information may make the prediction far off the correct value.

You are not expected to have any technical biological knowledge, but you may need to apply general experience of the real world to interpreting and criticising models.

There are many other possible criticisms of this model, so anything relevant would be acceptable.

Focus on ...

In focus on ... Modelling 1 you can explore some modifications to the population growth model.

WORKED EXAMPLE 8.7

A population of flies grows exponentially, so that its size can be modelled by the equation $N = Ae^{kt}$, where N is the number of flies after t weeks. At the time $t = 0$, the population size is 2400 and it is increasing at the rate of 80 flies per week. Find the values of A and k.

When $t = 0, N = A$,

so $A = 2400$

A is the initial value.

The rate of increase is kN, so:

$2400k = 80$

$k = \frac{1}{30}$

This is an exponential equation, so the rate of increase of N is kN.

EXERCISE 8C

1 An amount of £C is invested in an account giving $p\%$ annual interest.

a Find an expression for the value of the investment after 1 year in each case.

i The interest is compounded annually.

ii $\frac{p}{2}\%$ interest is compounded twice a year.

iii $\frac{p}{4}\%$ interest is compounded four times a year.

b If $\frac{p}{n}\%$ is compounded n times a year, explain why the value of the investment after one year is $V = C\left(1 + \frac{p}{100n}\right)^n$.

c Investigate the behaviour of the sequence $\left(1 + \frac{1}{n}\right)^n$ as n increases.

d For the case of a (not very realistic!) 100% interest rate, find an expression for the value of the investment after x years when the interest is compounded continuously.

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Explore

The procedure described in question 1 leads to one way to define the number e. What other definitions of e can you find?

2 In a yeast culture cell numbers are given by $N = 100e^{1.03t}$ where t is measured in hours after the cells are introduced to culture.

a What is the initial number of cells?

b How many cells will be present after 6 hours?

c How long will it take for the population to reach one thousand?

d At what rate will the population be growing at that point?

3 An algal population grows by 10% every day on the surface of a pond, and the area it covers can be modelled by the equation $y = k \times 1.1^t$ where t is measured in days. At 9 a.m. on Tuesday it covered $10\,\text{m}^2$. What area will it cover by 9 a.m. on Friday?

4 A technology company is interested in predicting the number of mobile phones in the world. The number of mobile phones in billions (N) in t years is predicted to follow the model $N = 2e^{0.1t} + 1$

a According to the model, how many mobile phones are currently in the world?

b How many mobile phones does the model predict will exist in 10 years' time? Give your answer to 3 significant figures.

Did you know?

π and e have many similar properties. Both are **irrational**, meaning that they cannot be written as a ratio of two whole numbers and both are **transcendental**, meaning that they cannot be written as the solution to a polynomial equation with integer coefficients. The proof of these facts is intricate but beautiful.

5 The mass of a piece of plutonium (M grams) after t seconds is given by $M = ke^{-0.01t}$.

a Sketch the graph of M against t.

b How long will it take to reach 25% of its original mass?

6 A population size is increasing according to an exponential model $N = N_0e^{at}$ where t is time measured in days. Initially the population size is 450 and is increasing at a rate of 90 per hour.

a Find the values of N_0 and a.

b At what rate is the population increasing when its size is 750?

c How long will the population size take to reach 2000?

7 A radioactive substance decays so that the rate of decay (measured in atoms per day) is numerically equal to 40% of the number of atoms remaining at that time. Initially there were 500 atoms. Write an equation to model the number of atoms N at time t days.

8 The value of a new car is £6800. One year later the value has decreased to £5440.

a Assuming the value continues to decrease by the same percentage every year, write the model for the price of the car in the form $V = Pe^{kt}$.

b What does this model predict the value of the car will be in 10 years' time?

9 The population of Great Britain is currently 70 million and is predicted to grow by 2% each year due to births and deaths.

a Write down a model for the population of Great Britain, P, at a time n years from now.

b Give two reasons why this model might not be valid when predicting the population of Great Britain in 2100.

10 A bowl of soup is served at a temperature 35°C above room temperature. Every 5 minutes, the temperature difference between the soup and the room air decreases by 30%. Assume that the room air temperature remains constant at 20°C. Then the temperature difference between the room and the soup, T°C, can be modelled by $T = \text{k}a^t$, where t is the time (minutes) since the soup was served.

a At what temperature will the soup be 7 minutes after serving?

b If the soup was put into a thermos flask instead of a bowl, determine how this would affect the value of:

i k

ii a.

11 The speed (V metres per second) of a parachutist t seconds after jumping from an aeroplane is modelled by the expression $V = 40\left(1 - 3^{-0.1t}\right)$.

a Find the initial speed.

b What speed does the model predict that he will eventually reach?

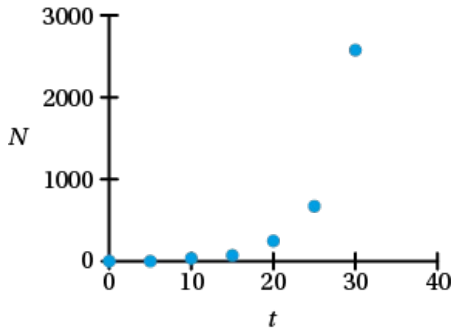
12 The model $I = 100e^{-2x}$ is used to estimate the intensity of light (I) at a distance x metres away from a bulb.

a By what factor has the light intensity dropped between 0 m and 1 m away from the bulb?

b Prove that every 1 m further from the bulb produces the same factor reduction in light intensity.

Section 4: Fitting models to data

In Worked example 8.6 you found the values of parameters A and k in the model $N = Ae^{kt}$ by using the information about the number of bacteria when $t = 0$ and $t = 5$. In many real-life situations, models are not exact, or there may be inaccuracies in the measurements. This means that if you used the number of bacteria when, say, $t = 10$ you would get slightly different values for N and k.



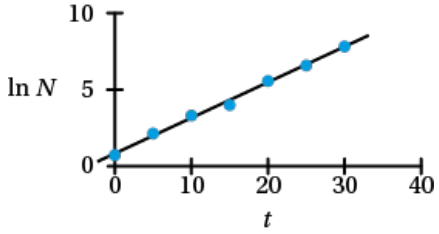
To get a more reliable result it may be a good idea to use data from more than two measurements. Suppose you measure the number of bacteria every 5 minutes for half an hour. If you plot your results you might get a graph like the one here. It is very difficult to draw an exponential curve that best fits the points. There is a clever trick that turns this curve into a straight line.

The equation for the population size is $N = Ae^{kt}$. You can take a logarithm of both sides:

$$\begin{aligned}\ln N &= \ln\left(Ae^{kt}\right)\\ &= \ln A + \ln e^{kt}\\ &= \ln A + kt\end{aligned}$$

If you write $y = \ln N$ (and remember that ln A and k are constants), you may notice that this is the equation of a straight line:

$$y = kt + \ln A$$



So, if you plot the data points with t on the x-axis and ln N on the y-axis, they should roughly follow a straight line with gradient k and y-intercept ln A.

But you know how to draw a line of best fit and find its gradient and y-intercept.

In the example, you can find from the graph that the y-intercept is 0.8 and the gradient is $\dfrac{8 - 0.8}{30} = 0.24$. So:

$k = 0.24$ and $\ln A = 0.8$

$\Rightarrow A = e^{0.8} = 2.2$

Therefore, the experimental data suggest that the model for the bacterial population growth is $N = 2.2e^{0.24t}$.

Rewind

See Chapter 6, Section 3 for a reminder of equations of straight lines.

You can perform a similar calculation when the base of the exponential is unknown. In that case you can take logarithms in any base (it is common to use base 10 or base e).

Fast forward

You will learn more about lines of best fit in Chapter 16, Section 4.

Key point 8.7

If $y = kb^x$ then $\log y = \log k + x \log b$.

- The graph of $\log y$ against x is a straight line with gradient $\log b$ and y-intercept $\log k$.

Did you know?

This type of graph, where you take a logarithm of one variable but not the other, is called a semi-log graph.

WORKED EXAMPLE 8.8

The mass of a piece of radioactive material decays exponentially, according to the model $M = Cb^t$, where M is the mass in grams, t is the time in seconds and C and b are constants. A physicist measures the mass several times and plots the points on a graph with t on the x-axis and $\log M$ on the y-axis. The line of best fit has equation $\log M = 1.3 - 1.8t$.

Estimate the values of C and b.

$\text{If } M = Cb^t \text{ then :}$
$\log M = \log C + \log(b^t)$
$= \log C + t \log b$

Take logs of both sides to transform to an equation of a straight line.

Use rules of logs.

So

$\log C + t \log b = 1.3 - 1.8t$

$\Rightarrow \log C = 1.3 \text{ and } \log b = -1.8$

$C = 10^{1.3} \approx 20$
$b = 10^{-1.8} \approx 0.016$

$\log C$ and $\log b$ are numbers that should match the coefficients in the straight line equation.

Remember that the logs are in base 10.

A variation on the method used in Worked example 8.7 can also be used for models of the form $y = ax^n$ where x is in the base of the exponential and the power is unknown. In this case, you need to take a logarithm of both variables in order to get a straight line graph. This is called a log–log graph.

Did you know?

Many natural and man-made phenomena follow so-called power laws of the form $y = ax^n$. Examples include the distribution of common words, the sizes of cities and corporations, and Kepler's law for planetary orbits.

Key point 8.8

If $y = ax^n$ then $\log y = \log a + n \log x$.

The graph of $\log y$ against $\log x$ is a straight line with gradient n and y-intercept $\log a$.

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WORKED EXAMPLE 8.9

A scientist thinks that variables x and y are related by an equation of the form $y = ax^n$.

She collects the data and plots a scatter graph with log x on the horizontal axis and log y on the vertical axis. The points follow a straight line with gradient 2.6 and y-intercept -0.9. Find the values of a and n, and hence write an equation for y in terms of x.

If $y = ax^n$ then

$\log y = \log a + \log(x^n)$

$= \log a + n \log x$

Take logs of both sides to identify the gradient and the intercept.

gradient: $n = 2.6$

intercept:

$\log a = -0.9 \Rightarrow a = 10^{-0.9} \approx 0.13$

This is a straight line with gradient n and y-intercept log a.

So: $y \approx 0.13x^{2.6}$

Fast forward

If you study the Statistics option of Further Mathematics you will learn how to use your calculator to find the equation of the line of best fit on a scatter graph. You will then be able to apply these techniques to real experimental data.

EXERCISE 8D

1 In each of the following examples variables x and y are related by $y = Ab^x$ You are given the equation of a straight line of $\ln(y)$ as a function of x. Find the values of the constants A and b.

a $\ln y = 1.2x + 0.7$

b $\ln y = 3.1 - 0.6x$

c $\ln y = 2.3x - 4$

2 In each of the following examples variables x and y are related by $y = Cx^n$. You are given the equation of a straight line of ln (y) as a function of ln (x). Find the values of the constants C and n.

a $\ln y = 0.7 \ln x + 1.2$

b $\ln y 2.1 \ln x - 4.7$

c $\ln y = 2 - 0.9 \ln x$

Worksheet

For more practice of questions like this, see Support sheet 8.

3 A zoologist is studying the growth of a population of fish in a lake. He thinks that the size of the population can be modelled by the equation $N = Ae^{kt}$ where N is the number of fish and t is the number of months since the fish were first introduced into the lake.

a The zoologist collected some data and wants to plot them on the graph in order to check whether his proposed model is suitable. Assuming his model is correct, state which of the

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following graphs will produce approximately a straight line.

A N against $\log t$

B $\log N$ against t

C $\log N$ against $\log t$

You may now assume that the proposed model is correct.

b Initially, 150 fish are introduced into the lake. Write down the value of A.

c After 10 months there are 780 fish in the lake. Find the value of k.

d Comment on the suitability of this model for predicting the number of fish in the long term.

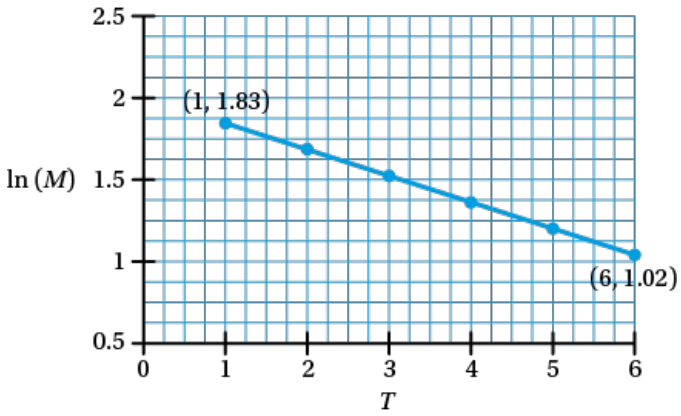
4 It is known that a population of bacteria can be modelled by the equation $N = 1500e^{bt}$, where N is the number of bacteria at time t hours.

a Explain the significance of the number 1500 in the equation.

b When $t = 0$, the population is growing at the rate of 500 bacteria per hour. Find the value of b.

c According to this model, how long will it take for the number of bacteria to reach 1 million?

5 A scientist is modelling exponential decay of the amount of substance in a chemical reaction. She proposes a model of the form $M = Kc^t$ where M is the mass of the substance in grams, t is the time in seconds since the start of the reaction, and K and c are constants. The mass of the substance is recorded for the first six seconds of the reaction. The graph of $\ln(M)$ against t is shown.

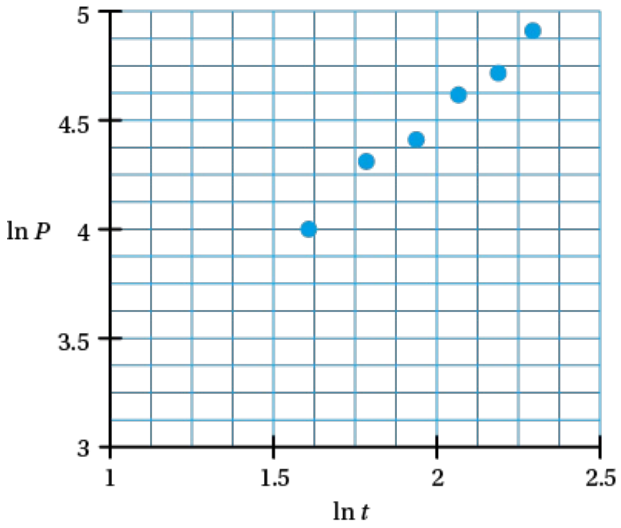


a The points are found to lie on a straight line. Find its equation, giving parameters to 2 significant figures.

b Hence find the values of K and c.

c How long, to the nearest second, will it take for the mass of the substance to fall below 1 gram?

6 A model for the size of the population of a city predicts that the population will grow according to the equation $P = Ct^n$ where P thousand is the number of people and t is the number of years since the measurements began. The graph shows $y = \ln(P)$ plotted against $x = \ln(t)$.



a Draw a line of best fit on the graph and find its equation in the form $y = mx + c$.

b Hence estimate the values of C and n.

c According to this model, after how many years will the population first exceed 200000?

7 A scientist is investigating the population of mice in a field. She collected some data over a period of time, and recorded it on a graph. Let N denote the number of mice at time t weeks. The graph of $y = \ln N$ against t has equation $y = 5.8 + 0.16t$.

a Find the equation for the size of the population at time t weeks.

b Find the rate at which the population is growing after 8 weeks.

8 **a** A common model used in population growth is called the logistic function. It predicts that the population N is related to the time t by the formula

$$N = \frac{1}{1 + Je^{-bt}}$$

Find an expression, in terms of N, which can be plotted against t to form a straight line if a population follows this model. Write down an expressions for the gradient and the intercept of the line in terms of b and J.

b By comparing the long-term predictions of the logistic function and normal exponential growth explain why the logistic function is a better model for population growth than normal exponential growth.

Worksheet

For more applications of logarithms see Extension sheet 8.

Checklist of learning and understanding

- For all the graphs $y = a^x$:
  - The y-intercept is always $(0, 1)$, because $a^0 = 1$.
  - The graph of the function lies entirely above the x-axis.
  - The x-axis is an asymptote.
- If $a > 1$, then as x increases so does y. This is called exponential growth.
- If $0 < a < 1$, then as x increases, y decreases. This is called exponential decay.
- Exponential functions are often used to model situations where the rate of growth is proportional to the amount present.
  - The gradient of e^{kx} equals ke^{kx}.

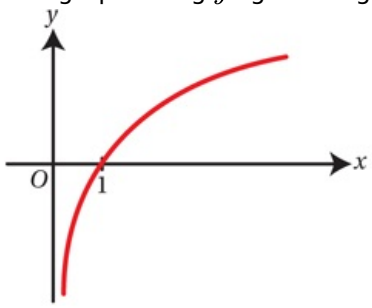
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- In a model of the form $y = Ae^{kt}$ the initial value is A and the rate of change equals ky.

- The graph of $y = \ln(x)$:
  - passes through $(1, 0)$
  - has the y-axis as an asymptote.
- Logarithms can be used to turn some curved graphs into straight lines. This is used to estimate parameters in models.
  - If $y = kb^x$ then $\log y = \log k + x \log b$

  The graph of log y against x is a straight line with gradient log b and y-intercept log k.
  - If $y = ax^n$ then $\log y = \log a + n \log x$

  The graph of log y against log x is a straight line with gradient n and y-intercept $\log a$.



Mixed practice 8

1 **a** Sketch the graph of $y = e^{0.8x}$.

b Find the gradient of your graph at the point where $x = 3$.

c Use your graph to determine the number of solutions of the equation $e^{0.8x} = \frac{1}{x}$.

2 The amount of substance in a chemical reaction is decreasing according to the equation $m = 32e^{-0.14t}$ where m grams is the mass of the substance t seconds after the start of the reaction.

a State the amount of the substance at the start of the reaction.

b At what rate is the amount of substance decreasing 3 seconds after the start of the reaction?

c How long will it take for the amount of substance to halve?

3 Use graphs to determine the number of solutions of the equation $\ln x = \frac{3}{x^2}$.

4 The volume of a blob of algae (V) in cm^3 in a jar is modelled by $V = 0.4 \times 2^{0.1t}$ where t is the time in weeks after the observation begins.

a What is the initial volume of the algae?

b How long does it take for the volume of algae to double?

c Give two reasons why the model would not be valid for predicting the volume in 10 years' time.

5 A rumour spreads exponentially through a school. When school begins (at 9 a.m.) 18 people know it. By 10 a.m. 42 people know it.

Let N be the number of people who know the rumour after t minutes.

a Find constants A and k so that $N = Ae^{kt}$.

b How many people know the rumour at 10:30?

c There are 1200 people in the school. According to the exponential model at what time will everyone know the rumour?

6 A patient is being treated for a condition by having insulin injected. The level of insulin (I) in the blood t minutes after the injection is given by $I = 10e^{-0.05t} + 2$, measured in microunits per millilitre ($\mu\text{U/ml}$).

a What is the level of insulin immediately after the injection?

b There is a danger of coma if insulin levels fall below 1.8 µU/ml. According to the model, will this level be reached? Justify your answer.

7 It is thought that the global population of tigers is falling exponentially. Estimates suggest that in 1970 there were 37000 tigers but by 1980 the number had dropped to 22000.

a A model of the form $T = ka^n$ is suggested, connecting the number of tigers (T) with the number of years (n) after 1970.

i Show that $22000 = ka^{10}$.

ii Write another similar equation and solve them to find k and a.

b What does the model predict the tiger population will be in 2020?

c When the population reaches 1000, the tiger population will be described as 'near extinction'. In which year will this happen?

8 A zoologist believes that the population of fish in a small lake is growing exponentially. He collects data about the number of fish every 10 days for 50 days. The data are given in this table:

Time (days)	0	10	20	30	40	50
Number of fish	35	42	46	51	62	71

The zoologist proposes a model of the form $N = Ae^{kt}$ where N is the number of fish and t is time in days. In order to estimate the values of the constant A and k he plots a graph with t on the horizontal axis and ln N on the vertical axis.

a Explain why, assuming the zoologist's model is correct, this graph will be approximately a straight line.

b Complete the table of values for the graph:

t	0	10	20	30	40	50
ln N	3.56	3.74	3.83	3.93		4.26

c Find the equation of the line of best fit for this table. (Do not draw the graph.) Hence estimate the values of A and k.

d Use this model to predict the number of fish in the lake when $t = 260$.

e The zoologist finds that the number of fish in the lake after 260 days is actually 720. Suggest one reason why the observed data does not fit the prediction.

Tip

You can use technology to find the equation of the line of best fit.

9 Quantities m and t are related by an equation of the form $m = at^p$ where a and p are constants. The graph of log m against log t is a straight line that passes through the points $(2, 5)$ and $(4, 0)$. Find the values of a and p.

10 A substance is decaying in such a way that its mass, m kg, at time t years from now is given by the formula $m = 240e^{-0.04t}$.

a Find the time taken for the substance to halve its mass.

b Find the value of t for which the mass is decreasing at a rate of 2.1 kg per year.

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[Question part reference style adapted]

11 The mass, M grams, of a certain substance is increasing exponentially so that, at time t hours, the mass is given by $M = 40e^{kt}$ where k is a constant. The following table shows certain values of t and M.

t	0	21	63
M		80	

a In either order:

i find the values missing from the table

ii determine the value of k.

b Find the rate at which the mass is increasing when $t = 21$.

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[Question part reference style adapted]

12 Radioactive decay can be modelled using an equation of the form $m = m_0 e^{-kt}$ where m is the mass of the radioactive substance at time t, m_0 is the initial mass and k is a positive constant. The half-life of a radioactive substance is the length of time it takes for half of the substance to decay. A particular radioactive substance has a half-life of 260 years. Find the value of k.

13 The speed, $V \mathrm{m} s^{-1}$, of a parachutist, t seconds after jumping from the aeroplane, is modelled by the equation $V = 42\left(1 - e^{-0.2t}\right)$.

a What is the initial speed of the parachutist?

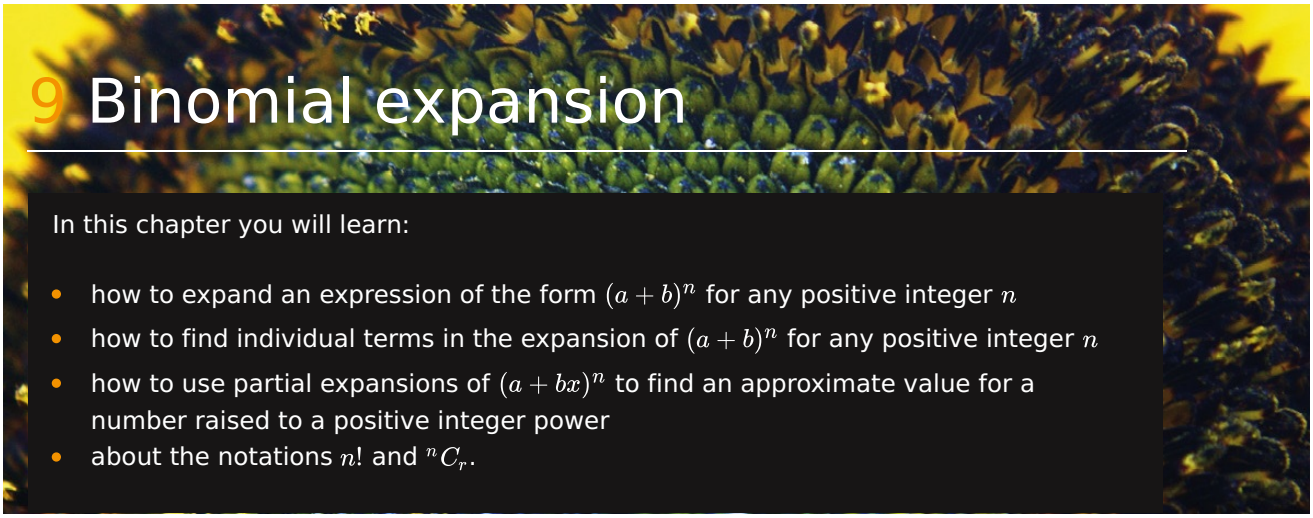
b What is the maximum speed that the parachutist could reach?

c When the parachutist reaches $22 \mathrm{m} s^{-1}$ he opens the parachute. For how long is he falling before he opens his parachute?

14 When a cup of tea is first made its temperature is $98^{\circ}\mathrm{C}$. After two minutes the temperature has reached $94^{\circ}\mathrm{C}$. The room temperature is $22^{\circ}\mathrm{C}$ and the difference between the temperature of the tea and the room temperature decreases exponentially.

a Let T be the temperature of the tea and t be the time, in minutes, since the tea was made. Find the constants C and t so that $T - 22 = Ce^{-kt}$.

b Find the time it takes for the tea to cool to $78^{\circ}\mathrm{C}$.



9 Binomial expansion

In this chapter you will learn:

- how to expand an expression of the form $(a+b)^n$ for any positive integer n
- how to find individual terms in the expansion of $(a+b)^n$ for any positive integer n
- how to use partial expansions of $(a+bx)^n$ to find an approximate value for a number raised to a positive integer power
- about the notations $n!$ and ${}^nC_r$.

Before you start...

GCSE	You should know how to evaluate expressions involving powers, including using the correct order of operations.	1 Evaluate 2×3^2.
GCSE	You should know how to use the rules of indices.	2 a Evaluate $\left(2x^3\right)^4$. b Simplify $x^4 \times x^7$. c Simplify $\frac{x^{12}}{x^3}$.
GCSE	You should know how to multiply out two brackets.	3 Expand $(2x+3)^2$.
Chapter 3	You should know how to solve quadratic equations using the formula or factorising.	4 Solve $x^2+5x+4=0$.

What is binomial expansion?

A **binomial expression** is one which contains two terms. For example, $a+b$.

Finding a power of such a binomial expression can be performed by expanding brackets. For example, $(a+b)^7$ can be found by calculating, at length:

$$(a+b)(a+b)(a+b)(a+b)(a+b)(a+b)(a+b)$$

This is time consuming and mistakes can easily be made, but fortunately there is a much quicker approach, called the **binomial expansion**.

Section 1: The binomial theorem

To see how you might rapidly expand an expression of the form $(a+b)^n$ for an integer power n first look at some expansions of $(a+b)^n$ done using the slow method of multiplying out brackets repeatedly. The table shows the results for $n = 0, 1, 2, 3$ and 4. In the right-hand column the coefficients and powers in the expansions are coloured to emphasise the pattern.

$(a+b)^0$	$= 1$	$= 1a^0b^0$
$(a+b)^1$	$= a + b$	$= 1a^1b^0 + 1a^0b^1$
$(a+b)^2$	$= a^2 + 2ab + b^2$	$= 1a^2b^0 + 2a^1b^1 + 1a^0b^2$
$(a+b)^3$	$= a^3 + 3a^2b + 3ab^2 + b^3$	$= 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$
$(a+b)^4$	$= a^2 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$	$= 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$

There are several patterns:

- The powers of a and b in each term (coloured blue) always add up to n.
- Each power of a from 0 up to n is present in one of the terms, with the corresponding complementary power of b.
- The pattern of coefficients (coloured in red) in each line is symmetrical.

Key point 9.1

The **binomial theorem** states that for any positive integer n,

$(a+b)^n = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_na^0b^n$

This will appear in your formula book.

For example, the calculation for the expansion of $(a+b)^4$ is:

$$(a+b)^4 = {}^4C_0a^4b^0 + {}^4C_1a^3b^1 + {}^4C_2a^2b^2 + {}^4C_3a^1b^3 + {}^4C_4a^0b^4$$

The numbers ${}^nC_r$ are called **binomial coefficients** because they are the constants in the expansions of expressions of the form $(a+b)^n$.

Key point 9.2

Binomial coefficients can be found using the ${}^nC_r$ button on your calculator or from Pascal's triangle.

${}^nC_r$ can also be written as $\binom{n}{r}$ or ${}_nC_r$.

Tip

You can use either ${}^nC_r$ or $\binom{n}{r}$, but you need to recognise both.

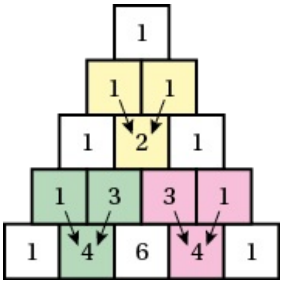
Pascal's triangle is constructed by adding two adjacent numbers to produce the number below them in the next row, as shown in the diagram.

To find the binomial coefficient ${}^4C_2$ in Pascal's triangle, look at the row starting $1, 4 \ldots$ The coefficients in this row are:

$${}^4\mathrm{C}_0 \ {}^4\mathrm{C}_1 \ {}^4\mathrm{C}_2 \ {}^4\mathrm{C}_3 \ {}^4\mathrm{C}_4$$

So ${}^4C_2$ is the third number in this row, which is 6.

In Key point 9.1, a and b can be replaced by any number, letter or expression.



Tip

The binomial coefficient ${}^nC_r$ is the $(r+1)$th number in the row starting 1, n ... in Pascal's triangle.

WORKED EXAMPLE 9.1

Expand and simplify $(2+x)^4$.

$(2+x)^4 = {}^4C_0(2)^4(x)^0 + {}^4C_1(2)^3(x)^1 + 4C_2(2)^2(x)^2 + {}^4C_3(2)^1(x)^3 + {}^4C_4(2)^0(x)^4$

Use the formula from Key point 9.1.

$= 1(16)(1) + 4(8)(x) + 6(4)(x^2) + 4(2)(x^3) + 1(1)(x^4)$

$= 16 + 32x + 24x^2 + 8x^3 + x^4$

Find the binomial coefficients (the red numbers) either from your calculator or by looking at Pascal's triangle: you want the row starting $1, 4$ ...

Remember that anything raised to the power of 0 is 1.

You need to be careful when there is a minus sign or a number in front of x.

WORKED EXAMPLE 9.2

a Expand and simplify $(3x-5)^3$.

b Hence find the expansion of $(3x^3-5)^3$.

a $(3x-5)^3 = (3x+(-5))^3$

Use the formula from Key point 9.1.

$= 3C_0(3x)^3(-5)^0 + 3C_1(3x)^2(-5)^1 + 3C_2(3x)^1(-5)^2 + 3C_3(3x)^0(-5)^3$

$= 1(27x^3)(1) + 3(9x^2) + 3(3x)(25) + 1(1)(-125)$

$= 27x^3 - 135x^2 + 225x - 125$

Find the binomial coefficients from your calculator or from Pascal's triangle.

Be careful with powers:

$(3x)^3 = 27x^3$

$(-5)^2 = 25$, but $(-5)^3 = -125$

b $(3x^3-5)^3 = 27(x^3)^3 - 135(x^3)^2 + 225(x^3) - 125$

Replace x by x^3 in the expansion in part **a**.

$= 27x^9 - 135x^6 + 225x^3 - 125$ Use laws of indices to simplify.

Sometimes you want to find a particular term, rather than the whole expansion.

WORKED EXAMPLE 9.3

Find the coefficient of x^5y^3 in the expansion of $(x + 2y)^8$.

The required term is ${}^8C_3(x)^5(2y)^3$. Write down the required term in the form ${}^nC_r(a)^{n-r}(b)^r$

$a = x, b = 2y, r = 3, n = 8$

${}^8C_3 = 56$

$(x)^5 = x^5$

$(2y)^3 = 8y^3$

Calculate the binomial coefficient and apply the powers to the bracketed terms.

The term is $56(x^5)(8y^3) = 448x^5y^3$

The coefficient is 448.

Combine the elements to calculate the coefficient.

Tip

A question might ask you to give the term or just the coefficient. Make sure that you answer the question posed.

WORK IT OUT 9.1

Expand $(1 + 2x)^4$.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$= 1^4 +^4 C_1 (2x) +^4 C_2(2x)^2 +^4 C_3(2x)^3 +^4 C_4(2x)^4$

$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$

Solution 2

$= 1 + 4 \times 2x + 6 \times 2x^2 + 4 \times 2x^3 + 2x^4$

$= 1 + 8x + 12x^2 + 8x^3 + 2x^4$

Solution 3

$= 1^4 + (2x)^0 + 1^3(2x)^1 + 1^2(2x)^2 + 1^1(2x)^3 + 1^0(2x)^4$

$= 1 + 2x + 4x^2 + 8x^3 + 16x^4$

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EXERCISE 9A

1 Use your calculator to find these binomial coefficients:

a **i** ${}^7C_3$

ii ${}^9C_2$

b **i** ${}^9C_0$

ii ${}^6C_0$

c **i** $\binom{8}{7}$

ii $\binom{10}{8}$

2 Expand and simplify the following:

a **i** $(2-x)^5$

ii $(3+x)^6$

b **i** $(2-3x)^4$

ii $(2x-7)^3$

c **i** $(3x+y)^5$ (first 3 terms only)

ii $(2c-d)^4$ (first 3 terms only)

d **i** $\left(2x^2-3x\right)^3$

ii $\left(2x^{-1}+5y\right)^3$

e **i** $\left(2x+\frac{1}{x}\right)^3$

ii $\left(x-\frac{3}{x}\right)^4$

3 **a** Find the coefficient of xy^3 in the expansion of $(x+y)^4$.

b Find the coefficient of x^3y^4 in the expansion of $(x+y)^7$.

c Find the coefficient of ab^6 in the expansion of $(a+b)^7$.

d Find the coefficient of a^5b^3 in the expansion of $(a+b)^8$.

4 **a** Find the coefficient of y^3 in:

i $(2-3y)^4$

ii $(5+y)^4$

b Find the term in y^4 in:

i $(x-2y)^7$

ii $(y-2x)^7$

c Find the coefficient of b^3 in:

i $\left(2-\frac{1}{2}b\right)^5$

ii $(17+3b)^5$

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5 Find the expansion of $(3+x)^3$.

6 Determine which term in the expansion of $(1-2y)^5$ has the given coefficient:

a 80

b -80

7 Find the coefficient of x^3 in the expansion of $(1-5x)^9$.

8 Find the term in x^2 in the expansion of $(3-2x)^7$.

9 Find the first four terms in the expansion of $(y+3y^2)^6$ in ascending powers of y.

10 **a** Find the expansion of $(1+2x)^4$.

b Expand $\left(\frac{x+2}{x}\right)^4$.

11 Find the coefficient of x^2 in the expansion of $\left(x+\frac{1}{x}\right)^8$.

12 The expansion of $(x+ay)^n$ contains the term $60x^4y^2$.

a Write down the value of n.

b Find the value of a.

13 Complete and simplify the expansion of $\left(2z^2+\frac{3}{z}\right)^4$ that begins $16z^8+96z^5$.

14 The expansion of $\left(3x^2y+\frac{5x}{y}\right)^n$ begins with $27x^6y^3+135x^5y$.

a Write down the value of n.

b Complete and simplify the expansion.

15 Find the constant term in the expansion of $(x-2x^{-2})^9$.

16 Find the term in x^5 in $\left(x^2-\frac{3}{x}\right)^7$.

17 Find the term that is independent of x in the expansion of $\left(2x-\frac{5}{x^2}\right)^{12}$.

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Section 2: Calculating the binomial coefficients

Although in many problems you can use your calculator or Pascal's triangle to find binomial coefficients, sometimes you may need to use a formula. This formula contains the **factorial** function $n!$

Key point 9.3

$n! = n \times (n-1) \times \ldots \times 2 \times 1$ for $n \in \mathbb{N}$

$0!$ is defined to be 1.

Using this function, it is possible to find a formula for the binomial coefficients.

Key point 9.4

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

This will appear in your formula book.

Fast forward

Binomial coefficients are also used to evaluate binomial probabilities, as you will learn in Chapter 17, Section 3.

You will see how to prove this formula if you study the Statistics option of Further Mathematics.

WORKED EXAMPLE 9.4

Show that $^6C_4 = 15$.

$$^6C_4 = \frac{6!}{4! \times (6-4)!} = \frac{6!}{4! \times 2!}$$

Use the formula with $n = 6$ and $r = 4$.

$$= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1) \times (2 \times 1)}$$

Use the definition of $n!$ to write out each term in full.

Then cancel factors from the numerator and denominator.

$$= \frac{6 \times 5}{2} = 15$$

Finally, evaluate what is left.

This method of cancelling factors that appear in the numerator and denominator can be used to provide an alternative (often more useful) version of the formula for ^nC_r:

$$\begin{aligned} ^nC_r &= \frac{n!}{r!\,(n-r)!} \\ &= \frac{(n) \times (n-1) \times \ldots \times (n-r+1) \times (n-r) \ldots \times 3 \times 2 \times 1}{r! \times ((n-r) \times \ldots \times 3 \times 2 \times 1)} \\ &= \frac{n \times (n-1) \times \ldots \times (n-r+1)}{r!} \end{aligned}$$

Tip

You can remember the last formula by noting that there are r numbers in both the numerator and the denominator. For example,

$$^7C_3 = \frac{7 \times 6 \times 5}{3 \times 2 \times 1}.$$

Key point 9.5 gives the expressions for $r = 0, 1$ and 2.

Key point 9.5

$${}^nC_0 = 1$$
$${}^nC_1 = n$$
$${}^nC_2 = \frac{n(n-1)}{2}$$

These expressions are useful when part of an expansion has been given and you need to find the power.

WORKED EXAMPLE 9.5

The first three terms of the expansion of $(1 + 2x)^n$ is given by $1 + ax + 112x^2$.

Find the values of n and a.

$$(1+2x)^n = \binom{n}{0}(1)^n + \binom{n}{1}(1)^{n-1}(2x) + \binom{n}{2}(1)^{n-2}(2x)^2 + \dots$$

Write out the expansion of the left-hand side in terms of n.

$$= 1 + n(2x) + \frac{n(n-1)}{2}4x^2 + \dots$$
$$= 1 + 2nx + 2n(n-1)x^2 + \dots$$

Now use the expressions from Key point 9.5 for $\binom{n}{0}$, $\binom{n}{1}$ and $\binom{n}{2}$.

Comparing coefficients of x^2:

You are given that the coefficient of x^2 is 112 (whereas you don't know the coefficient of x), so equating coefficients of x^2 allows you to solve for n.

$$2n(n-1) = 112$$
$$2n^2 - 2n - 112 = 0$$
$$n^2 - n - 56 = 0$$
$$(n-8)(n+7) = 0$$
$$n = 8, -7$$

$n = 8$ as n must be positive.

Comparing coefficients of x:

Now that you know n you can equate coefficients of x to find a.

$$2n = a$$
$$a = 16$$

EXERCISE 9B

1 Use the formula to evaluate these binomial coefficients:

a i ${}^7C_2$

ii ${}^5C_2$

b i ${}^{10}C_3$

ii ${}^{12}C_3$

c i $\binom{8}{1}$

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ii $\binom{15}{1}$

d **i** $\binom{15}{0}$

ii $\binom{9}{0}$

e **i** ${}^{12}C_{12}$

ii ${}^{8}C_{8}$

f **i** $\binom{7}{6}$

ii $\binom{12}{11}$

2 Find the value of n in each equation.

a **i** ${}^{n}C_{1} = 10$

ii ${}^{n}C_{1} = 13$

b **i** $\binom{n}{2} = 45$

ii $\binom{n}{2} = 66$

3 If $(2+x)^n = 32 + ax + \ldots$

a find the value of n

b find the value of a.

4 If $(1+2x)^n = 1 + 20x + ax^2 + \ldots$

a find the value of n

b find the value of a.

Worksheet

For more questions on identities involving binomial coefficients, see Extension sheet 9.

5 If $\left(1+\frac{x}{2}\right)^n = 1 + ax + \frac{33}{2}x^2 + \ldots$

a find the value of n

b find the value of a.

6 Given that ${}^{n}C_{k} = {}^{n}C_{k+1}$ write n in terms of k.

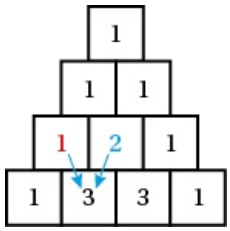
7 This question explores why binomial coefficients appear in Pascal's triangle.

Consider the expansions of $(a+b)^2$ and $(a+b)^3$:

$$\begin{aligned}(a+b)^2 &= 1a^2 + 2ab + 1b^2\\ (a+b)^3 &= (a+b)(1a^2 + 2ab + 1b^2)\\ &= 1a^3 + 2a^2b + 1ab^2\\ &\quad +1a^2b + 2ab^2 + 1b^3\end{aligned}$$

In the expansion of $(a+b)^3$, the coefficient of a^2b (which is 3) is found by adding $2a^2b$ and $1a^2b$. This exactly corresponds to the way that you get from one row of Pascal's triangle to the next

row:



a Multiply out $(a+b)(a+b)^3$ in a similar way to show which two coefficients add up to give the coefficient of a^2b^2 in the expansion of $(a+b)^4$.

b Try some other examples to see how multiplying out successive powers of $(a+b)$ leads to Pascal's triangle.

Explore

Find out about many interesting patterns that appear in Pascal's triangle. Which of them can you explain by using the binomial expansion?

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Section 3: Applications of the binomial theorem

There are two common applications of the binomial theorem:

- Simplifying a more complex expansion (usually multiplying a binomial expansion by another bracket).
- Making approximations to numbers raised to a positive integer power.

WORKED EXAMPLE 9.6

Use the binomial theorem to expand and simplify $(5-3x)(2-x)^4$.

$(2-x)^4 = 1(2)^4 + 4(2)^3(-x) + 6(2)^2(-x)^2 + 4(2)^1(-x)^3 + 1(-x)^4$
$= 16 - 32x + 24x^2 - 8x^3 + x^4$

First expand $(2-x)^4$

So $(5-3x)(2-x)^4 = (5-3x)(16-32x+24x^2-8x^3+x^4)$
$= 5[16-32x+24x^2-8x^3+x^4]$
$-3x[16-32x+24x^2-8x^3+x^4]$
$= 80 - 208x + 216x^2 - 112x^3 + 29x^4 - 3x^5$

Then multiply by $(5-3x)$ in the normal way.

Sometimes you are only interested in one of the terms, rather than the whole expansion.

WORKED EXAMPLE 9.7

a Find the first three terms in the expansion of $(1+2x)^6$.

b Hence find the coefficient of x^2 in the expansion of $(1-x+2x^2)(1+2x)^6$.

a $(1+2x)^6 = 1(1)^6 + 6(1)^5(2x)^1 + 15(1)^4(2x)^2 + \ldots$

You can stop after the first three terms.

$= 1 + 12x + 60x^2 + \ldots$

Be careful with the final term: $(2x)^2 = 4x^2$.

b $(1-x+2x^2)(1+12x+60x^2\ldots)$

You only need the first three terms of the binomial expansion, as after that all terms have powers of x^3 or higher.

The x^2 term is:

$60x^2 - 12x^2 + 2x^2 = 50x^2$

There are three ways to get the x^2 term: $1 \times 60x^2$, $-x \times 12x$ and $2x^2 \times 1$.

So the coefficient of x^2 is 50.

When x is very small, as the powers of x become larger their values become smaller. So the first few terms of a binomial expansion may be used to approximate the entire expansion.

Key point 9.6

If the value of x is close to zero, large powers of x will be extremely small.

WORKED EXAMPLE 9.8

a Find the first three terms in ascending powers of x of the expansion of $(2-x)^5$.

b Use your answer to a to find an approximate value of 1.99^5.

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a The first 3 terms of $(2-x)^5$ are

$1(2)^5 + 5(2)^4(-x)1 + 10(2)^3(-x)^2 = 32 - 80x + 80x^2$

You know the coefficients for $n = 5$.

b $2 - x = 1.99$

$x = 0.01$

You need to choose a value of x so that $(2-x)^5 = 1.99^5$, i.e. so that $2 - x = 1.99$.

$1.99^5 \approx 32 - 80(0.01) + 80(0.01^2)$

$= 32 - 0.8 + 0.008 = 31.208$

Now just evaluate the first three terms of the series at $x = 0.01$ to give an approximation for 1.99^5.

EXERCISE 9C

1 **a** Expand and simplify $(1+2x)^5$.

b Expand and simplify $(3-x)(1+2x)^5$.

2 **a** Find the first four terms, in ascending powers of x, in the expansion of $(2-5x)^7$.

b Hence find the coefficient of x^3 in the expansion of $(1+2x)(2-5x)^7$.

3 **a** Find the first three terms in the expansion of $(3-5x)^4$.

b Using a suitable value of x, use your answer to find a 5 significant figure approximation for 2.995^4.

4 **a** Find the first three terms in the expansion of $(8+2x)^5$.

b Use your answer, with a suitable value of x, to find an approximate value of 8.02^5.

Worksheet

For a further example of this type and more practice questions, see Support sheet 9.

5 **a** Find the first three terms in the expansion of $(2+3x)^7$.

b Hence find an approximation to:

i 2.3^7

ii 2.03^7.

c Which of your answers in part **b** provides a more accurate approximation? Justify your answer.

6 **a** Find the first three terms in the expansion of $(4-x)^7$.

b Hence find the coefficient of x^2 in the expansion of $\left(2+2x-x^2\right)(4-x)^7$.

7 **a** Expand $\left(\mathrm{e} + \frac{2}{\mathrm{e}}\right)^5$.

b Simplify $\left(\mathrm{e} + \frac{2}{\mathrm{e}}\right)^5 + \left(\mathrm{e} - \frac{2}{\mathrm{e}}\right)^5$.

8 **a** Write the expression $(1+x)^n(1-x)^n$ in the form $(\mathrm{f}(x))^n$.

b Find the first three non-zero terms of the expansion of $(1-x)^{10}\left(1+x\right)^{10}$ in ascending powers of x.

9 Find the coefficient of x^5 in the expansion of $(1+3x)(1+x)^7$.

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Checklist of learning and understanding

- The expansion of $(a+b)^n$ can be found directly using the formula:
  $(a+b)^n = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_n a^0b^n$
- The binomial coefficients, written ${}^nC_r$ or $\binom{n}{r}$, are given by the formula:
  ${}^nC_r = \dfrac{n!}{r!\,(n-r)!}$
  where $n! = n \times (n-1) \times$ ... $\times 3 \times 2 \times 1$ and $0! = 1$.
- Approximations for numbers raised to a positive integer power can be made using the first few terms of a binomial expansion $(a+bx)^n$. This is valid when bx is small, so that terms with higher powers are negligibly small.

Mixed practice 9

1. ${}^nC_2 = \dfrac{n!}{2!6!}$. What is the value of n?

2. Find the coefficient of x^5 in the expansion of $(2-x)^{12}$.

3. Fully expand and simplify $\left(2x^{-1}+5y\right)^3$.

4. Find the coefficient of x^2y^6 in $\left(3x+2y^2\right)^5$.

5. $a = 2-\sqrt{2}$. Using binomial expansion or otherwise, express a^5 in the form $m+n\sqrt{2}$.

6. Find the constant term in the expansion of $\left(x^3-2x^{-1}\right)^4$.

7. Fully expand and simplify $\left(x^2-\dfrac{2}{x}\right)^4$.

8. The constant term in the expansion of $\left(x^2+\dfrac{a}{x^4}\right)^9$ is $-\dfrac{28}{9}$. What is the value of a?

9. Find the coefficient of d^{11} in the expansion of $\left(2+5d\right)(1+d)^{14}$.

10. Find the coefficient of x^6 in the expansion of $\left(1-x^2\right)\ (1+x)^5$.

11. **a** Find the binomial expansion of $(2x+5)^4$ simplifying the terms.

    b Hence show that $(2x+5)^4-(2x-5)^4$ can be written as $320x^3+kx$ where the value of the constant k is to be stated.

    c Verify that $x=2$ is a root of the equation $(2x+5)^4-\left(2x-5\right)^4 = 3680x-800$ and find the other possible values of x.

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[Question part reference style adapted]

12. The expansion of $(2x+ay)^n$ contains the term $20x^3y^2$.

    a Write down the value of n.

    b Find the value of a where a is positive.

    c Find the first four terms in ascending powers of y.

    d Hence or otherwise, find 20.05^n correct to the nearest hundred. You do not need to justify the accuracy of your approximation.

13. Find the coefficient of x^2 in the expansion of $\left(2x+\dfrac{1}{\sqrt{x}}\right)^5$.

14. $(1+ax)^n = 1+10x+40x^2+\ldots$

    Find the values of a and n.

15. **a** Sketch the graph of $y=(x+2)^3$.

    b Find the binomial expansion of $(x+2)^3$.

    c Find the exact value of 2.01^3.

    d Solve the equation $x^3+6x^2+12x+16=0$.

16. In the binomial expansion of $(k+ax)^4$ the coefficient of x^2 is 24.

    a Given that a and k are both positive, show that $ak=2$.

    b Given also that the coefficient of x in the expansion is 128, find the values of a and k.

    c Hence find the coefficient of x^3 in the expansion.

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[Question part reference style adapted]

FOCUS ON … PROOF 1

From general to specific

In Chapter 4, Key point 4.2 states the factor theorem. One part of this theorem says:

> **Given that $f(x)$ is a polynomial, and that $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.**

This is actually a specific case of a more general result, called the remainder theorem:

> **Let $f(x)$ be a polynomial and suppose you can write $f(x) \equiv (x - a)q(x) + r$ for some polynomial $q(x)$. Then $r = f(a)$.**

Fast forward

A The remainder theorem can be interpreted as saying that, when $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$. You will study the remainder theorem in more detail in Student Book 2.

PROOF 5

Proof of the remainder theorem:

$f(x) \equiv (x - a)q(x) + r$	Start with the given expression.
$\Rightarrow f(a) = (a - a)q(a) + r$	You can substitute any value of x in the identity. $x = a$ is useful because it makes one of the terms equal to zero.
$f(a) = r$	

Questions

1. Where in the proof did you use the fact that $q(x)$ is a polynomial? Does $f(x)$ have to be a polynomial?
2. Adapt the given proof to find the remainder when $f(x)$ is divided by $(bx - a)$.
3. Use the remainder theorem to prove the factor theorem.
4. The full statement of the factor theorem is:

   $(x - a)$ is a factor of a polynomial $f(x)$ **if and only if** $f(a) = 0$.

   This statement comprises of two implications:

   - $f(a) = 0 \Rightarrow (x - a)$ is a factor of $f(x)$
   - $(x - a)$ is a factor of $f(x) \Rightarrow f(a) = 0$

   a Prove the second implication: If $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$.

   b Did you need to use the remainder theorem in your proof?

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FOCUS ON … PROBLEM SOLVING 1

Alternative approaches

When faced with an unfamiliar problem, you might be happy if you could just get the solution, by whatever method. However, once you have solved the problem, it is worth thinking about whether there is an alternative strategy. This helps confirm that your solution is correct but, more importantly, it gives you an opportunity to reflect on which approach is best for what type of question.

In Worked example 6.12 in Chapter 6, Section 5, you solved the following problem:

> **A circle has centre $(3, 2)$. Find the radius of the circle so that the circle is tangent to the line with equation $x + 5y = 20$.**

Strategy 1

This is the method used in Worked example 6.12:

- Write the equations of the circle with unknown radius:
  $(x-3)^2 + (y-2)^2 = r^2$
- The equation for the intersection of this circle with the line rearranges to:
  $26y^2 - 174 + (293 - r^2) = 0$
- If the circle is tangent to the line this has only one solution, so the discriminant is zero:
  $174^2 - 4(26)(293 - r^2) = 0$
- This gives the solution $r = 1.37$.

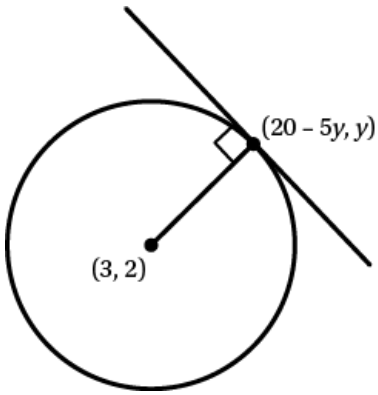
Strategy 2

This uses the fact that the tangent is perpendicular to the radius at the point of contact. You don't know the coordinates of the point where the line touches the circle, but you know it lies on the line $x + 5y = 20$.

This means that for every point on the line, $x = 20 - 5y$, so you can write the unknown coordinates as $(20 - 5y, y)$.

The line connecting this unknown point to the centre $(3, 2)$ needs to be perpendicular to $x + 5y = 20$.

- Write down the gradient of the line $x + 5y = 20$.
- Write down the gradient of a perpendicular line.
- Hence write an equation for the gradient of the line connecting $(3, 2)$ to $(20 - 5y, y)$.



You should find that the coordinates of the point of contact are $\left(\frac{85}{26}, \frac{87}{26}\right)$.

Now you can find the radius, which is the distance of this point from the centre.

Strategy 3

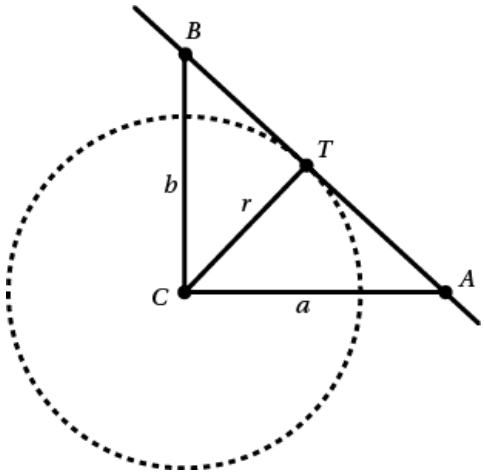
This is based on the same fact as strategy 2, but you find the coordinates of the point of contact by intersecting the line $x + 5y = 20$ with the line perpendicular to it and passing through the centre $(3, 2)$. See if you can carry out this strategy for yourself.

Rewind

This is the same method used to find the shortest distance from a point to a line in question 10 in Exercise 6C.

Strategy 4

This also uses the fact that the radius is perpendicular to the tangent, but you will find the length of the radius directly, without finding the coordinates of the point of contact first. Instead, you will create a right-angled triangle and find its area in two different ways.



Draw lines parallel to the coordinate axes to form a right-angled triangle ACB. You can calculate the area of this triangle in two different ways:

$$\text{Area} = \frac{1}{2}ab = \frac{1}{2}r\,(AB)$$

Points A and B are on the line $x + 5y = 20$ so you can find their coordinates:

- A has the same y-coordinate as $C(y = 2)$. Show that $a = 7$.
- B has the same x-coordinate as $C(x = 3)$. Show that $b = \frac{7}{5}$.
- Show that the length of AB is $\sqrt{\frac{1274}{25}}$.

So the area equation gives:

$$\frac{1}{2}(7)\left(\frac{7}{5}\right) = \frac{1}{2}r\sqrt{\frac{1274}{25}}$$
$$\Rightarrow r = 1.37$$

Questions

1 This is from question 7 from Exercise 6D:

A circle with centre $(3, -5)$ and radius 7 crosses the x-axis at points P and Q. Find the exact distance PQ.

- **a** Solve the problem by writing the equation of the circle and finding the coordinates of points P and Q.
- **b** Draw a diagram and label some lengths. Hence use a geometrical method to solve the problem.

2 How many different ways can you find to solve this problem?

Find the shortest distance from the origin to the line with equation $5x + 12y = 60$

3 This is question 10 from Mixed practice 6:

Find the exact values of k for which the line $y = kx + 3$ is tangent to the circle with centre $(6, 3)$ and radius 2.

- **a** Solve the problem by finding a quadratic equation for intersections and using the discriminant.
- **b** Draw a diagram and find a different way to solve the problem.
- **c** Would the second method work if the equation of the tangent was $y = kx + 2$ instead of $y = kx + 3$?

4 Circle c_1 has centre at the origin and radius 1. Circle c_2 has centre $(4, 3)$ and radius r. Find the value of r so that the two circles touch.

Tip

Remember how the gradient of a line is related to the angle it makes with the horizontal.

FOCUS ON … MODELLING 1

Using data to modify the model

In Chapter 8 you learnt that you can use an exponential function to model population growth. This basic model is based on the following assumption:

The population growth is proportional to the size of the population.

Fast forward

In Chapter 13, you will learn that this statement can be written as an equation: $\frac{\mathrm{d}N}{\mathrm{d}t} = kN$.

Questions

The simple population model leads to the equation $N = Ae^{kt}$ where N is the size of the population at time t and $k > 0$ is a constant.

1

a Plot the graph of N against t for various values of A and k.

b What do the constants A and k represent, in the context of this model?

c Suppose you observe the size of the population at two specific times. For example, $N = 35$ when $t = 5$ and $N = 152$ when $t = 11$. Use technology to find approximate values of A and k.

d Can you use algebra to find the values of A and k?

e At a later time, you acquire a third observation: $N = 340$ when $t = 18$. What does this tell you about your model?

In a modified model, the population is assumed to have a maximum capacity (for example, limited by the amount of food or space available). The rate of growth is proportional both to the size of the population and to the remaining space.

This is called a **logistic** model, and leads to the equation:

$$N = \frac{ACe^{kt}}{C - A + Ae^{kt}}$$

2

a Which of the parameters $(\mathrm{A}, \mathrm{C} \text{ or } \mathrm{k})$ represents the initial population size?

b Investigate this equation for various values of the parameters. Which parameter represents the maximum capacity?

c Use technology to find approximate values of the parameters which fit with the three observed data values: $(5,\ 35)$, $(11,\ 152)$ and $(18,\ 340)$.

Now consider the population model $N = Ae^{-kt}$ where k is a positive constant.

3

a What is the effect of the negative sign in the equation? How can you interpret this in the context of this problem?

b What does this model predict will happen to the population in the long term?

To counteract population decline, individuals can be added to the population. (For example, this could model controlled immigration in a country where the death rate is larger than the birth rate.)

In a simple immigration model, new individuals are added at a constant rate D.

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This model leads to the following equation:

$$N = \frac{D}{k} - \frac{1}{k}(D - kA)\,\mathrm{e}^{-kt}$$

4 **a** Which parameter represents the initial population?

b According to this model, what happens to the population in the long term?

c Explain what happens, and why, in the case when $D = kA$.

Tip

You should find that the behaviour depends on whether $D > kA$ or $D < kA$.

5 A population of a small country was $7\,430\,000$. Ten years later, the population has fallen to $7\,026\,000$.

a For a simple exponential model, $N = A\mathrm{e}^{-kt}$, find the value of the constants A and k.

b By what factor does the population decrease each year? Given that the annual birth rate is 4.7 babies per 1000 people, estimate the annual death rate.

c To counteract the population decline, the government proposes a controlled immigration programme (like the one you used in question 4). They want to aim for a stable long-term population of about $7\,200\,000$.

i What annual immigration target should they set?

ii What assumptions about the immigrant population need to be made for the model prediction (a long-term population of $7\,200\,000$) to be valid?

Explore

Find out about other modifications to the population growth models, for example those incorporating seasonal variation.

CROSS-TOPIC REVIEW EXERCISE 1

1 **Any calculations in your written solution to this question must show detailed reasoning.**

a Find the exact value of $3\log 5 - \log 20 + \log 16$.

b Given that $x = \ln 2$, $y = \ln 3$ and $z = \ln 5$, express $\ln\left(\frac{45}{4}\right)$ in terms of x, y and z.

c If ln $K = 2 - \ln$c, find and simplify an expression for K in terms of c.

2 **a** **i** Write $x^2 - 6x + 5$ in the form $(x-h)^2 - k$.

ii Describe a single transformation that transforms the graph of $y = x^2$ into the graph of $y = x^2 - 6x + 5$.

iii Sketch the graph of $y = x^2 - 6x + 5$.
Mark the coordinates of the axes intercepts and the minimum point.

b **i** Add the graph of $y = x - 1$ to your sketch.

ii Solve the equation $x^2 - 6x + 5 = x - 1$.

iii Using shading identify the region that satisfies $y \geqslant x^2 - 6x + 5$ and $y \leqslant x - 1$ (leave the area that satisfies the inequality unshaded).

3 **a** Show that $(x-3)$ is a factor of $\mathrm{p}(x) = 2x^3 - 5x^2 - 6x + 9$.

b Factorise $\mathrm{p}(x)$ completely.

c Hence sketch the graph of $y = \mathrm{p}(x)$.

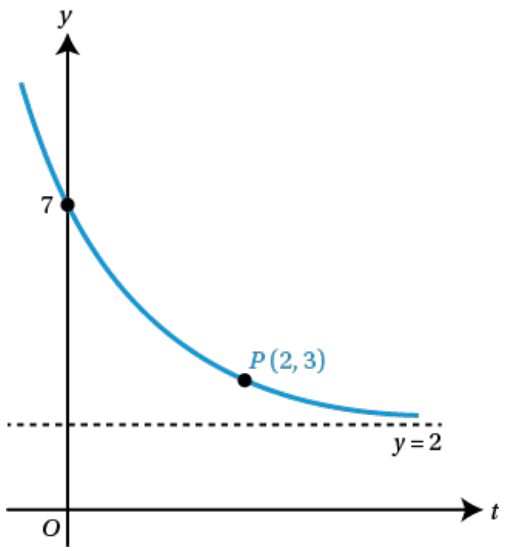
d Sketch the graph of $y = 2(x+2)^3 - 5(x+2)^2 - 6(x+2) + 9$.

4 The circle with centre $(12, 9)$ and radius $\sqrt{145}$ intersects the x-axis at points A and B and the y-axis at points C and D. Find the area of the quadrilateral ABDC.

5 **Any calculations in your written solution to this question must show detailed reasoning.**

Find the exact solutions of the equation $3e^{2x} - 7e^x + 2 = 0$.

6 The diagram shows the graph with equation $y = C + Ae^{-kt}$.
The graph passes through the point $P(2, 3)$.



a Write down the value of C and the value of A.

b Find the exact value of k.

7 **a** Expand and simplify $(1+x)^4 + (1-x)^4$.

b Hence show that $\left(\sqrt{2}+1\right)^4 + \left(\sqrt{2}-1\right)^4$ is an integer and find its value.

8 Number x satisfies the equation $x^2 = 3x - 1$.

a Show that $x + \frac{1}{x} = 3$.

i Expand $\left(x + \frac{1}{x}\right)^2$ and $\left(x + \frac{1}{x}\right)^3$.

ii Hence find the values of $x^2 + \frac{1}{x^2}$ and $x^3 + \frac{1}{x^3}$.

9 The graph of $y = \ln x$ can be transformed into the graph of $y = \ln kx$ using either a horizontal stretch or a vertical translation.

a State the stretch factor of the horizontal stretch.

b Find the vertical translation vector.

10 Given that $\binom{n}{2} = k$ express n in terms of k.

11 The population of a certain species in a particular locality is doubling every 9 years. The number of plants now is 42. The number of plants is treated as a continuous variable and is denoted by N. The number of years from now is denoted by t.

a Two equivalent expressions giving N in terms of t are $N = A \times 2^{kt}$ and $N = Ae^{mt}$
Determine the value of each of the constants A, k and m.

b Find the value of t for which $N = 100$, giving your answer to 3 s.f.

c Find the rate at which the number of plants will be increasing at a time 35 years from now.

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[Question and question part reference style adapted]

12 Show that the graph of $y = x^2 - (m+3)x + (m+1)$ crosses the x-axis for all values of m.

13 Given that $y = e^x + e^{-x}$, express x in terms of y.

a **i** Given that x is a real number, find the set of possible values of y.

ii For a fixed y from this set, show that the sum of all the possible values of x is zero.

14 **a** Sketch the graph of $y = 6b^x$ where b is a constant and $b > 1$. Label the coordinates of any points of intersection with the axes.

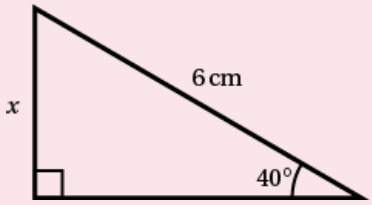
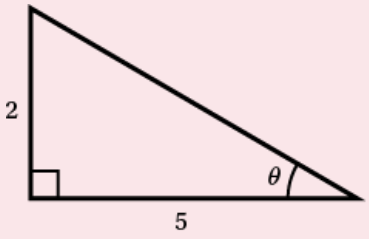
b The point P on the curve $y = 6b^x$ has its y-coordinate equal to $54b^2$. Show that the x-coordinate
of P can be written as $2(1 + \log_b 3)$.

10 Trigonometric functions and equations

In this chapter you will learn:

- the definitions of the sine, cosine and tangent functions, their basic properties and their graphs
- how to solve equations with trigonometric functions
- about relationships (called identities) between different trigonometric functions
- how to use identities to solve more complicated equations.

Before you start...

GCSE	You should know how to use trigonometry in right-angled triangles to find unknown lengths.	1 Find the value of x in the diagram. 
GCSE	You should know how to use trigonometry in right-angled triangles to find unknown angles.	2 Find the value of θ in the diagram. 
GCSE	You should know how to use Pythagoras' theorem in a right-angled triangle.	3 The two shorter sides of a right-angled triangle are 5 and 12. Find the length of the hypotenuse.
Chapter 3	You should know how to solve quadratic equations by using the formula or factorising.	4 Solve $x^2 - 2x + 1 = 0$.
Chapter 3	You should know how to solve equations that are quadratic in a function of the unknown.	5 Solve the equation $(\ln x)^2 - 2\ln x = 3$.

What are trigonometric functions?

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There are many real-life situations in which something repeats at regular Intervals. For example, the height of a fairground ride, the tides of the sea or the vibration of a guitar string. All of these can be modelled using the **trigonometric functions** sine, cosine and tangent.

You first met sine, cosine and tangent when working with angles in a triangle. In this chapter, you will find out how they can be used in a variety of other contexts.

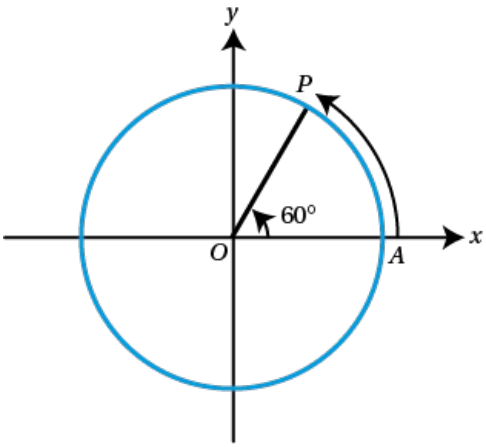
Section 1: Definitions and graphs of the sine and cosine functions

You've already used trigonometric functions in right-angled triangles but in these triangles no angle can exceed 90°. If we want to use trigonometric functions for other purposes it will be useful to have a more general definition.

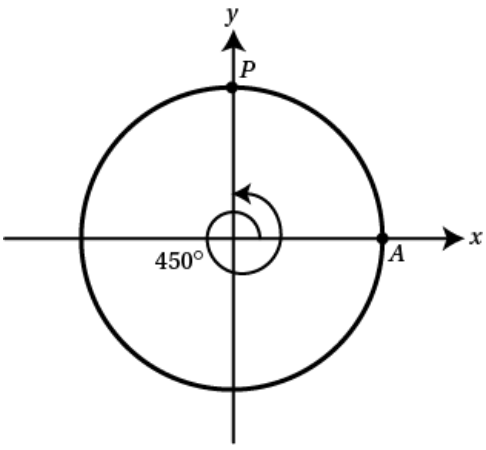
Gateway to A Level

For a reminder of using trigonometry in right-angled triangles see Gateway to A Level section P.

To do this, consider a circle of radius 1 centred at the origin (the 'unit circle'). As a point P moves anti-clockwise around the circumference, an angle is formed between OP and the horizontal, for example angle AOP in the diagram is 60°.



Any angle can be defined in this way. If the angle is greater than 360° the point P rotates more than a full turn, for example in the diagram point P has rotated one and a quarter turns and represents 450°.



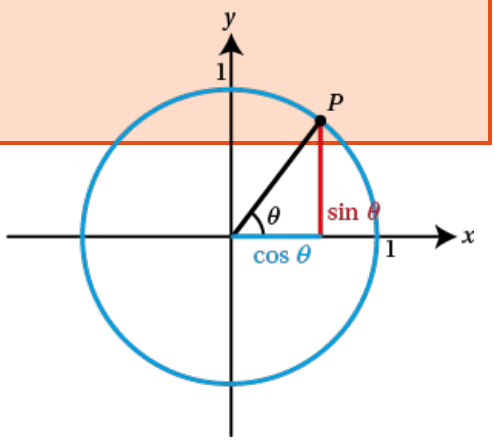
If the angle is negative then point P rotates clockwise.

For an angle θ, the sine and cosine functions are then defined in terms of the distance of the point P to the axes.

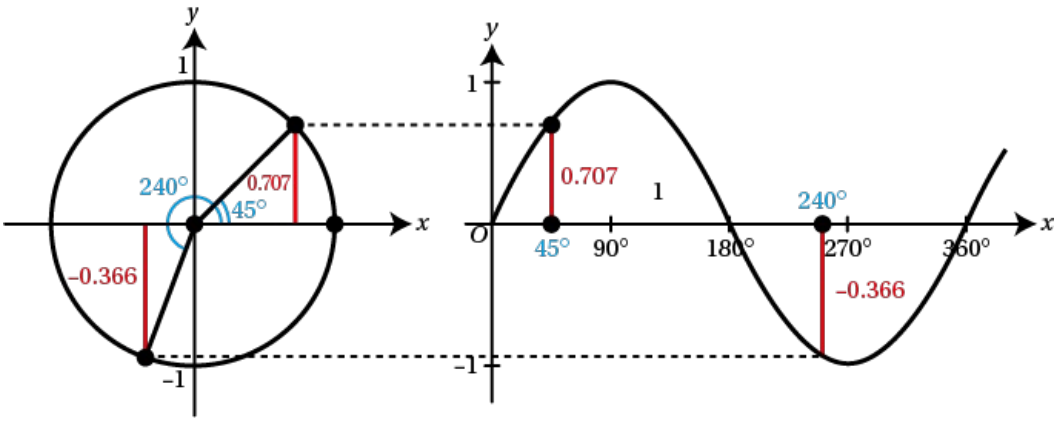
Key point 10.1

- $\sin\theta$ is the distance of the point P above the horizontal axis (its y-coordinate).
- $\cos\theta$ is the distance of the point P to the right of the vertical axis (its x-coordinate).

With this definition we can draw the graphs of $y = \sin x$ and $y = \cos x$ for any value of x.

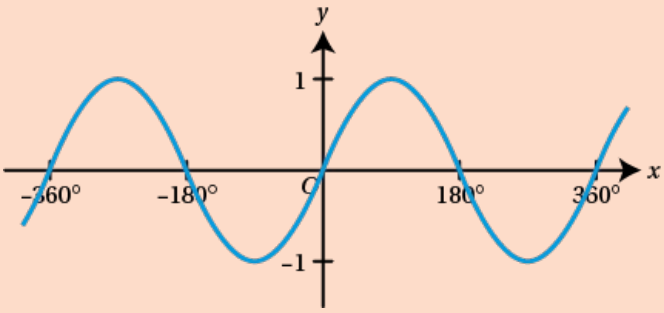


The graph of $y = \sin x$



Key point 10.2

This is the graph of $y = \sin x$.



The graph repeats after 360°. The sine function is **periodic** with **period** 360°, which means that $\sin(x + 360^\circ) = \sin(x)$ for all x.

You can also see that the minimum possible value of $\sin x$ is -1 and the maximum value is 1. The sine function has **amplitude** 1. Amplitude is the maximum 'height' of a periodic function, i.e. half of the distance from the minimum value of y to the maximum value of y.

Tip

For a periodic function with period p:

$f(x + p) = f(x)$ for all x.

You might think that considering negative angles or angles beyond 360° would not have any practical use. However, if you think about angles as measuring the amount of rotation you can see that they can be given a concrete meaning. By convention, positive angles represent anti-clockwise rotation and negative angles represent clockwise rotation.

You can use the symmetries of the sine graph to see how values of $\sin x$ for various angles are related to each other.

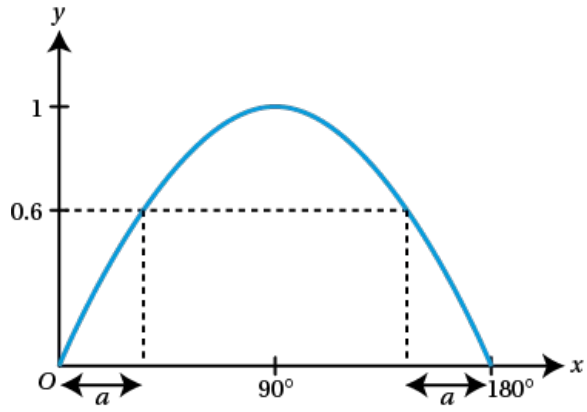
WORKED EXAMPLE 10.1

Given that $\sin a = 0.6$ find the value of:

a $\sin(180^\circ - a)$

b $\sin(180^\circ + a)$.

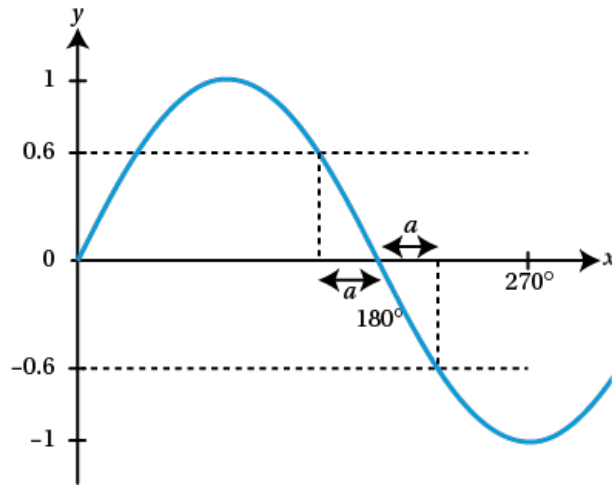
a



The graph has a vertical line of symmetry at $x = 90^\circ$, so $\sin(180^\circ - a)$ is the same as $\sin a$.

$\sin(180^\circ - a) = 0.6$

b



The part of the graph below the x-axis is the same shape as the part above. So $\sin(180^\circ + a)$ is the negative of $\sin a$.

$\sin(180^\circ + a) = -0.6$

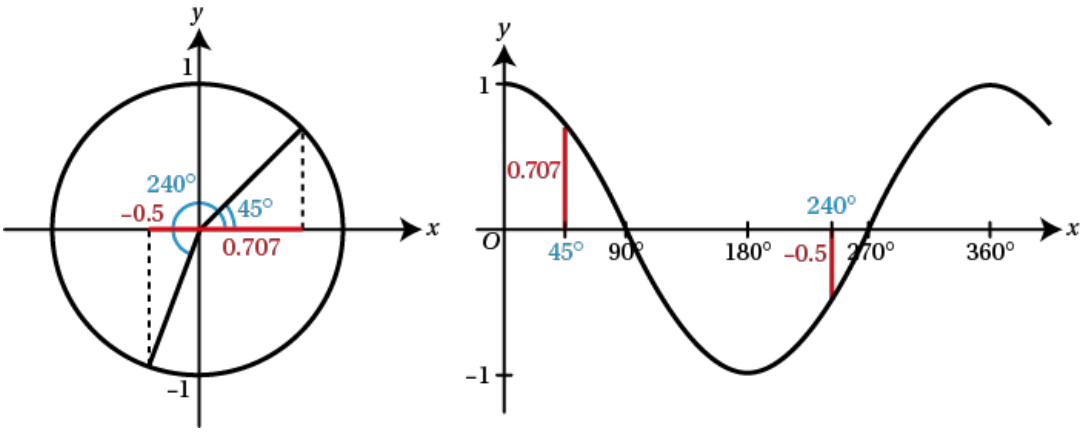
There are several other similar relationships that you can find from the graph. They will be useful when solving trigonometric equations.

Key point 10.3

- $\sin x \equiv \sin(180^\circ - x)$
- $\sin x \equiv \sin(x + 360^\circ)$
- $\sin(180^\circ + x) \equiv \sin(-x) \equiv -\sin x$

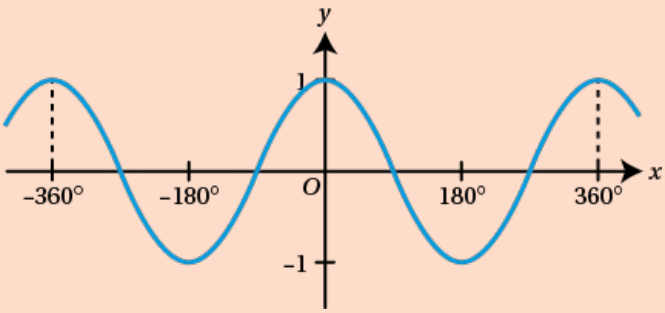
The graph of $y = \cos x$

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Key point 10.4

The graph of $y = \cos x$

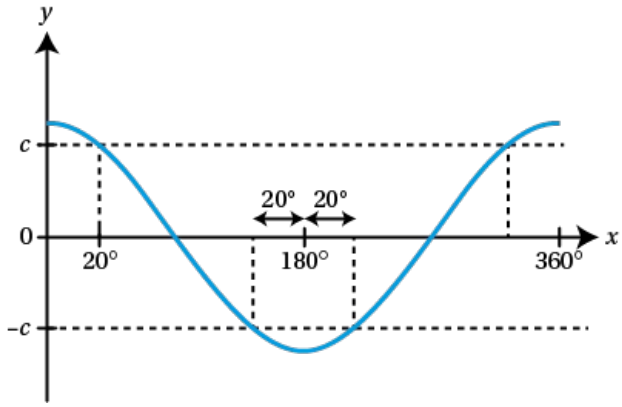


You can see that the cosine function is also periodic with period 360° and has amplitude 1.

You can use the symmetry of the graph to find relationships between values of $\cos x$ for different angles.

WORKED EXAMPLE 10.2

If $\cos 20^\circ = c$, find two values of x between 0° and 360° for which $\cos x = -c$.



$x_1 = 180^\circ - 20^\circ = 160^\circ$

$x_2 = 180^\circ + 20^\circ = 200^\circ$

The two values are 20° away from 180° on either side.

Key point 10.5

- $\cos x \equiv \cos(-x)$

- $\cos x \equiv \cos(x + 360^\circ)$
- $\cos(180^\circ - x) \equiv \cos(180^\circ + x) \equiv -\cos x$

The unit circle definition can be used to establish a connection between the sine and cosine functions.

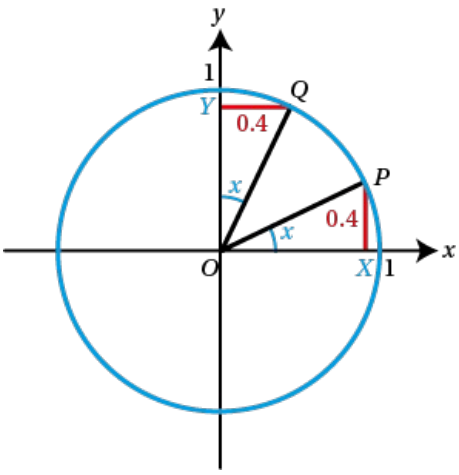
WORKED EXAMPLE 10.3

Given that $\sin x = 0.4$, find the value of

a $\cos(90^\circ - x)$

b $\cos(x + 90^\circ)$

a

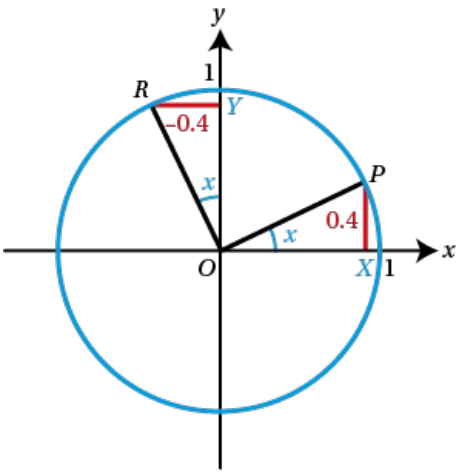


Let P be the point corresponding to x and Q the point corresponding to $90 - x$.

$\cos(90^\circ - x) = 0.4$

$\cos(90^\circ - x) = QY$

b



R is the point corresponding to $90 + x$.

$\cos(90^\circ + x) = -0.4$

$\cos(90 + x) = RY$, but this is in the negative direction on the x-axis.

Key point 10.6

- $\sin x \equiv \cos(90^\circ - x)$
- $\cos x \equiv \sin(90^\circ - x)$

Tip

These relationships can also be seen by noting that the graph of $y = \cos x$ is obtained from the

graph of $y = \sin x$ by translating it 90° to the left and vice versa.

WORKED EXAMPLE 10.4

Find the two values of x in the interval $0^\circ \leqslant x \leqslant 360^\circ$ for which $\sin x = \cos 20^\circ$.

$\cos\ (90^\circ - x) = \cos\ 20^\circ$	Use $\sin x = \cos\ (90^\circ - x)$ so that both sides are in terms of the same trigonometric function.
Also, $\cos\ (90^\circ - x) = \cos\ (-20^\circ)$	Since $\cos x = \cos(-x)$, $\cos 20^\circ = \cos(-20^\circ)$.
$\therefore 90^\circ - x = 20^\circ$ or -20°	Remove the cos function from both sides.
$x = 70^\circ$ or 110°	

EXERCISE 10A

1 Use your calculator to evaluate the following, giving your answers to 3 s.f.

a **i** $\sin 42^\circ$

ii $\cos 168^\circ$

b **i** $\sin(-50^\circ)$

ii $\cos(-227^\circ)$

2 Use graphs to find the value of:

a **i** $\sin 90^\circ$

ii $\sin 360^\circ$

b **i** $\cos 0^\circ$

ii $\cos(-180^\circ)$

c **i** $\sin(-90^\circ)$

ii $\cos 450^\circ$

3 Given that $\cos 40^\circ = 0.766$ (3 s.f.) find the value of:

a $\cos 400^\circ$

b $\cos 320^\circ$

c $\cos(-220)^\circ$

d $\cos 140^\circ$

4 Given that $\sin 130^\circ = 0.766$ (3 s.f.) find the value of:

a $\sin 490^\circ$

b $\sin 50^\circ$

c $\sin(-130)^\circ$

d $\sin 230^\circ$

5 **a** Sketch the graph of $y = \sin x$ for:

i $0^\circ \leqslant x \leqslant 180^\circ$

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ii $90^\circ \leqslant x \leqslant 360^\circ$

b Sketch the graph of $y = \cos x$ for:

i $-180^\circ \leqslant x \leqslant 180^\circ$

ii $0^\circ \leqslant x \leqslant 270^\circ$

6 Simplify $\sin(x + 360^\circ) + \sin(x + 540^\circ)$.

Worksheet

For more challenging questions on periodic functions, see Extension sheet 10.

7 Prove that $\cos(180^\circ + x) + \cos(180^\circ - x) = k \cos x$ where k is a constant to be found.

8 If $\cos x = a$, show that $\sin(x - 90^\circ) = ka$ where k is a constant to be found.

9 Simplify $\sin x + \sin\ (x + 90^\circ) + \sin\ (x + 180^\circ) + \sin\ (x + 270^\circ) + \sin\ (x + 360^\circ)$.

10 Find all values of x that satisfy $\cos x = \sin(\ln 3)$ for $-360^\circ < x < 360^\circ$, giving your answers in exact form.

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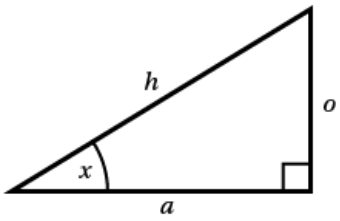
Section 2: Definition and graph of the tangent function

You can now define another trigonometric function, the **tangent** function. This is defined as the ratio between the sine and the cosine functions.

Key point 10.7

$$\tan x \equiv \frac{\sin x}{\cos x}$$

This is consistent with your previous knowledge of the tangent function. If $\sin x = \frac{o}{h}$ and $\cos x = \frac{a}{h}$ then $\frac{\sin x}{\cos x} = \frac{\frac{o}{h}}{\frac{a}{h}} = \frac{o}{a}$, which is how you previously defined $\tan x$.

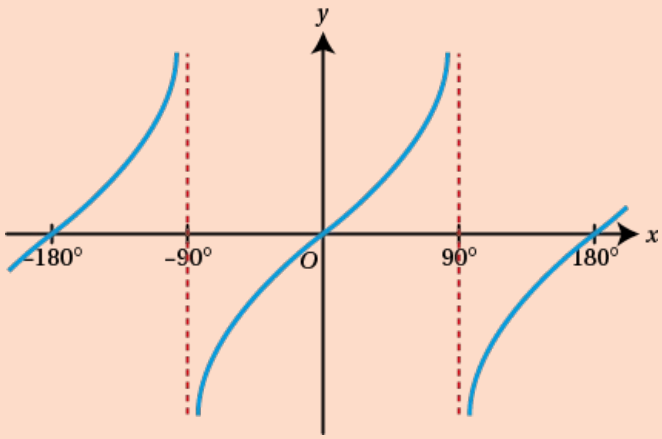


You may notice that there is a problem with this definition. When $\cos x$ is zero, you cannot divide by it. Thus the tangent function is undefined for values of x where $\cos x = 0$ (which is when $x = 90^\circ$, 270° and so on).

You can also see that $\tan x = 0$ whenever $\sin x = 0$, which is when $x = 0^\circ$, 180°, 360°, etc.

Key point 10.8

This is the graph of $y = \tan x$.



The tangent function is periodic with period 180°:

$\tan x = \tan(x + 180^\circ) = \tan(x + 360^\circ) = \ldots$

It is undefined for $x = 90^\circ$, 270°, etc. The lines $x = 90^\circ$, 270°, etc. are vertical **asymptotes**.

Rewind

Asymptotes were discussed in Chapter 5, Section 4.

EXERCISE 10B

1 Sketch the graph of $y = \tan x$ for:

a $0^\circ \leqslant x \leqslant 360^\circ$

b $-90^\circ \leqslant x \leqslant 270^\circ$

2 Use your calculator to evaluate the following, giving your answers to 3 s.f.

a **i** $\tan 32^\circ$

ii $\tan 168^\circ$

b **i** $\tan(-540^\circ)$

ii $\tan(-128^\circ)$

3 Given that $\tan 20^\circ = 0.364$ (3 s.f.) use the tangent graph to find the following:

a $\tan 380^\circ$

b $\tan(-160^\circ)$

c $\tan 160^\circ$

d $\tan(-200^\circ)$

4 Use the properties of sine and cosine to express the following in terms of $\tan x$:

a $\tan(90^\circ - x)$

b $\tan(x + 90^\circ)$

c $\tan(-270^\circ - x)$

d $\tan(540^\circ - x)$

5 Simplify $\tan(x + 360^\circ) - \tan(180^\circ - x)$.

6 Find the two exact values of x in the interval $-180 \leqslant x \leqslant 180$ for which $\tan x^\circ = \tan \mathrm{e}^\circ$.

7 Let $\tan x = t$.

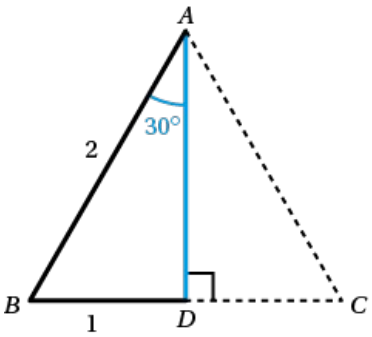
Express $\dfrac{\tan\ (360^\circ - x)}{\tan\ (270^\circ - x)}$ in terms of t.

Section 3: Exact values of trigonometric functions

Although values of trigonometric functions are generally difficult to find without a calculator, there are a few special numbers for which exact values are easily found. The method relies on properties of special right-angled triangles.

WORKED EXAMPLE 10.5

Find the exact values of $\sin 30^\circ$, $\cos 30^\circ$ and $\tan 30^\circ$.



If a right-angled triangle has a 30° angle, then the third angle is 60°, so this is half of an equilateral triangle.

You can choose any length for the side of the equilateral triangle. Let $AB = 2$; then $BD = 1$.

$AD^2 = 2^2 - 1^2$
$= 3$
$\therefore AD = \sqrt{3}$

To find AD use Pythagoras' theorem.

$\sin 30^\circ = \frac{o}{h} = \frac{1}{2}$
$\cos 30^\circ = \frac{a}{h} = \frac{\sqrt{3}}{2}$
$\tan 30^\circ = \frac{o}{a} = \frac{1}{\sqrt{3}}$

You can now use the definitions of $\sin$, $\cos$ and $\tan$.

You can also find exact values for certain angles greater than 90° by considering symmetries of the trigonometric graphs.

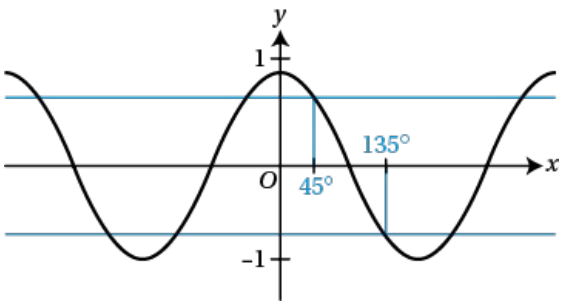
WORKED EXAMPLE 10.6

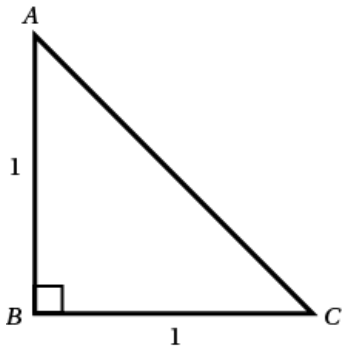
Show that $\cos\ 135^\circ = -\frac{\sqrt{2}}{2}$.

From the $\cos$ graph:

$\cos 135^\circ = -\cos 45^\circ$

You cannot have 135° in a right-angled triangle, so look at the $\cos$ graph.





Now use a right-angled triangle with one angle 45°. This triangle is isosceles.

Let $AB = 1$

Then $BC = AB = 1$

$AC = \sqrt{1^2 + 1^2}$
$= \sqrt{2}$

Use Pythagoras' theorem to find the hypotenuse.

So: $\cos\ 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

You can now find $\cos 45^\circ$, rationalising the denominator.

And so $\cos\ 135^\circ = -\frac{\sqrt{2}}{2}$

And hence $\cos 135^\circ$.

The results for other special values are summarised in Key point 10.9.

Key point 10.9

	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

Tip

Notice the pattern in the values of $\sin$ and $\cos$:

$0,\ \frac{1}{2},\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{3}}{2},\ 1\ \left(=\frac{\sqrt{4}}{2}\right)$.

You should understand how these exact values are derived, as illustrated in the previous examples, and need to be able to use them in calculations, but you can check the values on your calculator.

EXERCISE 10C

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Do not use your calculator in this exercise.

1 Find the exact value of:

a $\cos 45^\circ$

b $\sin 135^\circ$

c $\cos 225$

d $\tan 225$

e $\sin 210^\circ$

f $\cos 210^\circ$

g $\tan 210^\circ$

h $\tan 330^\circ$

2 Evaluate the following, simplifying as far as possible.

a $1 - \sin^2 30^\circ$

b $1 + \tan^2 30^\circ$

c $\sin 45^\circ + \sin 60^\circ$

d $\cos 60 - \cos 30^\circ$

3 Show that:

a $\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ = \sin 90^\circ$

b $\sin^2 45^\circ + \cos^2 45^\circ = 1$

c $\cos^2 30^\circ - \sin^2 30^\circ = \cos 60^\circ$

d $(1 + \tan 60^\circ)^2 = 4 + 2\sqrt{3}$

Tip

The notation $\sin^2 x$ means $(\sin x)^2$. This is not the same as $\sin(x^2)$.

4 **a** Show that $\sin\ 60^\circ + \tan\ 30^\circ = \dfrac{5\sqrt{3}}{6}$.

b Given that $1 - 2\sin^2 60^\circ = \cos A$, find one possible value of A.

5 **a** Find the exact value of $\cos 150^\circ - \cos 120^\circ$.

b Paul thinks that $\cos A - \cos B = \cos(A - B)$.

Use a counter example to disprove this statement.

6 Show that $\dfrac{\sin 30^\circ + \sin 45^\circ}{\cos 30^\circ + \cos 45^\circ} = \sqrt{6} + \sqrt{3} - \sqrt{a} + b$ where a and b are constants to be found.

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Section 4: Trigonometric identities

You have already seen one example of an identity in this chapter: $\frac{\sin x}{\cos x} = \tan x$. There are many other identities involving trigonometric functions. The most important of these is:

Key point 10.10

$\sin^2 x + \cos^2 x = 1$ for all x.

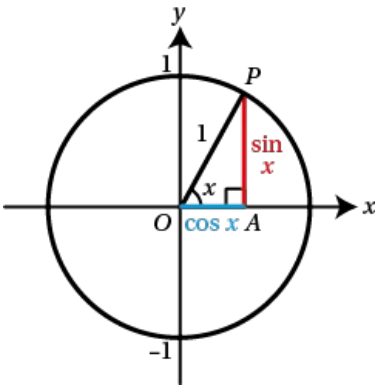
Because of the way it is derived, $\sin^2 x + \cos^2 x = 1$ is sometimes called the Pythagorean identity.

Tip

Many books write $\frac{\sin x}{\cos x} \equiv \tan x$ to emphasise that the expression is an identity (true for **all** values of x) rather than an equation (only true for some values of x which need to be found). However, mathematicians often use the equals sign in identities, unless there could be confusion.

PROOF 6

This result follows from considering the definitions of $\sin x$ and $\cos x$ on the unit circle.



If the point P represents the angle x, then $AP = \sin x$ and $OA = \cos x$.

The triangle OAP is right angled, with hypotenuse 1, so by Pythagoras' theorem $\sin^2 x + \cos^2 x = 1$.

On the previous diagram both $\sin x$ and $\cos x$ are positive. However, since they are being squared, the proof also applies when they have negative values.

You have already seen examples where you used the values of $\sin x$ and $\cos x$ to find the value of $\tan x$. Using the Pythagorean identity, you only need to know the value of one of the functions to find the possible values of the other two.

WORKED EXAMPLE 10.7

Given that $\sin x = \frac{1}{3}$ find the possible values of:

a $\cos x$

b $\tan x$.

a $\sin^2 x + \cos^2 x = 1$

Use the identity $\sin^2 x + \cos^2 x = 1$ to relate cos to sin.

$$\left(\frac{1}{3}\right)^2 + \cos^2 x = 1$$

$$\cos^2 x = 1 - \frac{1}{9}$$

$$\cos^2 x = \frac{8}{9}$$

$$\therefore \cos x = \pm\sqrt{\frac{8}{9}}$$

$$= \pm\frac{2\sqrt{2}}{3}$$

Remember $\pm$ when taking the square root.

b $\tan x = \dfrac{\sin x}{\cos x}$

Use the identity $\tan x = \dfrac{\sin x}{\cos x}$ to relate tan to sin and cos.

$$= \frac{\frac{1}{3}}{\pm\frac{2\sqrt{2}}{3}}$$

Substitute in the values of sin and cos.

$$= \pm\frac{\sqrt{2}}{4}$$

Notice that for a given value of $\sin x$, there are two possible values of $\cos x$ (positive and negative).

You can specify one of the two possible values by restricting x to a particular quadrant.

WORKED EXAMPLE 10.8

If $\tan x = -2$ and $90^\circ < x < 180^\circ$ find the value of $\cos x$.

$$\tan x = -2$$

$$\frac{\sin x}{\cos x} = -2$$

$$\sin x = -2\cos x$$

To introduce cos you need to use $\tan x = \dfrac{\sin x}{\cos x}$.

$$\sin^2 x = 4\cos^2 x$$

$$1 - \cos^2 x = 4\cos^2 x$$

To remove sin so that you only have cos, use $\sin^2 x + \cos^2 x = 1$.

$$5\cos^2 x = 1$$

$$\cos^2 x = \frac{1}{5}$$

$$\cos x = \pm\frac{1}{\sqrt{5}}$$

$$90 < 9 < x < 180^\circ \Rightarrow \cos x < 0$$

Since x is between 90° and 180°, $\cos x$ is negative. (You can see this from the graph.)

$$\therefore \cos x = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

You can use known identities to derive new ones. For example, multiplying both sides of the Pythagorean identity by 3 gives:

$$3\sin^2 x + 3\cos^2 x = 3$$

You can also rearrange an identity, just as you would with an equation. For example, you can rearrange the Pythagorean identity to get:

$$\cos^2 x = 1 - \sin^2 x$$

You can also substitute one identity into another. For example, you can write $\tan x = \dfrac{\sin x}{\cos x}$ as $\cos x = \dfrac{\sin x}{\tan x}$ and substitute it into the identity to get:

$$\frac{\sin^2 x}{\tan^2 x} = 1 - \sin^2 x$$

Finally, it is important to realise that in each of these identities the variable x can be replaced by any other variable or expression, as long as each occurrence of x is replaced by the same thing. This means that, for example:

$$\sin^2(2x+1) + \cos^2(2x+1) = 1 \text{ and } \frac{\sin 2x}{\cos 2x} = \tan 2x.$$

You can combine these ideas with algebraic manipulations, such as simplifying algebraic fractions or the difference of two squares, to prove even more identities.

Rewind

Proving new trigonometric identities is an example of proof by deduction, which you met in Chapter 1, Section 4, where you start from given assumptions and use known facts to reach a conclusion.

WORKED EXAMPLE 10.9

Prove the identity $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$.

$\text{LHS} = \sin^4 x - \cos^4 x$	You need to prove that the two sides are equal. One way to do this is to start with the left-hand side (LHS) and transform it until you make it look like the right-hand side (RHS).
$= (\sin^2 x - \cos^2 x)(\cos^2 x + \sin^2 x)$	In this case, you can start by using the difference of two squares.
$= (\sin^2 x - \cos^2 x)(1)$	The expression in the second bracket always equals 1.
$= \sin^2 x - (1 - \sin^2 x)$	The expression you want to get on the RHS contains $\sin x$ only, so use $\sin^2 x + \cos^2 x = 1$ to replace $\cos^2 x$ by $1 - \sin^2 x$.
$= 2\sin^2 x - 1$ $= \text{RHS}$	The final expression equals the required RHS, so you have completed the proof.

$\therefore$ proved

Trigonometric identities often involve fractions. When simplifying expressions with fractions, you should always look for common denominators.

WORKED EXAMPLE 10.10

Prove the identity $\sin x \cos x = \dfrac{\tan x}{1 + \tan^2 x}$.

$$\text{RHS} = \frac{\tan x}{1 + \tan^2 x}$$

$$= \frac{\frac{\sin x}{\cos x}}{1 + \frac{\sin^2 x}{\cos^2 x}}$$

The RHS looks more complicated, so start there and try to simplify. Since the required answer on the LHS involves $\sin$ and $\cos$, you should start by writing $\tan x$ as $\frac{\sin x}{\cos x}$.

$$= \frac{\frac{\sin x}{\cos x} \times \cos^2 x}{\cos^2 x + \frac{\sin^2 x}{\cos^2 x} \times \cos^2 x}$$

To simplify this 'double fraction', multiply top and bottom by the common denominator, which is $\cos^2 x$.

$$= \frac{\sin x \cos x}{\cos^2 x + \sin^2 x}$$

$$= \sin x \cos x$$

The expression in the denominator always equals 1.

$$= \text{LHS}$$

$\therefore$ proved

EXERCISE 10D

Do not use your calculator in this exercise, and give all your answers in surd form.

1 **a** Find the exact values of $\cos x$ and $\tan x$ given that:

i $\sin x = \frac{1}{3}$ and $0^\circ < x < 90^\circ$

ii $\sin x = \frac{4}{5}$ and $0^\circ < x < 90^\circ$

b Find the exact values of $\sin \theta$ and $\tan \theta$ given that:

i $\cos = -\frac{1}{3}$ and $180^\circ < \theta < 270^\circ$

ii $\sin = -\frac{3}{4}$ and $180^\circ < \theta < 270^\circ$

c Find the exact value of $\cos x$ if:

i $\sin x = \frac{1}{5}$ and $90^\circ < x < 180^\circ$

ii $\sin x = -\frac{1}{2}$ and $270^\circ < x < 360^\circ$

d Find the exact value of $\tan x$ if:

i $\cos x = \frac{3}{5}$ and $0^\circ < x < 180^\circ$

ii $\cos x = -1$ and $90^\circ < x < 270^\circ$

e **i** Find the possible values of $\cos x$ if $\tan x = \frac{2}{3}$.

ii Find the possible values of $\sin x$ if $\tan x = -\frac{1}{2}$.

2 Find the exact value of:

a $3\sin^2 x + 3\cos^2 x$

b $\sin^2 5x + \cos^2 5x$

c $-2\cos^2 2x - 2\sin^2 2x$

d $\frac{3}{2\sin^2 4x} - \frac{3}{2\tan^2 4x}$

3 **a** Express $3\sin^2 x + 4\cos^2 x$ in terms of $\sin x$ only.

b Express $\cos^2 x - \sin^2 x$ in terms of $\cos x$ only.

4 Prove the following identities:

a **i** $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$

ii $(2\sin x - \cos x)^2 + (\sin x + 2\cos x)^2 = 5$

b **i** $\sin\theta\tan\theta + \cos\theta = \dfrac{1}{\cos\theta}$

ii $\dfrac{\cos^2\theta}{\sin\theta} = \dfrac{1}{\sin\theta} - \sin\theta$

5 $\cos\theta = \dfrac{2}{3}$ and $270^\circ < \theta < 360^\circ$. Find the exact value of:

a $\sin\theta$

b $\tan\theta$

6 If $\tan\theta = 3$ find, in exact form, the possible values of $\sin\theta$.

7 If $s = \sin x$ and $90^\circ < x < 180^\circ$, express $\cos x$ in terms of s.

8 Express the following in terms of $\cos x$ only:

a $3 - 2\tan^2 x$

b $\dfrac{1}{1 + \tan^2 x}$

9 Simplify fully the expression $\left(\dfrac{1}{\sin x} - \dfrac{1}{\tan x}\right)\left(\dfrac{1}{\sin x} + \dfrac{1}{\tan x}\right)$.

10 Show that for all x, $2\tan^2 2x - \dfrac{2}{\cos^2 2x} = k$ stating the value of the constant k.

11 If $t = \tan x$ express the following in terms of t:

a $\cos^2 x$

b $\sin^2 x$

c $\cos^2 x - \sin^2 x$

d $\dfrac{2}{\sin^2 x} + 1$

12 Prove the following identities:

a $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\sin\theta\cos\theta}$

b $\dfrac{1}{\cos\theta} + \tan\theta = \dfrac{\cos\theta}{1 - \sin\theta}$

13 Prove the identity $\dfrac{1}{\cos\theta} - \cos\theta = \sin\theta\tan\theta$.

Section 5: Introducing trigonometric equations

In order to solve equations it will be important that you can undo trigonometric functions. If you were told that the sine of a value is 0.6 the original value is not easy to find. To do this you need to undo the sine function using the inverse function of sine, written as arcsin x or $\sin^{-1} x$.

The inverse cosine function is called arccos x or $\cos^{-1} x$. The inverse tangent function is called arctan x or $\tan^{-1} x$.

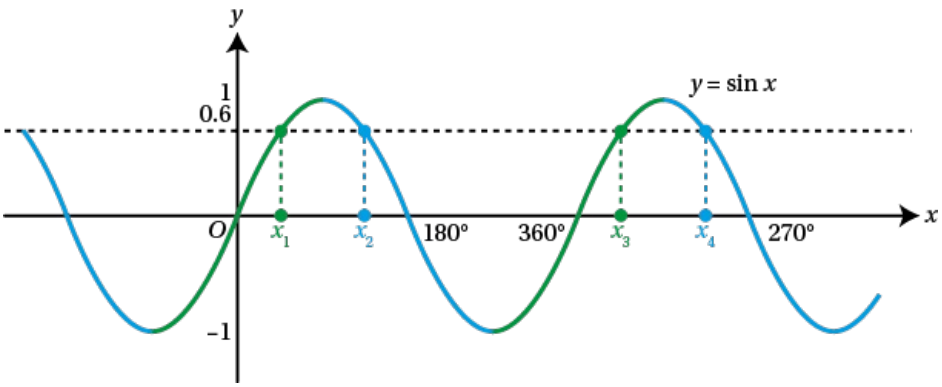
Rewind

You already know how to use $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ to find angles in a triangle.

Suppose you want to find the x values which satisfy $\sin x = 0.6$. By applying inverse sine to both sides of the equation you get

$$\begin{aligned} \sin x &= 0.6 \\ x &= \sin^{-1} 0.6 \\ &= 36.9^\circ \text{ (to 3s. f.)} \end{aligned}$$

The inverse sine function only gives you one solution. However, looking at the graph of $y = \sin x$ you can see that there are many values that satisfy this equation.



Tip

Calculators usually do not have a button labelled arcsin; use the $\boxed{\sin^{-1}}$ button (usually found as $\boxed{\text{SHIFT}} + \boxed{\sin}$) instead.

The solutions come in pairs – one solution in the green section of the graph and one in the blue section (for example, x_1 and x_2). The inverse sine function will always give you a solution in the green section that passes through the origin. To get from the first pair (x_1 and x_2) to the second pair (x_3 and x_4) you add 360°.

Tip

The curve between 0° and 180° has line symmetry about the line $x = 90^\circ$. If $x_1 = 36.9^\circ$, what will x_2 be?

Key point 10.11

To find the possible values of x satisfying $\sin x = a$:

- use your calculator to find $x_1 = \sin^{-1} a$
- the second solution is given by $x_2 = 180^\circ - x_1$
- other solutions are found by adding or subtracting 360° to any solution already found.

Tip

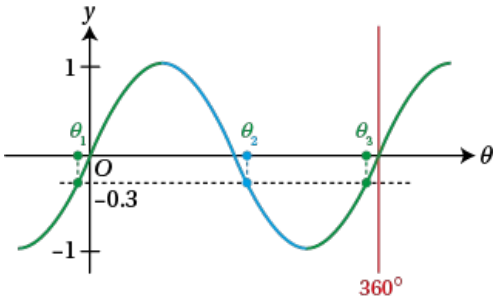
Always use the ANS button or the stored value rather than the rounded answer when doing subsequent calculations.

WORKED EXAMPLE 10.11

Find all possible values of $\theta \in [0^\circ, 360^\circ]$ for which $\sin\theta = -0.3$. Give your answers to 1 decimal place.

$\sin^{-1}(-0.3) = -17.5^\circ$ — Start by taking the inverse sine (which gives a solution that is not in the required interval).

$180^\circ - (-17.5^\circ) = 197.5^\circ$ — Another solution is given by $180^\circ - \theta$ (which is in the required interval).

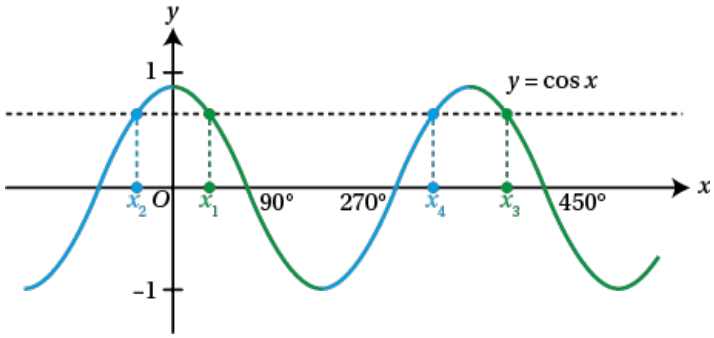


There are two solutions. — Sketch the graph to see how many solutions there are in the required interval.

$-17.5^\circ + 360^\circ = 342.5^\circ$ — The first solution (-17.5°) is not in the required interval. However, you can see from the graph that adding 360° produces another solution that is.

$\theta = 197.5^\circ, 342.5^\circ$ — State the complete list of solutions.

You can apply a similar analysis to the equation $\cos x = k$.



The solutions again come in pairs – one solution in the green section of the graph and one in the blue section (for example, x_1 and x_2). The inverse cosine function will always give you a solution in the green section closest to the origin.

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Key point 10.12

To find the possible values of x satisfying $\cos x = a$:

- use your calculator to find $x_1 = \cos^{-1} a$
- the second solution is given by $x_2 = -x_1$
- other solutions are found by adding or subtracting 360° to any solution already found.

It can be difficult to foresee how many times to add or subtract 360° to make sure that you have found all the solutions in a given interval. Drawing a graph can help here. You can then see how many solutions you are looking for and where they are.

Tip

If you are looking for positive solutions, then the second solution can be found directly as $x_2 = 360^\circ - x_1$.

WORKED EXAMPLE 10.12

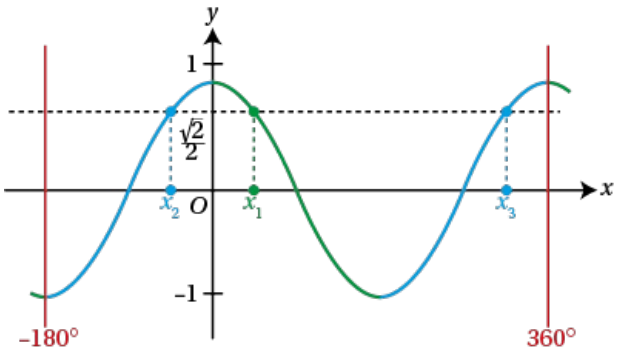
Find the values of $-180^\circ < x < 360^\circ$ for which $\cos x = \dfrac{\sqrt{2}}{2}$.

$x_1 = \cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = 45^\circ$

Start by taking the inverse cosine.

$x_2 = -45^\circ$

The second solution is given by $-x_1$.



Sketch the graph to see how many solutions there are in the required interval.

There are 3 solutions.

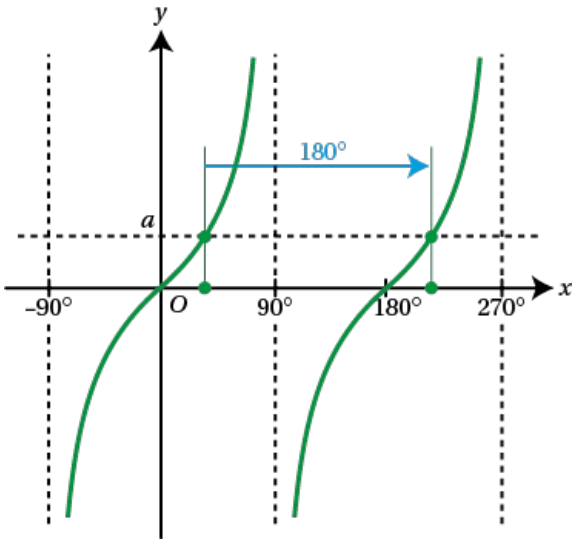
$x_3 = -45^\circ + 360^\circ = 315^\circ$

Add 360° to x_2 to find the other solution. Notice that adding 360° to x_1 would take you outside the interval.

$\therefore x = -45^\circ, 45^\circ, 315^\circ$

State the complete list of solutions.

The procedure for solving equations of the type $\tan x = a$ is slightly different, because the tangent function has period 180° rather than 360°.



Key point 10.13

To find the possible values of x satisfying $\tan x = a$:

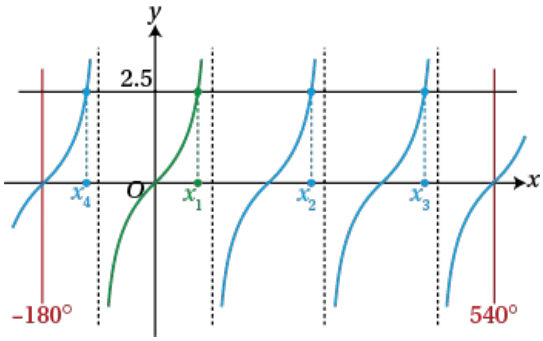
- use your calculator to find $x_1 = \tan^{-1} a$
- other solutions are found by adding or subtracting multiples of 180°.

WORKED EXAMPLE 10.13

Solve the equation $\tan x = 2.5$ for $-180^\circ < x < 540^\circ$. Give your answers to 3 s.f.

$x_1 = \tan^{-1} 2.5 = 68.2^\circ$

Start by taking the inverse tangent.



Sketch the graph to see how many solutions there are in the required interval.

There are four solutions.

$x_2 = x_1 + 180^\circ = 248^\circ$

$x_3 = x_2 + 180^\circ = 428^\circ$

$x_4 = x_1 - 180^\circ = -112^\circ$

The other solutions are found by adding or subtracting 180°.

$\therefore x = -112^\circ, 68.2^\circ, 248^\circ, 428^\circ$

List the complete set of solutions.

EXERCISE 10E

In this exercise, you must show detailed reasoning. This means that you can't, for example, use graphs or the equation solver function on your calculator. You need to use algebraic rearrangement and inverse trigonometric functions.

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1 Evaluate the following, giving your answer in degrees correct to one decimal place.

a **i** $\sin^{-1} 0.7$

ii $\sin^{-1} 0.3$

b **i** $\cos^{-1}(-0.62)$

ii $\cos^{-1}(-0.75)$

c **i** $\tan^{-1} 6.4$

ii $\tan^{-1}(-7.1)$

2 Find the values of x between 0° and 360° for which:

a **i** $\sin x = \frac{1}{2}$

ii $\sin x = \frac{\sqrt{2}}{2}$

b **i** $\cos x = \frac{1}{2}$

ii $\cos x = \frac{\sqrt{3}}{2}$

c **i** $\tan x = 1$

ii $\tan x = \sqrt{3}$

d **i** $\sin x = -\frac{\sqrt{3}}{2}$

ii $\sin x = -\frac{1}{2}$

e **i** $\cos x = -\frac{1}{\sqrt{2}}$

ii $\cos x = -1$

f **i** $\tan x = -\frac{1}{\sqrt{3}}$

ii $\tan x = -1$

Rewind

Question 2 involves exact values of trigonometric functions, which are summarised in Key Point 10.9.

3 Solve these equations in the given interval. Give your answers to one decimal place.

a **i** $\sin x = 0.45$ for $x \in [0^\circ, 360^\circ]$

ii $\sin x = 0.7$ for $x \in [0^\circ, 360^\circ]$

b **i** $\cos x = -0.75$ for $-180^\circ \leqslant x \leqslant 180^\circ$

ii $\cos x = -0.2$ for $-180^\circ \leqslant x \leqslant 180^\circ$

c **i** $\tan \theta = \frac{1}{3}$ for $0^\circ \leqslant \theta \leqslant 720^\circ$

ii $\tan \theta = \frac{4}{3}$ for $0^\circ \leqslant \theta \leqslant 720^\circ$

d **i** $\sin t = -\frac{2}{3}$ for $t \in [-180^\circ, 360^\circ]$

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ii $\sin t = -\frac{1}{4}$ for $t \in [-180^\circ, 360^\circ]$

4 Solve the following equations.

a **i** $2\sin\theta + 1 = 1.2$ for $0^\circ < \theta < 360^\circ$

ii $4\sin x + 3 = 2$ for $-90^\circ < x < 270^\circ$

b **i** $3\cos x - 1 = \frac{1}{3}$ for $0^\circ < x < 360^\circ$

ii $5\cos x + 2 = 4.7$ for $0^\circ < x < 360^\circ$

c **i** $3\tan t - 1 = 4$ for $-180^\circ < t < 180^\circ$

ii $5\tan t - 3 = 8$ for $0^\circ < t < 360^\circ$

5 Find the values of x between -180° and 180° for which $2\sin x + 1 = 0$.

6 Solve the equation $3\tan x + 5 = 0$ for $0^\circ \leqslant x \leqslant 360^\circ$.

7 Solve $2\cos x = \sqrt{3}$ for $x \in [-360^\circ, 360^\circ]$.

8 Find all values of θ in the interval $0^\circ \leqslant \theta \leqslant 360^\circ$ for which $\sin\theta^\circ\ (3\cos\theta^\circ - 2) = 0$.

9 Solve $(\tan x + 1)(5\sin x - 2) = 0$ for $0^\circ \leqslant x \leqslant 360^\circ$.

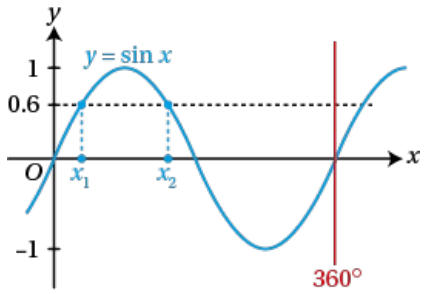
10 Show by a counter example that $\tan^{-1}x \not\equiv \frac{\sin^{-1}x}{\cos^{-1}x}$.

11 Show by a counter example that $\sin^{-1}(\sin x)$ is **not** always x.

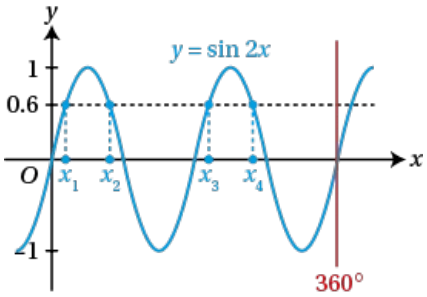
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Section 6: Transformations of trigonometric graphs

If you solve the equation $\sin x = 0.6$ for $0^\circ \leqslant x \leqslant 360^\circ$, you can see from the graph that there are two solutions:



If you solve the equation $\sin 2x = 0.6$ for $0^\circ \leqslant x \leqslant 360^\circ$ you can see from the graph that there are four solutions:



You need to extend your methods to deal with equations like this. A substitution is a useful way of doing this.

Rewind

Remember that replacing x by $2x$ results in the graph being 'squashed' in the horizontal direction.

WORKED EXAMPLE 10.14

Solve $3\sin(2x) + 1 = 0$ for $x \in [0^\circ, 360^\circ]$. Give your answers to one decimal place.

$\mathrm{A} = 2x$

Make a substitution: replace $2x$ by a single letter A.

If $x \in [0^\circ, 360^\circ]$ then $\mathrm{A} \in [0^\circ, 720^\circ]$.

Rewrite the interval in terms of A.

$3\sin(\mathrm{A}) + 1 = 0$

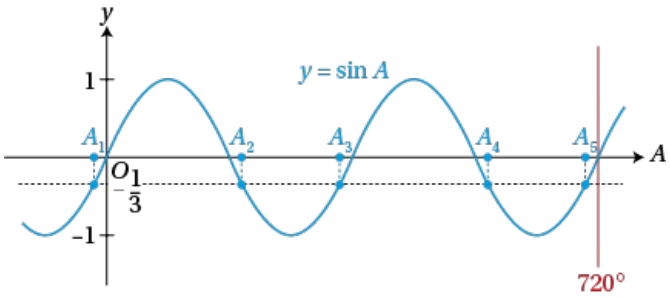
$\sin(\mathrm{A}) = -\frac{1}{3}$

Rearrange into the form $\sin(\mathrm{A}) = k$.

$\sin \mathrm{A} = -\frac{1}{3}$

$\sin^{-1}\left(-\frac{1}{3}\right) \approx -19.47^\circ$ (outside interval)

Find the solutions to 2 d.p., then round at the end.



There are four solutions.

Solve the equation for A.

$\mathrm{A}_2 = 180^\circ - (-19.47^\circ) = 199.47^\circ$

$\mathrm{A}_3 = -19.47^\circ + 360^\circ = 340.53^\circ$

$\mathrm{A}_4 = 199.47^\circ + 360^\circ = 559.47^\circ$

$\mathrm{A}_5 = 340.53^\circ + 360^\circ = 700.53^\circ$

Transform the solutions back into x.

$$x = \frac{\mathrm{A}}{2}$$
$$= 99.7^\circ, 170.3^\circ, 279.7^\circ, 350.3^\circ \text{ to 1 d.p.}$$

A summary of this procedure is given in the four-step process.

Key point 10.14

1. Make a substitution (such as $\mathrm{A} = 2x$).
2. Change the interval for x into the interval for A.
3. Solve the equation in the usual way.
4. Transform the solutions back into the original variable.

Tip

Make sure you do not round your answers until the end.

WORK IT OUT 10.1

Solve $\tan 3x = 1$ for $-180^\circ < x < 180^\circ$.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

Let $\mathrm{A} = 3x$

$\mathrm{A}_1 = \tan^{-1}(1) = 45^\circ$

$\mathrm{A}_2 = 45 - 180 = -135^\circ$

$$x = \frac{\mathrm{A}}{3}$$
$$= 15^\circ, -45^\circ$$

Solution 2

Let $\mathrm{A} = 3x$

$-180^\circ < x < 180^\circ$

$\Rightarrow -540^\circ < \mathrm{A} < 540^\circ$

$\mathrm{A}_1 = \tan^{-1}(1) = 45^\circ$

$\therefore x_1 = 15^\circ$

$x_2 = 15 + 180 = 195$
$x_3 = 195 + 180 = 375$
$x_4 = 15 - 180 = -165$
$x_5 = -165 - 180 = -345$
$x_6 = -345 - 180 = -525$

$x = -525^\circ, -345^\circ, -165^\circ, 15^\circ, 195^\circ, 375^\circ$

Solution 3

Let $\mathrm{A} = 3x$

$-180^\circ < x < 180^\circ$

$\Rightarrow -540^\circ < \mathrm{A} < 540^\circ$

$\mathrm{A}_1 = \tan^{-1}(1) = 45^\circ$
$\mathrm{A}_2 = 45 + 180 = 225$
$\mathrm{A}_3 = 225 + 180 = 405$
$\mathrm{A}_4 = 45 - 180 = -135$
$\mathrm{A}_5 = -135 - 180 = -315$
$\mathrm{A}_6 = -315 - 180 = -495$

$x = \dfrac{\mathrm{A}}{3}$
$= -165^\circ, -105^\circ, -45^\circ, 15^\circ, 75^\circ, 135^\circ$

Worked example 10.15 shows this method in a more complicated situation.

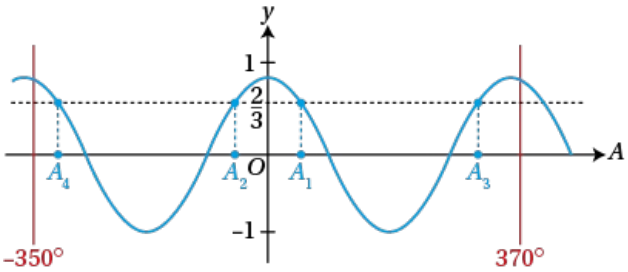
WORKED EXAMPLE 10.15

Solve the equation $3\cos(2x + 10^\circ) = 2$ for $-180^\circ \leqslant x \leqslant 180^\circ$. Give your answers to 3 s.f.

$\mathrm{A} = 2x + 10$ — Make a substitution.

If $-180^\circ \leqslant x \leqslant 180^\circ$ *then* — Rewrite the interval.
$-360^\circ \leqslant 2x \leqslant 360^\circ$
$-350^\circ \leqslant 2x + 10^\circ \leqslant 370^\circ$
$So\ -350^\circ \leqslant \mathrm{A} \leqslant 370^\circ$

$3\cos(\mathrm{A}) = 2$ — Write the equation in the form $\cos(\mathrm{A}) = k$.
$\cos(\mathrm{A}) = \dfrac{2}{3}$



There are four solutions:

$\mathrm{A}_1 = \cos^{-1}\left(\frac{2}{3}\right) = 48.19^\circ$ Solve the equation for A.

$\mathrm{A2} = -48.19^\circ$

$\mathrm{A3} = 48.19^\circ + 360^\circ = 408.19^\circ$ - not in range

$\mathrm{A4} = -48.19^\circ + 360^\circ = 311.81^\circ$

$\mathrm{A5} = 48.19^\circ$ - $360^\circ = -311.81^\circ$

$x = \frac{\mathrm{A} - 10}{2}$ Transform the solutions back into x.

$= -161^\circ, -29.1^\circ, 19.1^\circ, 151^\circ (3\ s.\mathrm{f.})$

Worked example 10.16 revisits the tangent function and exact values.

WORKED EXAMPLE 10.16

Solve the equation $3\tan\left(\frac{1}{2}\theta - 30^\circ\right) = \sqrt{3}$ for $0^\circ \leqslant \theta \leqslant 720^\circ$.

$\mathrm{A} = \frac{1}{2}30^\circ$ Make a substitution.

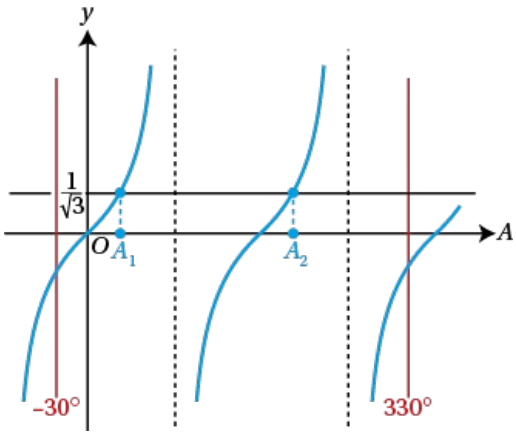
If $0^\circ \leqslant \theta \leqslant 720^\circ$ then Rewrite the interval.

$0^\circ \leqslant \frac{1}{2} \leqslant 360^\circ$

So $-30^\circ \leqslant \frac{1}{2} - 30^\circ \leqslant 330^\circ$

$3\tan\left(\frac{1}{2} - 30^\circ\right) = \sqrt{3}$ Rearrange the equation into the form $\tan(\mathrm{A}) = k$.

$\tan\left(\frac{1}{2} - 30^\circ\right) = \frac{\sqrt{3}}{3}$



There are two solutions:

$A_1 = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = 30^\circ$ Solve the equation for A.

$A_2 = 30^\circ + 180^\circ = 210^\circ$

$\theta = 2(A + 30^\circ)$ Transform the solutions back into θ.
$= 120^\circ, 480^\circ$

EXERCISE 10F

In this exercise, you must show detailed reasoning. This means that you can't, for example, use graphs or the equation solver function. You need to use algebraic rearrangement and inverse trigonometric functions.

1 Solve the following equations in the interval $0^\circ \leqslant x \leqslant 360^\circ$ giving your answers to 3 s.f.

a i $\sin 2x = 0.7$

ii $\sin 3x = -0.2$

b i $\cos 3x = -0.4$

ii $\cos 4x = 1$

c i $\tan 4x = 1.5$

ii $\tan 2x = -2$

2 Solve the following equations in the interval $-180^\circ \leqslant \theta \leqslant 180^\circ$ giving your answers to 3 s.f.

a i $\sin(\theta + 40^\circ) = 0.25$

ii $\sin(\theta - 25^\circ) = -0.75$

b i $\cos(\theta - 50^\circ) = -0.9$

ii $\sin(\theta + 10^\circ) = 0.3$

c i $\tan(\theta - 45^\circ) = 2$

ii $\tan(\theta + 60^\circ) = -3$

3 Find the exact solutions of the equation $\tan 3x = \sqrt{3}$ for $0^\circ < x < 180^\circ$.

4 Solve $2\ \cos\ (2x) + 1 = 0$ for $x \in [-180^\circ,\ 180^\circ]$.

5 Find the values of θ in the interval $-360^\circ < \theta < 360^\circ$ for which $3\sin\left(\frac{\theta}{2}\right) = -2$.

6 Solve $2\tan\left(\frac{\theta}{3}\right) = 5$ for $0^\circ < \theta < 540^\circ$.

7 Find the values of x in the interval $-360^\circ < x < 360^\circ$ for which $2\ \sin\ (2x + 30^\circ) + \sqrt{3} = 0$.

8 Solve $2\ \cos\ (3x - 50^\circ) - \sqrt{2} = 0$ for $-90^\circ < x < 90^\circ$.

9 Solve the equation $\dfrac{\cos\ 2x + 0.5}{1 - \cos\ 2x} = 2$ for $x \in (-180^\circ,\ 180^\circ)$.

10 Find the values of x in the interval $-\sqrt{180^\circ} < x < \sqrt{180^\circ}$ for which $\sin\left(x^2\right) = \dfrac{1}{2}$.

Section 7: More complex trigonometric equations

You have already seen how to solve equations that are in the form 'trigonometric function of unknown = constant'. However, sometimes you need to get the equation into this form first. There are three tactics that are often used:

- Look for disguised quadratics.
- Take everything over to one side and factorise.
- Use trigonometric identities.

Tip

Remember that $\cos^2\theta$ means $(\cos\theta)^2$.

WORKED EXAMPLE 10.17

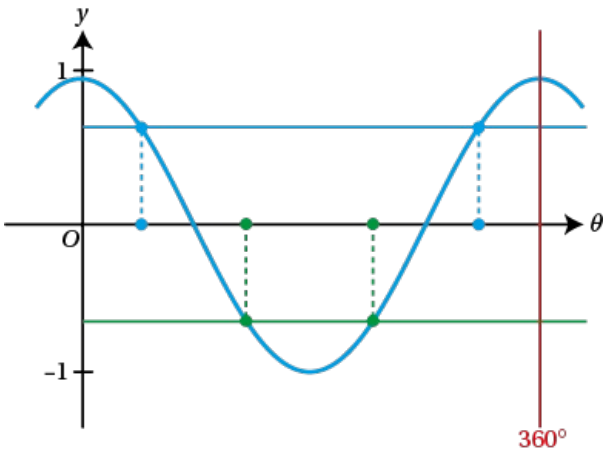
Solve the equation $\cos^2\theta = \frac{4}{9}$ for $\theta \in [0^\circ,\ 360^\circ]$. Give your answers correct to one decimal place.

$\cos^2\theta = \frac{4}{9}$

$\cos\theta = \pm\frac{2}{3}$

First find possible values of $\cos\theta$.

Remember $\pm$ when taking the square root.



Sketch the graph to see how many solutions there are in the required interval.

There are two solutions to each.

When $\cos = \frac{2}{3}$:

$\cos^{-1}\left(\frac{2}{3}\right) = 48.2^\circ$

$\theta_1 = 48.2^\circ$

$\theta_2 = 360^\circ - 48.2^\circ = 311.8^\circ$

Solve each equation separately.

When $\cos = -\frac{2}{3}$:

$\cos^{-1}\left(-\frac{2}{3}\right) = 131.8^\circ$

$\theta_1 = 131.8^\circ$

$\theta_2 = 360^\circ - 131.8^\circ = 228.2^\circ$

$\theta = 48.2^\circ, 131.8^\circ, 228.2^\circ, 311.8^\circ$

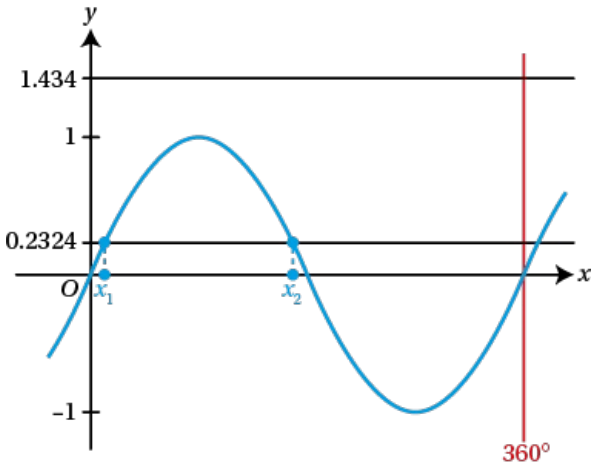
List all the solutions.

WORKED EXAMPLE 10.18

Solve the equation $3\sin^2 x - 5\sin x + 1 = 0$ for $0^\circ < x < 360^\circ$.

$$\sin x = \frac{5 \pm \sqrt{5^2 - 4 \times 3 \times 1}}{2 \times 3}$$
$$= 1.434 \text{ or } 0.2324$$

This is a disguised quadratic equation in $\sin x$. If you cannot factorise it, use the quadratic formula.



Sketch the graph to see how many solutions there are.

There are two solutions.

$\sin x = 1.434 > 1$ is impossible.

$\sin x$ is always between -1 and 1, so only one of the values is possible.

$\sin x = 0.2324$

$\sin^{-1}(0.2324) = 13.4^\circ$

$x_1 = 13.4^\circ$

$x_2 = 180 - 13.4 = 167^\circ$

$\therefore x = 13.4^\circ, 167^\circ$

You can now solve the equation as usual.

Rewind

You may want to revisit Chapter 3, Section 5 on disguised quadratics.

WORKED EXAMPLE 10.19

Solve the equation $3\sin x \cos x = 2\sin x$ for $-180^\circ \leqslant x \leqslant 180^\circ$.

$3\sin x \cos x = 2\sin x$

$3\sin x \cos x - 2\sin x = 0$

$\sin x(3\cos x - 2) = 0$

This equation contains both $\sin$ and $\cos$. However, both sides have a factor of $\sin x$, so you can make one side of the equation equal to zero and factorise.

$\sin x = 0 \quad \text{or} \quad \cos x = \frac{2}{3}$

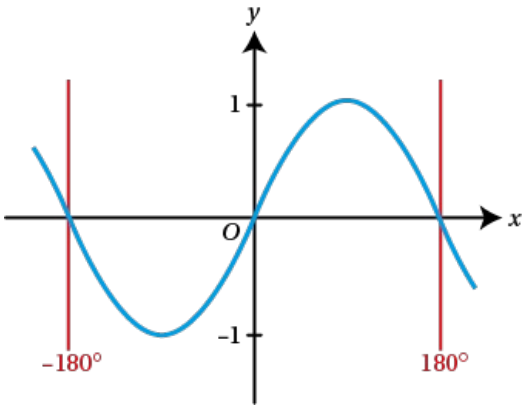
You then have two separate equations, each containing only one trig function.

When $\sin x = 0$:

Solve each equation separately.

$\sin^{-1} 0 = 0$

Remember to sketch the graph for each equation to see how many solutions there are.



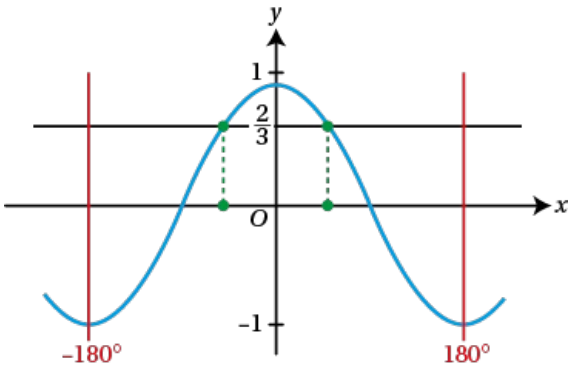
$x_1 = 0^\circ$

$x_2 = 180^\circ - 0^\circ = 180^\circ$

$x_3 = 180^\circ - 360^\circ = -180^\circ$

When $\cos x = \frac{2}{3}$:

$\cos^{-1}\left(\frac{2}{3}\right) = 48.2^\circ$



$x_1 = 48.2^\circ$

$x_2 = -48.2^\circ$

$\therefore x = -180^\circ, -48.2^\circ, 0^\circ, 48.2^\circ, 180^\circ$

List all the solutions

EXERCISE 10G

1 Solve the following equations in the interval $0^\circ \leqslant x \leqslant 360^\circ$ giving your answers to 3 s.f.

a i $3\sin^2\theta = 2$

ii $3\tan^2\theta = 5$

b i $\tan^2 x - \tan x - 6 = 0$

ii $3\cos^2 x + \cos x - 2 = 0$

c i $4\cos^2 x - 11\cos x + 6 = 0$

ii $5\sin^2 x + 6\sin x - 8 = 0$

d i $3\sin^2 x + \sin x = 0$

ii $4\tan^2 x + 5\tan x = 0$

2 Solve the following equations in the interval $-180^\circ \leqslant \theta \leqslant 180^\circ$ giving your answers to 3 s.f.

a i $2\sin\theta - 5\sin\theta\cos\theta = 0$

ii $4\cos\theta + 5\sin\theta\cos\theta = 0$

b i $4\sin\theta\cos\theta = \cos\theta$

ii $3\sin\theta = 5\sin\theta\cos\theta$

3 Find the values of $x \in (-360^\circ, 360^\circ)$ for which $2\sin x\cos x = \cos x$.

4 **a** Given that $2\sin^2 x - 3\sin x = 2$, find the exact value of $\sin x$.

b Hence solve the equation $2\sin^2 x - 3\sin x = 2$ for $0^\circ < x < 360^\circ$.

5 Solve the equation $\tan^2 x = \tan x + 12$ for $-180^\circ \leqslant x \leqslant 180^\circ$. Give your answers to 1 decimal place.

6 Find the values of θ in the interval $0^\circ < \theta < 360^\circ$ for which $2\sin 2\theta = \sqrt{3}\sin 2\theta\cos 2\theta$.

7 Solve the equation $3\cos^2 3x + 7\cos 3x + 2 = 0$ for $x \in [0^\circ,\ 180^\circ]$.

Using identities to solve equations

When there is more than one trigonometric function in an equation, it is often useful to use an identity to eliminate one of the functions.

WORKED EXAMPLE 10.20

Find all values of θ in the interval $[-180^\circ, 180^\circ]$ that satisfy the equation $2\sin^2\theta + 3\cos\theta = 1$.

$2\sin^2\theta + 3\cos\theta = 1$

$2(1 - \cos^2\theta) + 3\cos\theta = 1$

The equation contains both $\sin$ and $\cos$. Because you have $\sin^2$ you can use $\sin^2 x + \cos^2 x = 1$ to replace $\sin^2\theta$ by $1 - \cos^2\theta$.

$2 - 2\cos^2\theta + 3\cos\theta = 1$

$2\cos^2\theta - 3\cos\theta - 1 = 0$

This is a disguised quadratic equation in $\cos\theta$, so write it in the standard form.

$\cos\theta = 1.78$ or -0.281

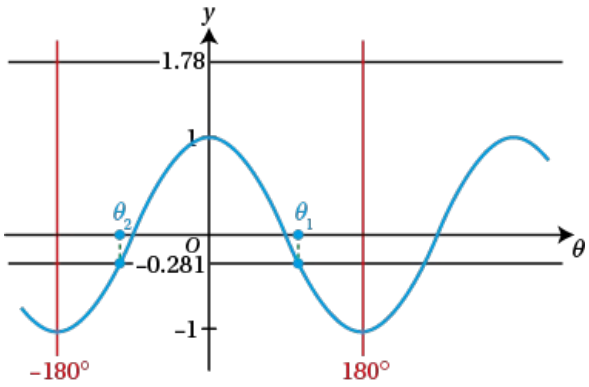
Solve it using your calculator or the quadratic formula.

$\cos\theta = 1.78$ is impossible.

$\cos$ is always between -1 and 1.

$\cos\theta = -0.281$:

Solve the equation as normal.



There are two solutions:

$\theta_1 = \cos^{-1}(-0.281) = 106^\circ$

$\theta_2 = -106^\circ$

$\therefore \theta = -106^\circ, 106^\circ$

WORKED EXAMPLE 10.21

Solve the equation $\sin\theta + \sqrt{3}\cos\theta = 0$ for $-360^\circ < \theta < 360^\circ$.

$\sin\theta + \cos\theta = 0$

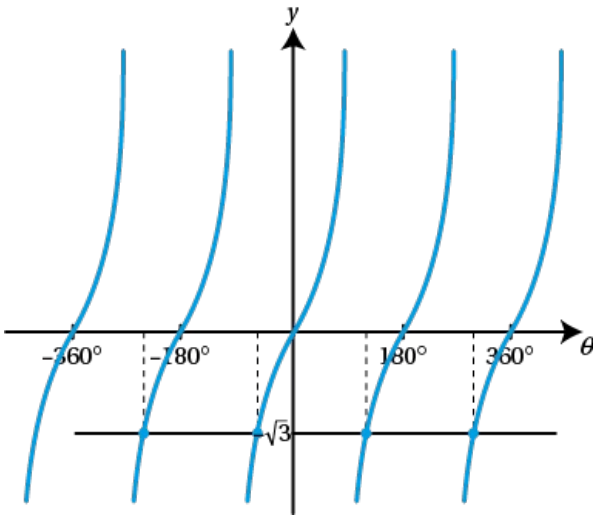
$\sin\theta = -\cos\theta$

The equation involves both $\sin$ and $\cos$. It is inefficient to use $\sin^2 x + \cos^2 x = 1$ to eliminate one of them as neither is squared.

$\frac{\sin\theta}{\cos\theta} = -\sqrt{3}$

$\tan\theta = -\sqrt{3}$

Instead rearrange and use $\frac{\sin\theta}{\cos\theta} = \tan\theta$.



$\theta_1 = \tan^{-1}(-\sqrt{3}) = -60^\circ$

$\theta_2 = -60^\circ + 180^\circ = 120^\circ$

$\theta_3 = 120^\circ + 180^\circ = 300^\circ$

$\theta_4 = -60^\circ - 180^\circ = -240^\circ$

$\therefore \theta = -240^\circ, -60^\circ, 120^\circ, 300^\circ$

Solve this equation as normal.

WORKED EXAMPLE 10.22

Solve the equation $4\sin 2x = \tan 2x$ in the interval $0^\circ \leqslant x \leqslant 180^\circ$.

$4\sin 2x = \tan 2x$

The only identity you can use here is the one for $\tan$.

$4\sin 2x = \dfrac{\sin 2x}{\cos 2x}$

$4\sin 2x \cos 2x = \sin 2x$

Multiply both sides by $\cos 2x$.

$4\sin 2x \cos 2x - \sin 2x = 0$

$\sin 2x(4\cos 2x - 1) = 0$

Although you have both $\sin$ and $\cos$, both sides contain $\sin 2x$, so you can make one side of the equation equal to zero and factorise.

$\sin 2x = 0$ or $\cos \mathrm{A} = \dfrac{1}{4}$

We now have two equations, each of which only has one trig function.

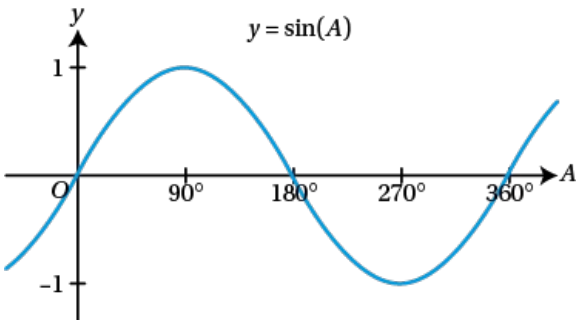
Let $\mathrm{A} = 2x$

$0^\circ \leqslant x \leqslant 180^\circ$

$\Rightarrow 0^\circ \leqslant \mathrm{A} \leqslant 360^\circ$

Now solve each equation separately.

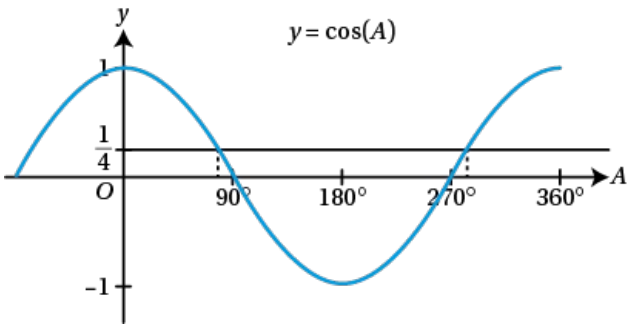
When $\sin \mathrm{A} = 0$:



$\mathrm{A} = 0^\circ, 180^\circ, 360^\circ$

$x = 0^\circ, 90^\circ, 180^\circ$

When $\cos \mathrm{A} = \dfrac{1}{4}$:



$\cos^{-1}\left(\dfrac{1}{4}\right) = 75.5$

$\mathrm{A} = 75.5^\circ$ or $360 - 75.5^\circ = 284^\circ$

$x = 37.8^\circ, 142^\circ$

$\therefore x = 0^\circ, 37.8^\circ, 90^\circ, 142^\circ, 180^\circ$

List all the solutions.

WORK IT OUT

10.2

Solve the equation $\tan x = 2\sin x$ for $x \in [0, 360^\circ]$.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$\frac{\sin x}{\cos x} = 2\sin x$

$\sin x = 2\sin x \cos x$

$\sin x = 0 \text{or} 1 = 2\cos x$

$x = 0 \text{ or } \cos^{-1}\left(\frac{1}{2}\right) = 60°$

Solution 2

$\frac{\sin x}{\cos x} = 2\sin x$

$\sin x = 2\sin x \cos x$

$\sin x(1 - 2\cos x) = 0$

$\sin x = 0 \quad or \quad cosx = \frac{1}{2}$

$x = 0, 180^\circ, 360^\circ, 60^\circ, 300^\circ$

Solution 3

$\frac{\sin x}{\cos x} = 2\sin x$

$\sin x = 2\sin x \cos x$

$1 = 2\cos x$

$\cos x = \frac{1}{2}$

$x = 60^\circ, 300^\circ$

EXERCISE 10H

1 By using the identity $\tan x = \dfrac{\sin x}{\cos x}$ solve the following equations for $0^\circ \leqslant x \leqslant 180^\circ$.

a i $3\sin x = 2\cos x$

ii $3\sin x = 5\cos x$

b i $7\cos x - 3\sin x = 0$

ii $\sin x - 5\cos x = 0$

2 Use the identity $\sin^2 x + \cos^2 x = 1$ to solve the following equations for x in the interval $[0^\circ, 360^\circ]$.

a i $7\sin^2 x + 3\cos^2 x = 5$

ii $\sin^2 x + 4\cos^2 x = 2$

b i $3\sin^2 x - \cos^2 x = 1$

ii $\cos^2 x - \sin^2 x = 1$

3 Find the values of x in the interval $0^\circ < x < 360^\circ$ for which $\sin 2x + \cos 2x = 0$.

4 Solve $\dfrac{\cos\theta}{\sin\theta} - 2 = 0$ for $\in [-180^\circ,\ 180^\circ]$.

Worksheet

For a further example of this type and more practice questions, see Support sheet 10.

5 Solve $\sin x + \dfrac{\sin^2 x}{\cos x} = 0$ for $0^\circ \leqslant x \leqslant 360^\circ$.

6 Solve the equation $\sin x \tan x = \sin^2 x$ for $-180^\circ \leqslant x \leqslant 180^\circ$.

7 Solve the equation $5\sin^2\theta = 4\cos^2\theta$ for $-180^\circ \leqslant \theta \leqslant 180^\circ$. Give your answers to the nearest 0.1°.

8 Solve the equation $2\cos^2 t - \sin t - 1 = 0$ for $0^\circ \leqslant t \leqslant 360^\circ$.

9 Find the values of x in the interval $-180^\circ \leqslant x \leqslant 180^\circ$ that satisfy $4\cos^2 x - 5\sin x - 5 = 0$.

10 **a** Given that $\cos^2 t + 5\cos t = 2\sin^2 t$, find the exact value of $\cos t$.

b Hence solve the equation $\cos^2 t + 5\cos t = 2\sin^2 t$ for $t \in [0^\circ,\ 360^\circ]$.

11 **a** Given that $6\sin^2 x + \cos x = 4$, find the exact values of $\cos x$.

b Hence solve the equation $6\sin^2 x + \cos x = 4$ for $0^\circ \leqslant x \leqslant 360^\circ$ giving your answers to 3 s.f.

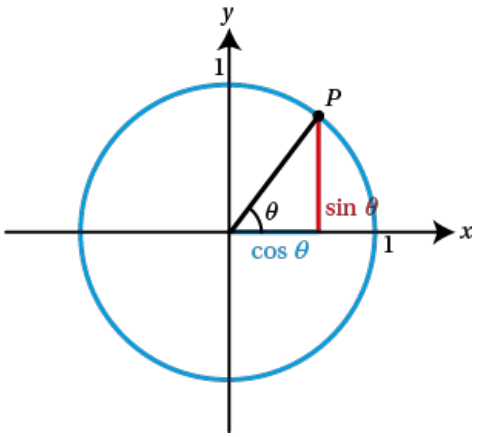
12 **a** Show that the equation $2\sin^2 x - 3\sin x\cos x + \cos^2 x = 0$ can be written in the form $2\tan^2 x - 3\tan x + 1 = 0$.

b Hence solve the equation $2\sin^2 x - 3\sin x\cos x + \cos^2 x = 0$ giving all solutions in the interval $-180^\circ < x < 180^\circ$.

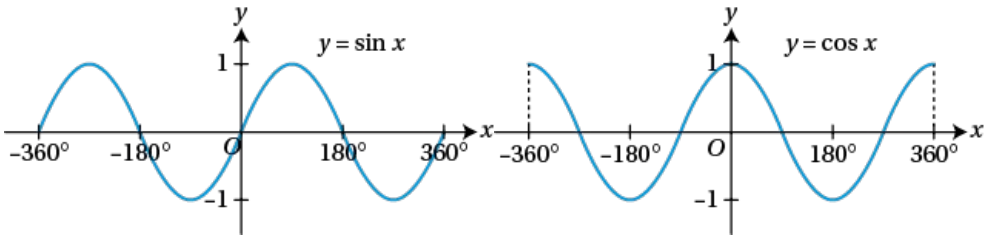
Checklist of learning and understanding

- The sine and cosine functions can be defined using the unit circle:

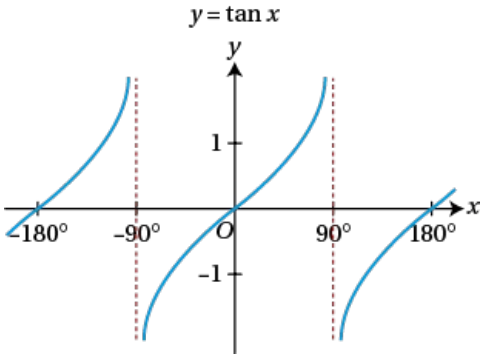
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- The sine and cosine functions are periodic with period 360°.



- These relationships can be seen from the unit circle or the graph:
  - $\sin x \equiv \cos(90^\circ - x)$
  - $\cos x \equiv \sin(90^\circ - x)$
- The tangent function is defined by the identity: $\tan x \equiv \dfrac{\sin x}{\cos x}$. It is periodic with period 180°.
- To solve trigonometric equations, you should use the following procedure:
  - first rearrange into the form $\sin x = k$, $\cos x = k$ or $\tan x = k$
  - always draw a graph to see how many solutions there are
  - find one solution (for $\tan$) or the first two solutions (for $\sin$ and $\cos$)
    - $\sin x = k : x_1 = \sin^{-1} k, x_2 = 180^\circ - x_1$
    - $\cos x = k : x_1 = \cos^{-1} k, x_2 = -x_1$
    - $\tan x = k : x_1 = \tan^{-1} k, x_2 = x_1 + 180^\circ$
  - other solutions are found by adding or subtracting multiples of 180° (for $\tan$) or 360° (for $\sin$ and $\cos$).



- If the angle in the function in the trigonometric equation has been transformed:
  - make a substitution (such as $\mathrm{A} = 2x$)
  - change the interval for x into the interval for A
  - solve the equation in the usual way
  - transform the solutions back into the original variable.
- The identities $\sin^2 x + \cos^2 x \equiv 1$ and $\tan x \equiv \dfrac{\sin x}{\cos x}$ can be used to solve some trigonometric equations. This often leads to disguised quadratic equations.

Mixed practice 10

1. If $\cos(x + 180^\circ) = a$ what is the value of $\cos x$?
2. Solve the equation $\tan x = -0.62$ for $x \in (-90^\circ,\ 270^\circ)$ giving your answers to the nearest 0.1°.
3. Solve the equation $\sqrt{2}\sin\theta + 1 = 0$ for $-360^\circ < \theta < 360^\circ$.
4. Find the values of x in the interval $0^\circ < x < 720^\circ$ for which $2\cos\left(\frac{1}{2}x + 45^\circ\right) = \sqrt{3}$.
5. Solve, to 3 s.f., the equation $7\sin^2\theta = 9\cos^2\theta$ for $-180^\circ \leqslant \theta \leqslant 180^\circ$.
6. **a** Show that the equation $2\sin^2 x = 5\cos x - 1$ can be expressed in the form $2\cos^2 x + 5\cos x - 3 = 0$.

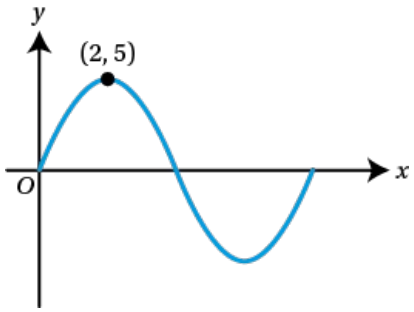
   b Hence solve the equation $2\sin^2 x = 5\cos x - 1$ giving all values of x between 0° and 360°.

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7. **a** Show that the equation $\cos\theta - 2\sin^2\theta + 2 = 0$ can be expressed in the form $2\cos^2\theta + \cos\theta = 0$.

   b Hence find all values of $\in [0^\circ,\ 360^\circ]$ for which $\cos\theta - 2\sin^2\theta + 2 = 0$.

8. How many solutions are there to the equation $\sin^2 2x = \frac{1}{4}$ in the interval $-180^\circ < x < 180^\circ$?
9. The diagram shows the graph of the function $f(x) = a\sin(bx)$. Find the values of a and b.



10. Solve the equation $6\sin^2 x + \cos x = 4$ for $0^\circ \leqslant x \leqslant 360^\circ$.

    Give your answers to 3 s.f.

11. Prove the identity $\frac{2}{\cos^2 x} - \tan^2 x = 2 + \tan^2 x$.
12. **a** Show that the equation $2\sin x = \frac{4\cos x - 1}{\tan x}$ can be expressed in the form $6\cos^2 x - \cos x - 2 = 0$.

    b Hence solve the equation $2\sin x = \frac{4\cos x - 1}{\tan x}$ giving all values of x between 0° and 360°.

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13. Prove the identity $\frac{1}{1+\cos x} + \frac{1}{1-\cos x} = \frac{2}{\sin^2 x}$.
14. **a** Show that $\frac{\sin^2 x - \cos^2 x}{1 - \sin^2 x} \equiv \tan^2 x - 1$.

    b Hence solve the equation $\frac{\sin^2 x - \cos^2 x}{1 - \sin^2 x} = 5 - \tan x$ for $0^\circ \leqslant x \leqslant 360^\circ$.

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[Question part reference style adapted]

15 Show that $\tan x + \dfrac{1}{\tan x} = \dfrac{\tan x}{\sin^2 x}$.

16 Find all values of x in the interval $-90^\circ < x < 90^\circ$ that satisfy $6\cos^2 2x = \sin 2x + 4$.

17 **a** Find the values of k for which the equation $4x^2 - kx + 1 = 0$ has a repeated root.

b Show that the equation $4\sin^2\theta = 5 - k\cos\theta$ can be written as $4\cos^2\theta - k\cos\theta + 1 = 0$.

c Let $\mathrm{f}_k(\theta) = 4\cos^2 - k\cos\ + 1$.

i State the number of values of $\cos\theta$ that satisfy the equation $\mathrm{f}_4(\theta) = 0$.

ii Find all the values of $\in [-360^\circ,\ 360^\circ]$ that satisfy the equation $\mathrm{f}_4(\theta) = 0$.

iii Find the value of k for which $x = 1$ is a solution of the equation $4x^2 - kx + 1 = 0$.

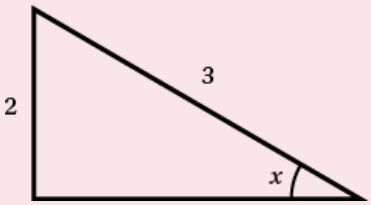
iv For this value of k, find the number of solutions of the equation $\mathrm{f}_k(\theta) = 0$ for interval $\theta \in [-360^\circ, 360^\circ]$.

11 Triangle geometry

In this chapter you will learn how to:

- use the sine rule to find sides and angles of any triangle
- use the cosine rule to find sides and angles of any triangle
- use a formula for the area of a triangle when you don't know the perpendicular height.

Before you start...

GCSE	You should know how to use trigonometry in right-angled triangles.	1 Find the angle marked x in the diagram. 
GCSE	You should know how to use three-figure bearings.	2 Point A is a on a bearing of 290° from B. Find the bearing of B from A.
Chapter 3	You should know how to solve quadratic equations using the formula or factorising.	3 Solve $x^2 + 5x + 4 = 0$.
Chapter 10	You should know how to solve trigonometric equations.	4 Solve the equation $\sin x = 0.15$ for $0^\circ < x < 180^\circ$.

Trigonometry and triangles

The first steps in developing trigonometry were made by Babylonian astronomers as early as the 2nd millennium BCE. It is thought that Egyptians used some trigonometric calculations when building the pyramids. It was further developed by Greek, Islamic and Indian mathematicians to solve problems in land measurement, building and astronomy.

In this chapter you will use what you already know about trigonometric functions, as well as develop some new results to enable you to calculate lengths and angles in triangles.

Gateway to A Level

For a reminder of using trigonometry in right-angled triangles see Gateway to A Level section P.

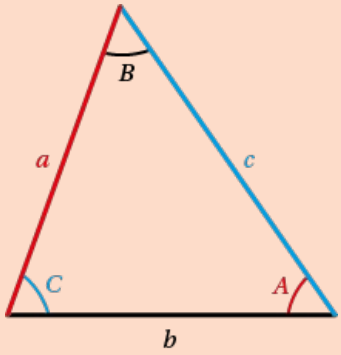
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Section 1: The sine rule

You can use trigonometry to calculate sides and angles in triangles without a right angle.

Key point 11.1

The sine rule is: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$



To use the sine rule you need to know an angle and the opposite side.

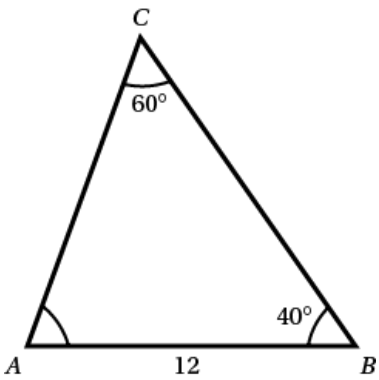
When using the sine rule you will normally use only two of the three ratios. To decide which ratios to use, you need to look at what information is given in the question.

Focus on ...

Focus on ... Proof 2 shows you how to prove the sine rule for any triangle.

WORKED EXAMPLE 11.1

Find the length of side AC.



$$\frac{12}{\sin 60^\circ} = \frac{AC}{\sin 40^\circ}$$

$$AC = \frac{12 \sin 40^\circ}{\sin 60^\circ}$$

$$= 8.91 \text{ to } 3 \text{ s.f.}$$

You are given length AB and the angles opposite sides AB and AC, so use the sine rule with those two sides.

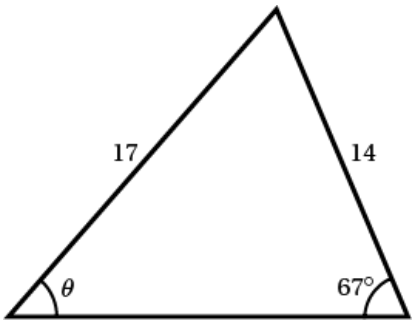
Tip

Give your answer to three significant figures, unless the question requests otherwise.

You can also use the sine rule to find angles:

WORKED EXAMPLE 11.2

Find the size of the angle marked θ. Give your answer to one decimal place.



$$\frac{17}{\sin 67^\circ} = \frac{14}{\sin \theta}$$

You have an angle opposite one of the sides, and want the angle opposite the other known side, so use the sine rule.

$$\sin\theta = \frac{14 \sin 67^\circ}{17} = 0.758 \ldots$$

$$\therefore \theta = \sin^{-1} 0.758 \ldots = 49.3^\circ (1\,\text{d.p.})$$

Although you can write down this intermediate answer to 3s.f., you should use the unrounded value in subsequent calculations.

Notice that you can also find the third angle even though you do not know the length of the side opposite. Having found θ using the sine rule, you could deduce that the final angle must equal $180^\circ - 67^\circ - \theta = 63.7^\circ$.

The ambiguous case

You should remember from your work on trigonometric equations that there is more than one value of θ with $\sin\theta = 0.758$. Does this mean that Worked example 11.2 has more than one possible answer? Another solution of the equation $\sin\theta = 0.758$ is $180^\circ - 49.3^\circ = 130.7^\circ$.

However, as one of the other angles is 67°, this is impossible, because all three angles of the triangle must add up to 180°, and $130.7 + 67 = 197.7 > 180$.

All other possible values of θ are outside the interval $(0^\circ, 180^\circ)$, so cannot be angles of a triangle. In this example, there is only one possible value of angle θ.

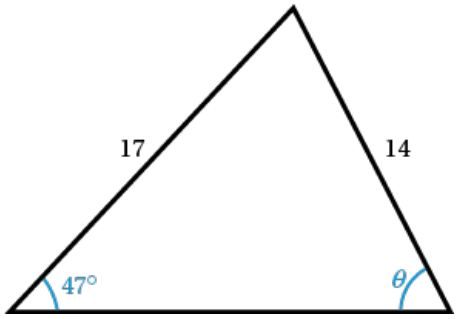
Worked example 11.3 shows that this is not always the case.

Rewind

See Chapter 10, Section 5 for more on solving trigonometric equations.

WORKED EXAMPLE 11.3

Find the size of the angle marked θ, giving your answer to the nearest degree.



$\frac{17}{\sin\theta} = \frac{14}{\sin 47^\circ}$

Use the sine rule with the two given sides.

$\therefore \sin\theta = \frac{17\sin 47^\circ}{14} = 0.888\ldots$

$\sin^{-1} 0.888\ldots = 62.6^\circ$ (nearest degree)

Find the two possible values of θ.

$\therefore \theta = 62.6^\circ \text{ or } 180^\circ - 62.6^\circ$
$= 117.4^\circ$

$62.6 + 47 = 109.6 < 180$

$117.4 + 47 = 164.4 < 180$

Check whether both solutions are possible: do the two angles add up to less than 180°?

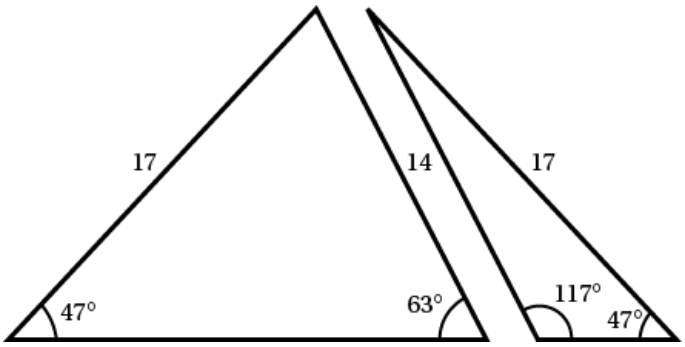
$\therefore \theta = 63^\circ \text{ or } 117^\circ$

Both solutions are possible.

Tip

The question will often alert you to look for two possible answers. For example by specifying that θ is obtuse. However, if it doesn't, you should check whether the second solution is possible by finding the sum of the known angles.

This diagram shows the two possible triangles. In both triangles, length 14 is opposite angle 47°, with another side having length 17. As illustrated, if the two triangles are placed adjacent to each other, together they form an isosceles triangle with base angle 47° and matched sides of length 17.



Key point 11.2

When using the sine rule to find an angle, there may be two possible solutions.

WORK IT OUT 11.1

In triangle *ABC*, $AB = 10\text{cm}$, $AC = 12\text{cm}$ and angle $ABC = 70^\circ$. Find the size of angle *ACB*.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$$\frac{\sin 70^\circ}{12} = \frac{\sin x}{10}$$

$$\sin x = \frac{10 \sin 70^\circ}{12} = 0.783$$

$$x = \sin^{-1}(0.783) = 51.5^\circ$$

Solution 2

$$\frac{\sin x}{12} = \frac{\sin 70^\circ}{10}$$

$$\sin x = 1.13$$

There are no solutions.

Solution 3

$$\frac{\sin 70^\circ}{12} = \frac{\sin x}{10}$$

$$\sin x = \frac{10 \sin 70^\circ}{12} = 0.783$$

$$\sin^{-1}(0.783) = 51.5^\circ$$

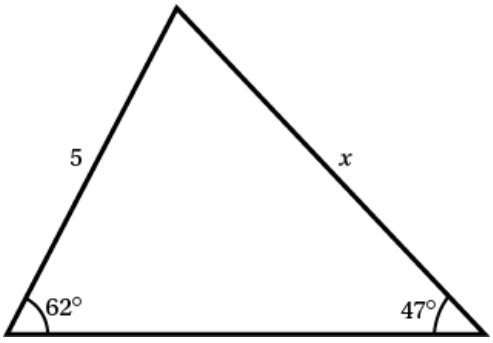
So $x = 51.5^\circ$ or 128°

EXERCISE 11A

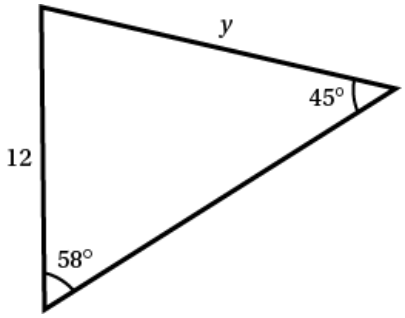
Unless stated otherwise, give your answers to three significant figures.

1 Find the lengths of the sides marked with letters.

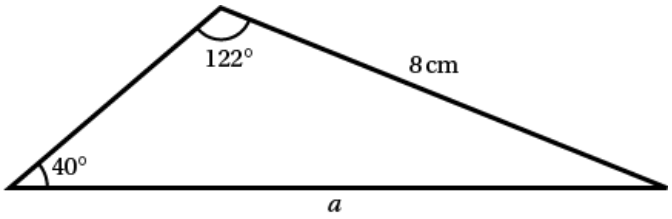
a i



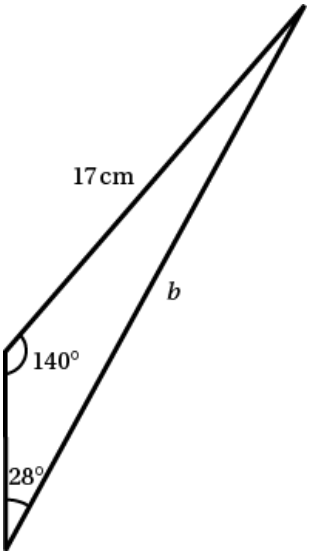
ii



b i

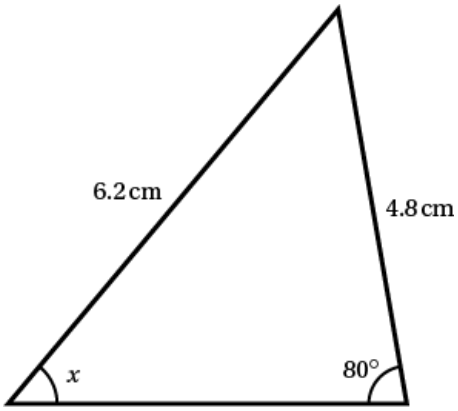


ii

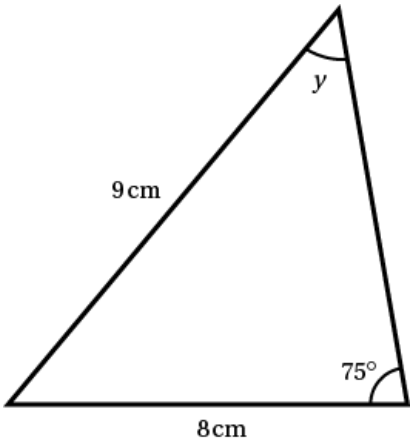


2 Find the angles marked with letters, checking whether there is more than one solution.

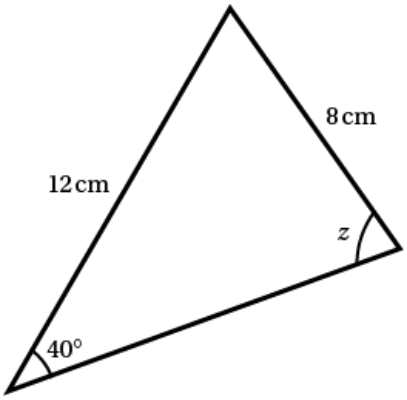
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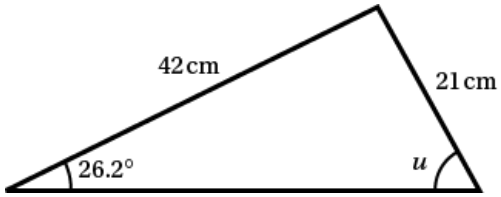
ii


y
9 cm
75°
8 cm

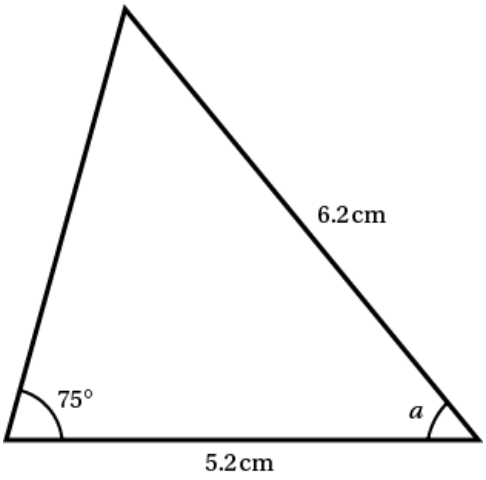
b i


8 cm
12 cm
z
40°

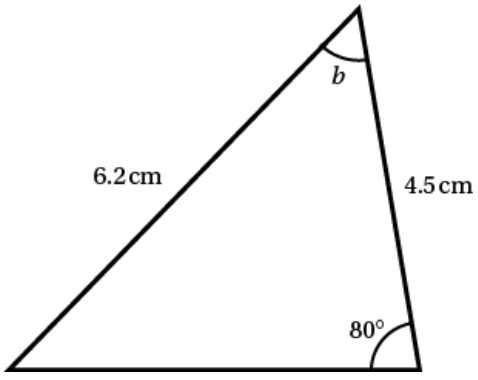
ii


42 cm
21 cm
26.2°
u

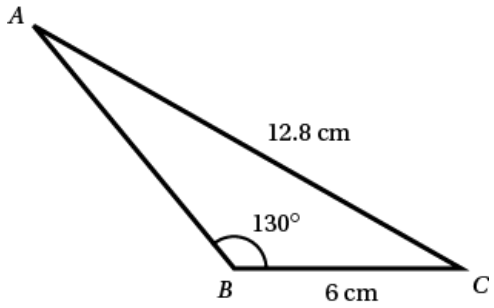
c i


6.2 cm
75°
a
5.2 cm

ii

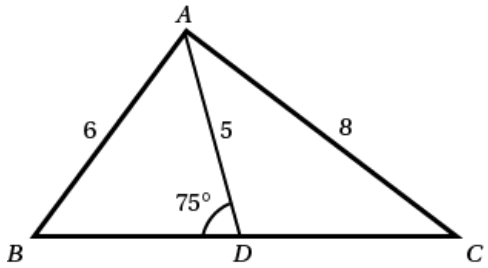

b
6.2 cm
4.5 cm
80°

3 Find all the unknown sides and angles of triangle *ABC*.



4 In triangle *ABC*, $AB = 6$ cm, $BC = 8$ cm, $ACB = 35^\circ$. Show that there are two possible triangles with these measurements and find the remaining side and angles for each.

5 In the triangle shown in the diagram, $AB = 6$, $AC = 8$, $AD = 5$ and angle $ADB = 75^\circ$. Find the length of the side BC.

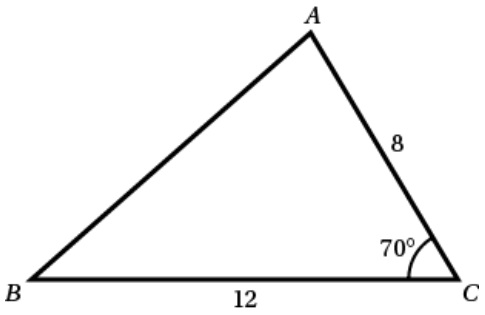


6 A balloon is tethered to a peg in the ground by a $20\ \mathrm{m}$ string, which makes an angle of 72° to the horizontal. An observer on the same side of the peg as the balloon notes that the angle of elevation from him to the balloon is 41° and his angle of depression to the peg is 10°. Find the horizontal distance of the observer from the peg.

7 Show that it is impossible to draw a triangle *ABC* with $AB = 12$ cm, $AC = 8$ cm and angle $ABC = 47^\circ$.

Section 2: The cosine rule

If you have two sides and the angle between them or all three side lengths, you **cannot** use the sine rule. For example, can you find the length of the side AB in the triangle shown in the diagram?

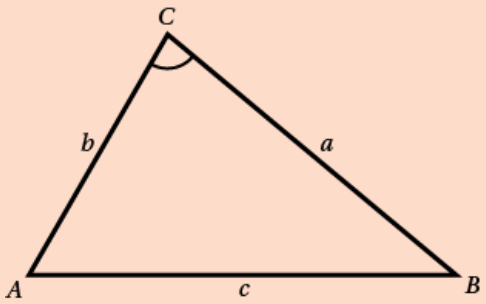


The sine rule for this triangle says: $\dfrac{AB}{\sin 70^\circ} = \dfrac{8}{\sin B} = \dfrac{12}{\sin C}$

But you do not know either of the angles B or C, so it is impossible to find AB from this equation. We need a different strategy.

Key point 11.3

The cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$



There is nothing special about the letters a, b and c in the formula in Key point 11.3. You can use any variables you like. The important thing is that the angle in 'cos C' is opposite the side on the left-hand side of the equation.

Tip

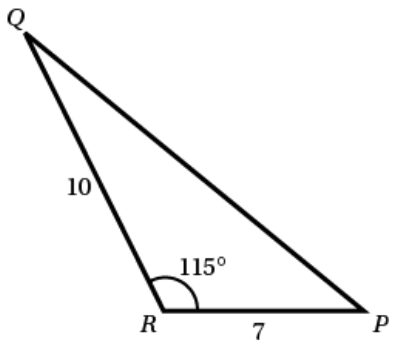
See Focus on … Proof 2 for the proof of the cosine rule.

Focus on …

See Focus on … Proof 2 for the proof.

WORKED EXAMPLE 11.4

Find the length of the side PQ.



$PQ^2 = 7^2 + 10^2 - 2 \times 7 \times 10 \times \cos 115^\circ$

When you are given two sides and the angle between them, use the cosine rule.

$PQ^2 = 208.16\ldots$

$PQ = \sqrt{208.16}\ldots = 14.4 (3\,\text{s.f.})$

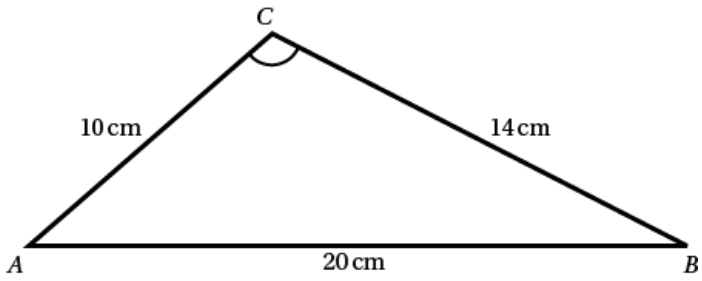
You can also use the cosine rule to find an angle if you know all three sides of a triangle. To help with this, there is a rearrangement of the cosine rule.

Key point 11.4

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

WORKED EXAMPLE 11.5

Find the size of the angle *ACB* correct to the nearest degree.



$$\cos C = \frac{196 + 100 - 400}{2 \times 14 \times 10}$$
$$= -\frac{104}{280}$$
$$= -0.371\ldots$$

As you don't know any angles, use the cosine rule.

$C = \arccos(-0.371\ldots) = 112^\circ$ (nearest degree)

Tip

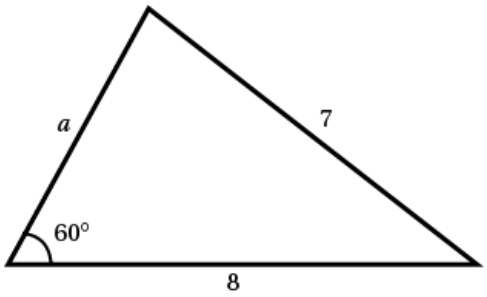
Remember that arccos (x) is another way of writing $\cos^{-1} x$.

When using the cosine rule there is no second solution for the angle. This is because the equation cos $x = c$ has only one solution between 0° and 180°.

It is possible to use the cosine rule even when the given angle is not opposite the required side. You may need to solve a quadratic equation.

WORKED EXAMPLE 11.6

Find the possible lengths of the side marked a.



$7^2 = a^2 + 8^2 - 2a \times 8 \cos 60^\circ$	As all three sides are involved in the question, you can use the cosine rule. The known angle is opposite the side marked 7.
$49 = a^2 + 64 - 8a$	$\cos 60^\circ = \frac{1}{2}$
$a^2 - 8a + 15 = 0$	Solve the quadratic equation.
$(a - 3)(a - 5) = 0$	
$a = 3$ or 5	

It is also possible to answer Worked example 11.6 using the sine rule twice, first to find the angle opposite the side marked 8, and then to find side a. Try to see if you can get the same answers.

Worked example 11.7 illustrates how to select which of the two rules to use. For both the sine and cosine rules, you need to know three measurements in a triangle to find a fourth one.

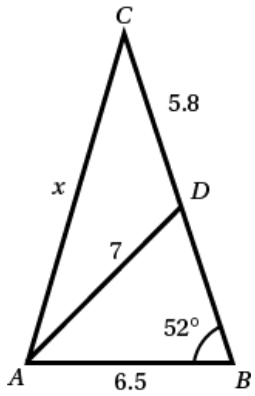
Worksheet

For a further example of choosing between the sine and cosine rules, and for more practice questions, see Support sheet 11.

WORKED EXAMPLE 11.7

In the triangle shown in the diagram, $AB = 6.5$ cm, $AD = 7$ cm, $CD = 5.8$ cm, angle $ABC = 52°$ and $AC = x$. Find the value of x correct to one decimal place.

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Sine rule in triangle ABD; let angle $ADB = \theta$:

$$\frac{6.5}{\sin\theta} = \frac{7}{\sin 52^\circ}$$

$$\sin\theta = \frac{6.5\sin 52^\circ}{7} = 0.7317\ldots$$

$$\theta = \sin^{-1} 0.7317\ldots = 47.0^\circ (1\text{ d. p.})$$

The only triangle in which you know three measurements is *ABD*. You know two side lengths and an angle opposite one of these, so use the sine rule to find angle *ADB*.

$180^\circ - 47.0^\circ = 133^\circ, 133^\circ + 52^\circ > 180^\circ$

$\therefore$ there is only one solution: $\theta = 47.0^\circ$

Are there two possible solutions?

$ADC = 180^\circ - 47.0^\circ = 133^\circ$

In triangle *ADC*, you know two sides and want to find the third. If you knew angle *ADC*, you could use the cosine rule. But you can find this angle easily.

Cosine rule in triangle ADC:

$$x^2 = 7^2 + 5.8^2 - 2 \times 7 \times 5.8\cos 133^\circ$$

$$x^2 = 138$$

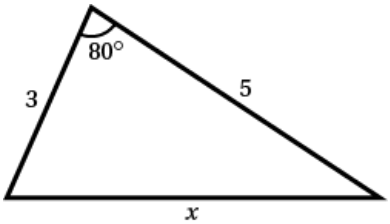
$$x = \sqrt{138} = 11.7\text{cm}(2\text{ d.p.})$$

Now use the cosine rule.

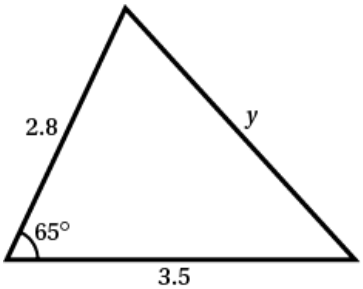
EXERCISE 11B

1 Find the lengths of the sides marked with letters.

a i

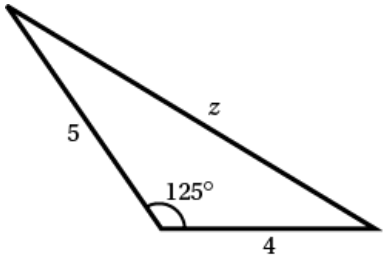


ii

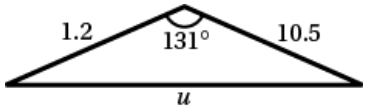


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b i

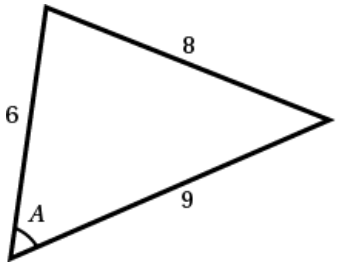


ii

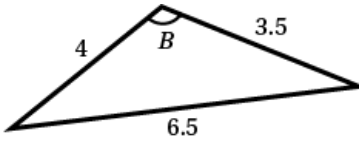


2 Find the angles marked with letters.

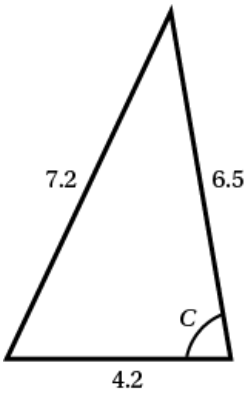
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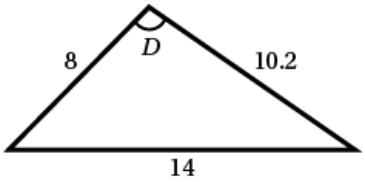
ii



b i



ii



3 **a** Triangle *PQR* has sides $PQ = 8$ cm, $QR = 12$ cm, $RP = 7$ cm. Find the size of the largest angle.

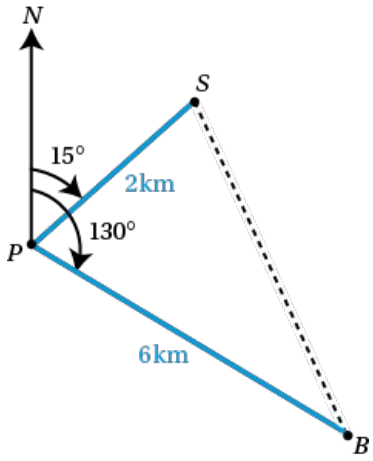
b Triangle *ABC* has sides $AB = 4.5$ cm, $BC = 6.2$ cm, $CA = 3.7$ cm. Find the size of the smallest angle.

Tip

The largest angle in a triangle is opposite the longest side.

4 Ship S is 2 km from the port on a bearing of 15° and boat B is 6 km from the port on a bearing of 130°, as illustrated in the diagram.

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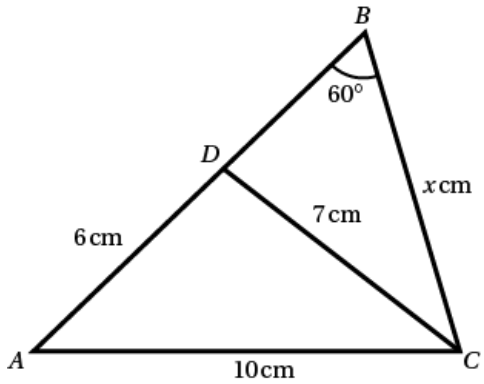
Find the distance between the ship and the boat.

Gateway to A Level

For a reminder of bearings, see Gateway to A Level section N.

5 A cyclist rides from H for 15 km on a bearing of 55° until she reaches A. She then changes direction and rides for 25 km on a bearing of 160 from A to B. Find her distance from H when she is at B.

6 Find the value of x in this diagram.



7 In triangle *ABC*, $AB = (x - 3)$ cm, $BC = (x + 3)$ cm, $AC = 8$ cm and $\angle BAC = 60^\circ$. Find the value of x.

8 The longest side of a triangle has length $(3x - 10)$ cm. The other two sides have lengths $(x - 2)$ cm and $(x + 2)$ cm. The largest angle is 120°. Find the value of x.

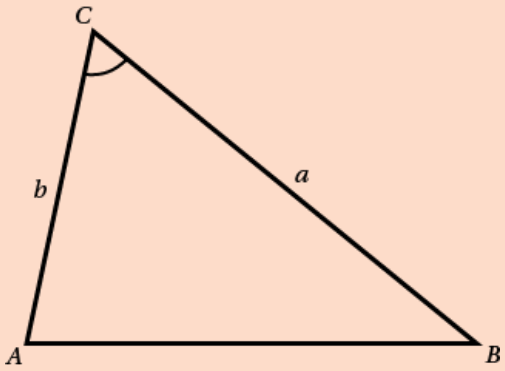
9 In triangle *KLM*, $KL = 4$, $LM = 7$ and angle $LKM = 45^\circ$. Find the exact length of KM.

Section 3: Area of a triangle

One way to calculate the area of a triangle is $\frac{1}{2}$ base × height. There is another formula using two sides and the angle between them.

Key point 11.5

The area of the triangle is given by: $\text{Area} = \frac{1}{2}ab\sin C$

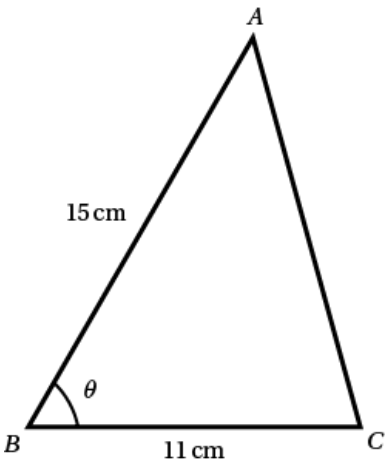


Explore

There is also a formula for the area of a triangle if you know all three of its sides. Find out about Heron's formula.

WORKED EXAMPLE 11.8

The area of the triangle shown in the diagram is $52\ \text{cm}^2$. Find the two possible values of angle *ABC*, correct to one decimal place.



$\frac{1}{2}(11 \times 15)\ \sin\theta = 52$

Use formula for the area of a triangle.

$\sin\theta = \frac{2 \times 52}{11 \times 15} = 0.630\ldots$

$\theta = \sin^{-1} 0.630\ldots = 39.1^\circ (1\,\text{d.p.})$

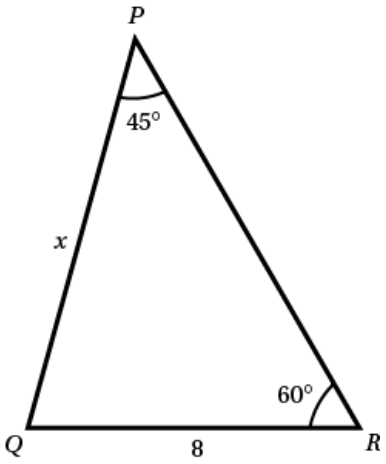
$\theta = 39.1^\circ \text{ or } 180^\circ - 39.1^\circ = 140.9^\circ (1\,\text{d.p.})$

Remember that with sin there are two possible answers.

Worked example 11.9 combines the sine rule with the area of the triangle, working with exact values.

WORKED EXAMPLE 11.9

For triangle *PQR* shown in the diagram:



a calculate the exact value of x

b find the area of the triangle.

$\frac{8}{\sin\ 45^\circ} = \frac{x}{\sin\ 60^\circ}$

Since you know two angles and a side, use the sine rule.

$\frac{8}{\frac{\sqrt{2}}{2}} = \frac{x}{\frac{\sqrt{3}}{2}}$

$\frac{16}{\sqrt{2}} = \frac{2x}{\sqrt{3}}$

$x = \frac{16\sqrt{3}}{2\sqrt{2}}$

$x = 4\sqrt{6}$

You need the exact value of x, so use $\sin 45^\circ = \frac{\sqrt{2}}{2}$ and $\sin\ 60^\circ = \frac{\sqrt{3}}{2}$.

$PQR = 180 - 60 - 45 = 75^\circ$

$\text{Area} = \frac{1}{2}(8 \times 4\sqrt{6})\ \sin\ 75^\circ$

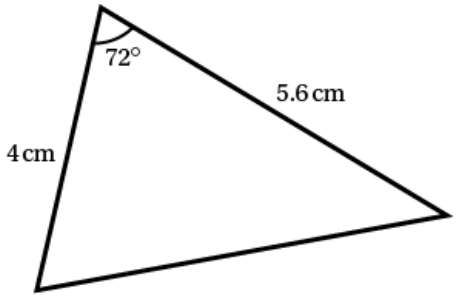
$= 37.9 (3\ s.\ f.)$

To use the formula for the area of the triangle, you need angle *PQR*.

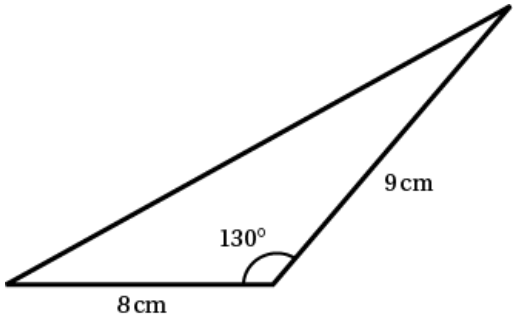
EXERCISE 11C

1 Calculate the areas of these triangles.

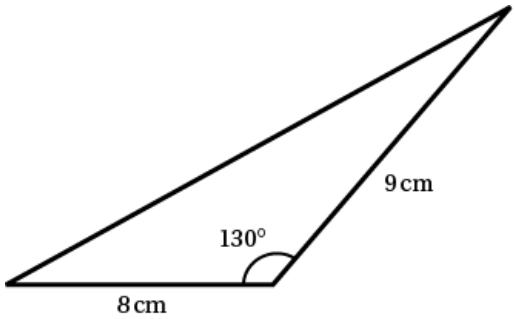
a i



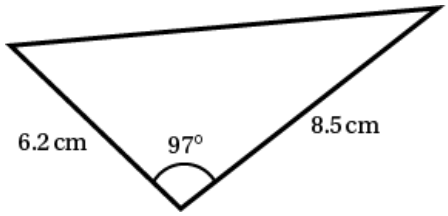
ii



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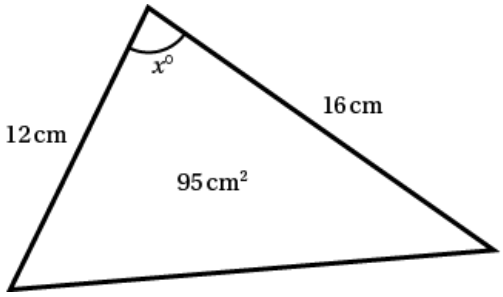


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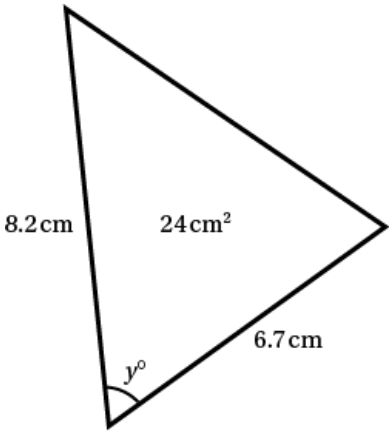


2 Each triangle has the area shown. Find two possible values of each marked angle.

a



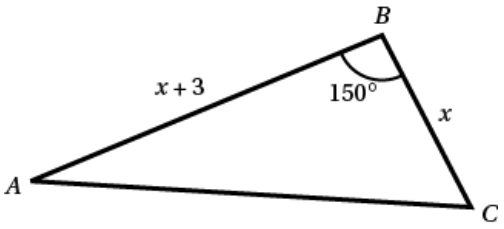
b



3 In triangle *LMN*, $LM = 12\,\text{cm}, MN = 7\,\text{cm}$, and $LMN = 135^\circ$. Find the length of the side LN and the area of the triangle.

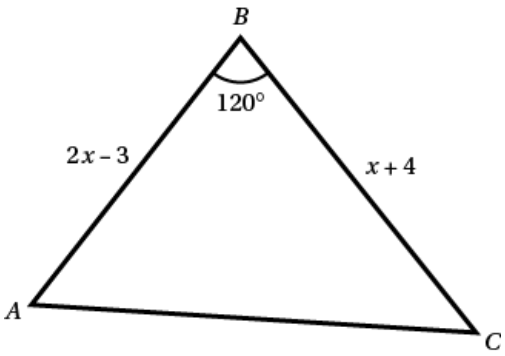
4 An equilateral triangle has area $25\sqrt{3}\,\text{cm}^2$. Find the length of each side.

5 In triangle *ABC*, $AB = x + 3$, $BC = x$ and angle $ABC = 150^\circ$.



The area of the triangle is 10. Find the value of x.

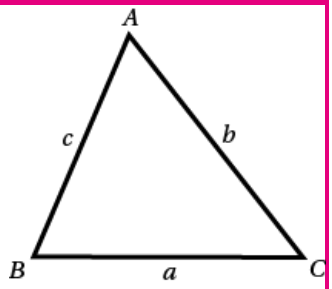
6 In triangle *ABC*, $AB = 2x - 3$, $BC = x + 4$ and angle $ABC = 120^\circ$.



The area of the triangle is $39\sqrt{3}$. Find the value of x.

7 In triangle *PQR*, $PQ = 8\,\text{cm}$, $RQ = 7\,\text{cm}$, and $RPQ = 60^\circ$. Find the exact difference in areas between the two possible triangles.

Checklist of learning and understanding

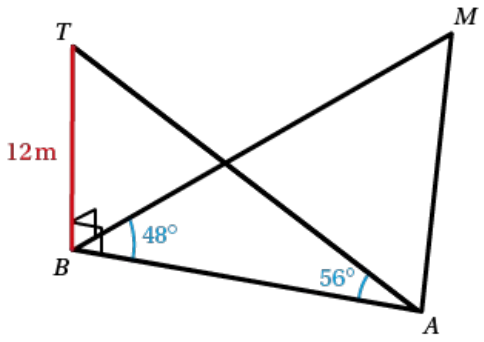


- To find a side when two angles and a side are given, or an angle when two sides and a non-included angle are given, use the **sine rule**:
  - $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
- When using the sine rule to find an angle, there may be two possible answers: A and $180 - A$.
- To find a side when two sides and the angle between them are given, or an angle when all three sides are given, use the cosine rule:
  - $c^2 = a^2 + b^2 - 2ab\cos C$
  - $\cos C = \frac{a^2+b^2-c^2}{2ab}$
- To find the area of a triangle when two sides and the included angle are known, use the formula:
  - $\text{Area} = \dfrac{1}{2}ab\sin C$

Mixed practice 11

1 In triangle *ABC*, $AB = 6.2$ cm, $CA = 8.7$ cm and angle $ACB = 37.5°$. Find the two possible values of *ABC*.

2 A vertical tree of height 12 m stands on horizontal ground. The bottom of the tree is at the point B. Observer A, standing on the ground, sees the top of the tree at an angle of elevation of 56°.

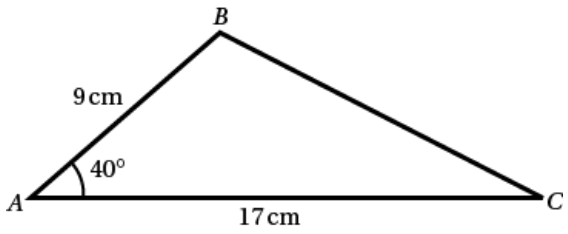


a Find the distance of A from the bottom of the tree.

Another observer, M, stands the same distance away from the tree, and $ABM = 48°$

b Find the distance AM.

3 The diagram shows triangle *ABC*, with $AB = 9$ cm, $AC = 17$ cm and angle $BAC = 40°$.



a Find the length of BC.

b Find the area of triangle *ABC*.

c D is a point on AC such that angle $BDA = 63°$. Find the length of BD.

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4 The lengths of the three sides of a triangle are 6.4 cm, 7.0 cm and 11.3 cm.

a Find the largest angle in the triangle.

b Find the area of the triangle.

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5 In triangle *ABC*, $AB = 2\sqrt{3}, AC = 10$ and angle $BAC = 150^{\circ}$. Find the exact length of BC.

6 In the obtuse angled triangle *KLM*, $LM = 6.1$ cm, $KM = 4.2$ cm and angle $KLM = 42°$.

Find the area of the triangle.

7 In triangle *ABC*, angle $AB = 10$ cm, $BC = 8$ cm and $CA = 7$ cm.

a Find the exact value of $\cos(\angle ABC)$

b Find the exact value of $\sin(\angle ABC)$.

c Find the exact value of the area of the triangle.

8 In triangle *ABC*, $AB = 5$, $AC = x$ and the angle at A is θ. M is the midpoint of the side AC.

a Use the cosine rule to find an expression for MB2 in terms of x and θ.

b Given that $BC = MB$, show that $\cos\theta = \dfrac{3x}{20}$.

c If $x = 5$, find the value of the angle θ such that $MB = BC$.

9 Two radar stations, A and B, are 20 km apart. B is due east of A. Station B detects a ship on a bearing of 310°. The same ship is 15 km from station A.

a Find the two possible bearings of the ship from station A.

b Hence find the distance between the two possible positions of the ship.

10 A regular pentagon has area $200\,\text{cm}^2$. Find the length of each side.

11 In triangle *ABC*, $AB = 10$, $BC = 5$, $CA = x$ and angle $CAB = \theta$.

a Show that $x^2 - 20x\cos\theta + 75 = 0$.

b Find the range of values of $\cos\theta$ for which the equation in part a has real roots.

c Hence find the set of values of θ for which it is possible to construct triangle *ABC* with the given measurements.

Worksheet

See Extension sheet 11 for challenge questions involving triangle geometry.

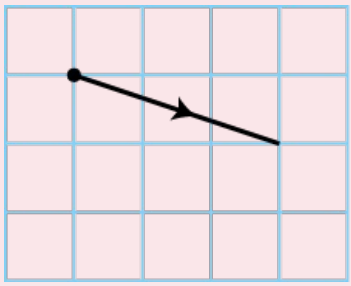
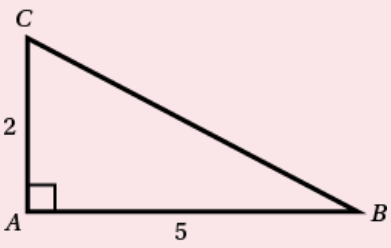
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12 Vectors

In this chapter you will learn how to:

- represent vectors using the base vectors i and j
- find the magnitude and direction of a vector
- perform algebraic operations on vectors given in different forms
- recognise when two vectors are parallel
- find unit vectors
- work with positions and displacements of points in the plane
- use vectors to solve problems about geometrical figures.

Before you start...

GCSE	You should know how to represent vectors on a grid and write them as column vectors.	1 Write this as a column vector: 
GCSE	You should know how to use Pythagoras' theorem and trigonometry in right-angled triangles.	2 Find the length of the side BC and the size of the angle ABC. 
Chapter 3	You should know how to solve quadratic equations, and recognise when a quadratic equation has no solutions.	3 State the number of solutions of the equation $6x^2 + 9x + 1 = 0$.

Where can we use vectors?

A **vector** is a quantity that has both **magnitude** (size) and **direction**. You may be familiar with examples such as velocity and force. By contrast, **scalar** quantities (quantities such as mass, that have magnitude but no direction) can be used fully described by a single number. In pure mathematics, vectors can be used to represent displacement from one point to another, and thus to describe

geometrical figures. They also have many applications in spatial modelling problems, for example describing flight paths or positions of characters in a computer game.

In this chapter you will learn about different ways to represent vectors and use them to solve geometrical problems.

Fast forward

In Chapters 21 and 22 you will use vectors to work with forces.

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Section 1: Describing vectors

You already know that there are two ways to describe a vector. You can draw an arrow, or write it as a column vector.

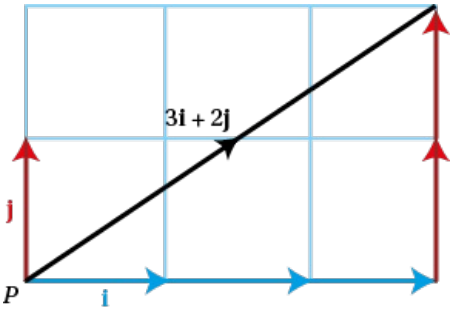
- The arrow shows the magnitude and the direction of the vector explicitly.
- The numbers in a column vector are called the **components** of the vector. For example, the column vector $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$ has horizontal component 5 and vertical component -3.

Gateway to A Level

For a reminder of column vector notation see Gateway to A Level section Q.

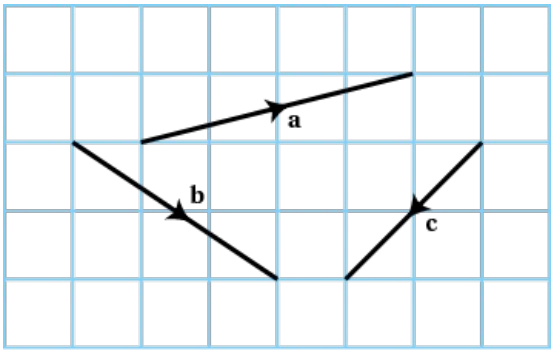
Another way of writing a vector in **component form** is using the $\mathbf{i}$ and $\mathbf{j}$ notation. Vectors of length one unit in the horizontal and vertical directions are denoted by $\mathbf{i}$ and $\mathbf{j}$ respectively. These vectors are called **base vectors**. You can then express any vector in terms of $\mathbf{i}$ and $\mathbf{j}$, as shown in the diagram. $2\mathbf{i}$ and $3\mathbf{j}$ are called **component vectors** of $2\mathbf{i} + 3\mathbf{j}$.

To emphasise that something is a vector, rather than a scalar (number), we use bold type (for example, $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$). When writing by hand, you should **underline** the letters representing vectors.



WORKED EXAMPLE 12.1

a Write vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ using $\mathbf{i}, \mathbf{j}$ notation.



b Write the following as column vectors: $\mathbf{p} = 4\mathbf{i} - 4\mathbf{j}$ $\mathbf{q} = 2\mathbf{j}$ $\mathbf{r} = \mathbf{j} - 4\mathbf{i}$

a $\underline{a} = 4\underline{i} + \underline{j}$

$\underline{b} = 3\underline{i} - 2\underline{j}$

$\underline{c} = -2\underline{i} - 2\underline{j}$

The coefficient of $\mathbf{i}$ represents the number of horizontal units and the coefficient of $\mathbf{j}$ the number of vertical units. Positive directions are to the right and up.

b $\underline{p} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$

This means 4 units in the horizontal direction and -4 in the vertical direction.

$\underline{q} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

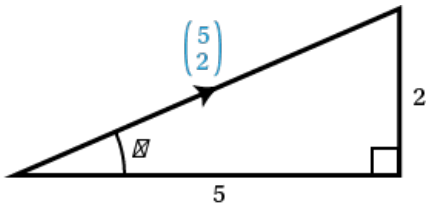
There are zero units in the $\mathbf{i}$-direction, so the horizontal component is 0.

$\underline{\mathrm{r}} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

Be careful: $\mathbf{i}$ and $\mathbf{j}$ are given in the opposite order to normal!

Magnitude and direction

If you are given a vector in component form (written to show its components, either as a column vector or using $\mathbf{i}, \mathbf{j}$ base vectors) you can find its magnitude and direction.



The magnitude of the vector is its length, which and you can find by using Pythagoras' theorem. The direction is described by giving an angle.

Tip

When finding the direction of a column vector it is important to draw a diagram to be sure that the correct angle is being calculated.

Key point 12.1

The **magnitude (modulus)** of a vector is its length.

If $\mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$ then $|\mathbf{a}| = \sqrt{p^2 + q^2}$.

The **direction** of a vector is the angle it makes with the positive horizontal direction. It can be found from the right-angled triangle formed by the vector and its components.

The magnitude of a vector can never be negative. The only vector with zero magnitude is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ called the zero vector.

Tip

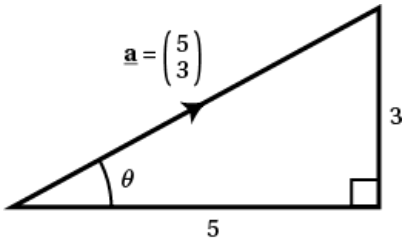
By convention, the x-axis is called the horizontal axis and the y-axis the vertical axis, even if they are both in the horizontal plane.

WORKED EXAMPLE 12.2

a Find the magnitude of the vector $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j}$ and the angle it makes with the direction of $\mathbf{i}$.

b Find the direction of the vector $\mathbf{b} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$.

a



Draw a diagram and mark the required angle. The direction of $\mathbf{i}$ is to the right.

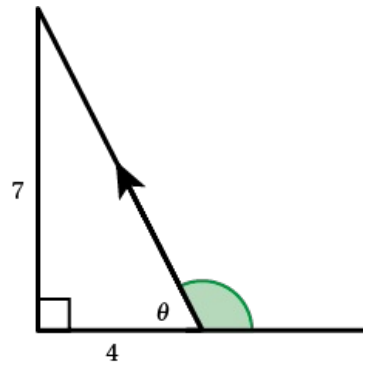
$|\underline{a}| = \sqrt{25+9}$
$= \sqrt{34}$

Use Pythagoras' theorem to find the length (magnitude).

$\tan\theta = \frac{3}{5} (3 \text{ s.f.})$
$\theta = 31.0^\circ$

The direction is at 31°.

b



The required angle is marked in green.

$\tan\theta = \frac{7}{4}$
$\theta = 60.3^\circ$

Use a right-angled triangle to find θ first.

The direction is at $180 - \theta = 120^\circ$.

WORK IT OUT 12.1

Find the direction of the vector $-5\mathbf{i} + 2\mathbf{j}$.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$\tan^{-1}\left(\frac{2}{-5}\right) = -21.8^\circ$

So the angle is 21.8°.

Solution 2

$\tan^{-1}\left(\frac{2}{5}\right) = 21.8^\circ$

$180 - 21.8 = 158$

So the angle is 158°.

Solution 3

$\tan^{-1}\left(\frac{5}{2}\right) = 68.2^\circ$

$180 - 68.2 = 112$

So the angle is 112°.

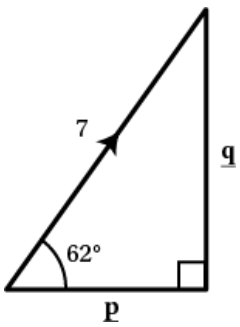
You can also use trigonometry to find the components of a vector if you know its magnitude and direction.

Fast forward

A This method will be particularly useful when you learn about resolving forces in the mechanics section in Student Book 2.

WORKED EXAMPLE 12.3

Vector $\mathbf{v}$ has magnitude 7 and has direction 62°. Write $\mathbf{v}$ as a column vector.



Draw a diagram showing the given information and the components (labelled $\mathbf{p}$ and $\mathbf{q}$).

$\underline{v} = \begin{pmatrix} \underline{p} \\ \underline{q} \end{pmatrix}$

You have a right-angled triangle with a known hypotenuse and one angle, so use trigonometry.

$\underline{p} = 7\cos 62^\circ = 3.29$

$\underline{q} = 7\sin 62^\circ = 6.18$

$\therefore \underline{v} = \begin{pmatrix} 3.29 \\ 6.18 \end{pmatrix}$

Unit vectors

Vectors of length one are often very useful, so are given a special name.

Key point 12.2

A **unit vector** is a vector whose magnitude is one.

Tip

You have already met two special examples of unit vectors: the base vectors $\mathbf{i}$ and $\mathbf{j}$.

WORKED EXAMPLE 12.4

a Which of the following is a unit vector?

$$\mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$$

b Find two possible values of t such that $\mathbf{c} = (12t)\mathbf{i} - (9t)\mathbf{j}$ is a unit vector.

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a $|\underline{a}| = \sqrt{1+4}$
$= \sqrt{5} \neq 1$

Use Pythagoras' theorem to find the magnitudes.

$|\underline{b}| = \sqrt{\frac{1}{3} + \frac{2}{3}}$
$= 1$

$\underline{b}$ is a unit vector.

A unit vector has magnitude one.

b $|\underline{c}| = \sqrt{144t^2 + 81t^2}$
$= \sqrt{225t^2}$

Find the magnitude of $\mathbf{c}$ in terms of t ...

$|\underline{c}| = 1$

... and make it equal to 1.

$\Rightarrow \sqrt{225t^2} = 1$

Square both sides of the equation.

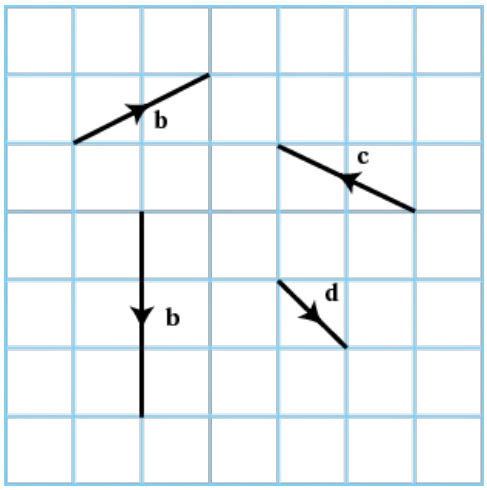
$\Rightarrow 225t^2 = 1$

$\Rightarrow t^2 = \frac{1}{225}$

$\Rightarrow t = \pm\frac{1}{15}$

EXERCISE 12A

1 Write the following vectors using $\mathbf{i}, \mathbf{j}$ notation:



2 Represent the following vectors on a grid:

a i $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$

ii $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$

b i $-4\mathbf{i} + 3\mathbf{j}$

ii $2\mathbf{i} - \mathbf{j}$

c i $6\mathbf{j}$

ii $-2\mathbf{i}$

3 For each vector from question 2 find:

A the magnitude

B the direction (anticlockwise angles are positive)

C the angle it makes with the direction of $\mathbf{j}$.

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4 Decide which of the following are unit vectors:

a $\frac{1}{2}\mathbf{i} + \frac{1}{3}\mathbf{j}$

b $-\mathbf{j}$

c $(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$

d $(3\cos\theta)\mathbf{i} + (3\sin\theta)\mathbf{j}$

5 Find the values of c such that $(2c)\mathbf{i} + (5c)\mathbf{j}$ is a unit vector.

6 $\begin{pmatrix} 3k \\ -2k \end{pmatrix}$ is a unit vector. Find the possible values of k.

7 Find the values of t such that the vector $\begin{pmatrix} 4t+10 \\ -2k \end{pmatrix}$ has magnitude $2\sqrt{10}$.

8 Vector $\mathbf{v}$ has magnitude 12 and is 27° clockwise from the $\mathbf{j}$-direction. Find:

a the direction of $\mathbf{v}$

b the components of $\mathbf{v}$.

9 Vector $\mathbf{b}$ has magnitude $2\sqrt{3}$ and direction 150°. Write $\mathbf{b}$ as a column vector, giving your answers in surd form.

Section 2: Operations with vectors

In order to use vectors to solve problems you need to be able to perform some algebraic operations with them: adding, subtracting and multiplying by a scalar.

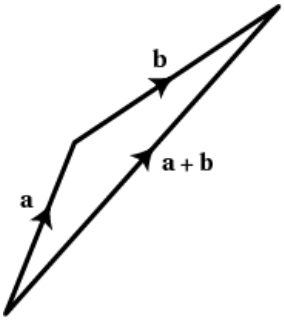
Fast forward

If you study Further Mathematics, in Pure Core Student Book 1 you will learn two different ways of multiplying one vector by another.

Adding vectors

If two vectors are represented by arrows then you can perform vector addition by joining the starting point of the second vector to the end point of the first. If the vectors are given in component form, you just add the corresponding components. The sum of two vectors is also called the **resultant vector**.

What if the two vectors are not in this position? Remember that vectors represent a length in a given direction, but don't tell you anything about **where** this length actually is. So vectors can be 'moved around' as long as the magnitude and direction remain unchanged.



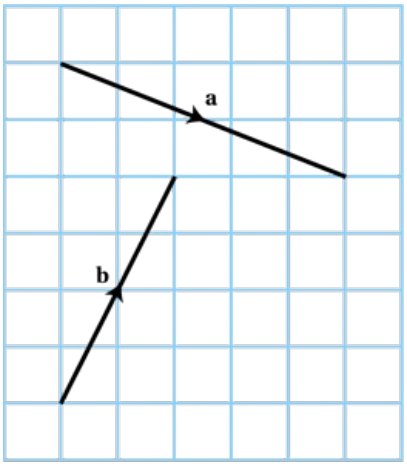
Tip

Two vectors do not have to start and finish at the same points to be equal. They just need to have the same length and point in the same direction.

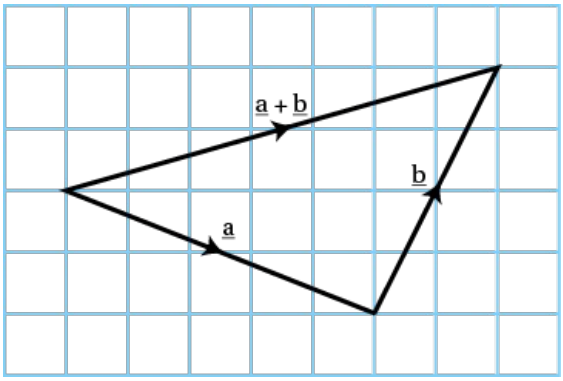
WORKED EXAMPLE 12.5

a Add vectors $\mathbf{a}$ and $\mathbf{b}$ as arrows on the grid.

b Write $\mathbf{a}$ and $\mathbf{b}$ as column vectors and hence find $\mathbf{a} + \mathbf{b}$ as a column vector.



a



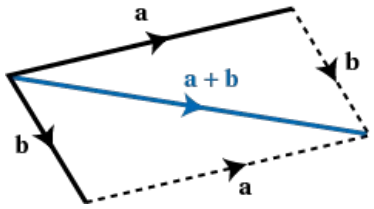
b $\underline{a} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$\underline{a} + \underline{b} = \begin{pmatrix} 5+2 \\ -2+4 \end{pmatrix}$

$= \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

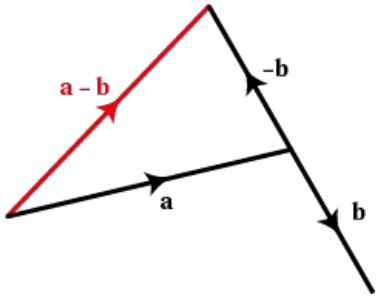
Add the components.

Another way of visualising vector addition is as a diagonal of the parallelogram formed by the two vectors. In this case the vectors are moved so that they have a common starting point.



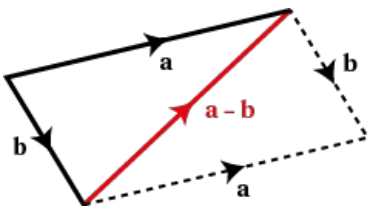
Subtracting vectors

Making a vector negative reverses the direction of the vector. Subtracting a vector is the same as adding its negative. So to subtract two vectors you need to reverse the direction of the second vector and then add it to the first. In component form you subtract corresponding components.



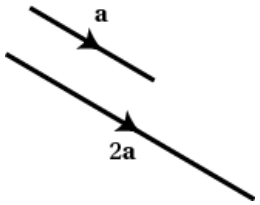
$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

The difference of two vectors can also be represented by the diagonal of the parallelogram formed by the two vectors: the arrow points from the end of $\mathbf{b}$ towards the end of $\mathbf{a}$.



Scalar multiplication (multiplying a vector by a number) changes the magnitude (length) of the vector, leaving the direction the same. If the scalar is negative the vector will point in the opposite direction (but

still along the same line). In component form, each component is multiplied by the scalar.



$$2\begin{pmatrix}5\\-3\end{pmatrix}=\begin{pmatrix}10\\-6\end{pmatrix}$$

Tip

In this paragraph the word 'direction' seems ambiguous: in the first instance it refers to the line along which the arrow lies, and in the second to the way the arrow is pointing. You need to be clear which meaning is being used.

WORKED EXAMPLE 12.6

Given the vectors $\mathbf{a}=\begin{pmatrix}4\\-3\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}-1\\1\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}11\\-9\end{pmatrix}$, find scalars k and m such that $k\mathbf{a}-m\mathbf{b}=\mathbf{c}$.

$k\underline{a}-m\underline{b}=\begin{pmatrix}4k\\-3k\end{pmatrix}-\begin{pmatrix}-m\\m\end{pmatrix}$
$=\begin{pmatrix}4k+m\\-3k-m\end{pmatrix}$

Write $k\mathbf{a}-m\mathbf{b}$ in component form ...

$k\underline{a}-m\underline{b}=c$
$\begin{pmatrix}4k+m\\-3k-m\end{pmatrix}=\begin{pmatrix}11\\-9\end{pmatrix}$

... and compare it to $\mathbf{c}$.

$\begin{cases}4k+m=11\\-3k-m=-9\end{cases}$

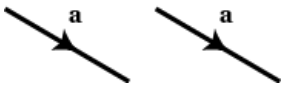
Both components have to be equal.

$\Rightarrow k=2, m=3$

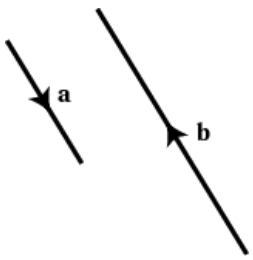
Solve simultaneous equations.

Equal and parallel vectors

Two vectors are equal if they have the same magnitude and same direction. All their components are equal. They may have different start and end points.



Parallel vectors are scalar multiples of each other.



$\begin{pmatrix}3\\-2\end{pmatrix}$ is parallel to $\begin{pmatrix}-9\\6\end{pmatrix}$ since $\begin{pmatrix}-9\\6\end{pmatrix}=-3\begin{pmatrix}3\\-2\end{pmatrix}$

Key point 12.3

If vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel then you can write $\mathbf{b} = t\mathbf{a}$ for some scalar t.

Tip

Remember that multiplying by a negative scalar will change the way the arrow is pointing.

WORKED EXAMPLE 12.7

Given vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, find the value of scalar p such that $\mathbf{a} + p\mathbf{c}$ is parallel to the vector $\begin{pmatrix} 10 \\ 23 \end{pmatrix}$.

$\underline{a} + p\underline{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -2p \\ 5p \end{pmatrix}$

$= \begin{pmatrix} 1 - 2p \\ 2 + 5p \end{pmatrix}$

You can write $\mathbf{a} + p\mathbf{c}$ as a single column vector.

Parallel to $\begin{pmatrix} 10 \\ 23 \end{pmatrix}$:

$\begin{pmatrix} 1 - 2p \\ 2 + 5p \end{pmatrix} = t \begin{pmatrix} 10 \\ 23 \end{pmatrix}$ for some scalar t.

If two vectors are parallel, then they are scalar multiples of each other.

$\Rightarrow \begin{cases} 1 - 2p = 10t \\ 2 + 5p = 23t \end{cases}$

If two vectors are equal then all their components are equal.

$\Rightarrow \begin{cases} 10t + 2p = 1 \\ 23t - 5p = 2 \end{cases}$

$\Rightarrow \begin{cases} 230t + 46p = 23 \\ 230t - 50p = 20 \end{cases}$

Solve simultaneous equations: you only need to find p, so eliminate t.

$\Rightarrow 96p = 3$

$\Rightarrow p = \frac{1}{32}$

You can use scalar multiplication to find a vector with a given magnitude in the same direction as a given vector. In particular, you can find a unit vector in a given direction.

WORKED EXAMPLE 12.8

Find two unit vectors parallel to $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Let $\underline{a} = t \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3t \\ 4t \end{pmatrix}$

If a vector is parallel to $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ you can write it as $t \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Then

A unit vector has magnitude 1.

$|\underline{a}| = 1$

$\sqrt{(3t)^2 + (4t)^2} = 1$

You can find the magnitude of a vector using Pythagoras' theorem.

$9t^2 + 16t^2 = 1^2$

$25t^2 = 1$

$t = \pm\sqrt{\frac{1}{25}}$

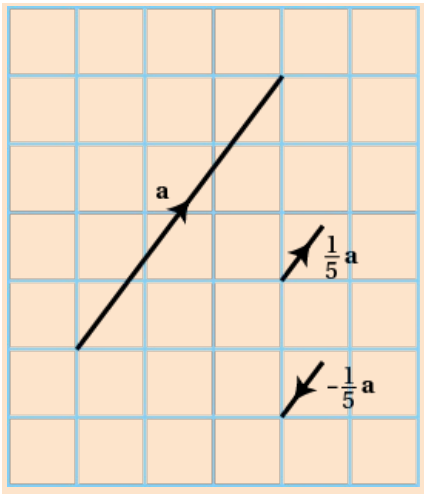
$= \pm\frac{1}{5}$

$\therefore \ \underline{a} = \pm\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$ or $\begin{pmatrix} -0.6 \\ -0.8 \end{pmatrix}$

Don't forget $\pm$ when taking a square root.

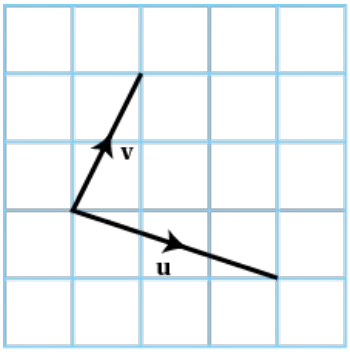
The two possible vectors are shown in this diagram.



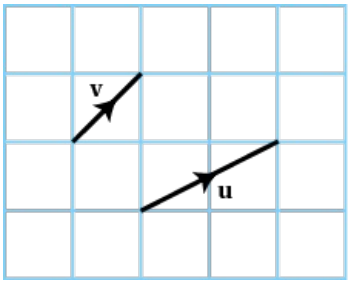
EXERCISE 12B

1 Represent vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ by arrows on the grid.

a $\mathbf{a} = \mathbf{u} + \mathbf{v}, \mathbf{b} = -2\mathbf{u}, \mathbf{c} = \mathbf{v} - \mathbf{u}$



b $\mathbf{a} = 3\mathbf{v}, \mathbf{b} = \mathbf{u} - \mathbf{v}, \mathbf{c} = \mathbf{u} + \mathbf{v}$



2 Let $\mathbf{a} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find the following vectors:

a **i** $3\mathbf{a}$

ii $4\mathbf{b}$

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b **i** $\mathbf{a}-\mathbf{b}$

ii $\mathbf{b}+\mathbf{c}$

c **i** $2\mathbf{b}+\mathbf{c}$

ii $\mathbf{a}-2\mathbf{b}$

d **i** $\mathbf{a}+\mathbf{b}-2\mathbf{c}$

ii $3\mathbf{a}-\mathbf{b}+\mathbf{c}$

3 Let $a=\mathbf{i}+2\mathbf{j}, b=\mathbf{i}$ and $c=2\mathbf{i}-\mathbf{j}$. Find the following vectors:

a **i** $-6\mathbf{b}$

ii $4\mathbf{a}$

b **i** $\mathbf{c}-\mathbf{a}$

ii $\mathbf{a}-\mathbf{b}$

c **i** $\mathbf{a}-\mathbf{b}+2\mathbf{c}$

ii $4\mathbf{c}-3\mathbf{b}$

4 Given that $a=4\mathbf{i}-2\mathbf{j}$, find the vector $\mathbf{b}$ such that:

Tip

A zero vector is a vector with magnitude equal to zero.

a $\mathbf{a}+\mathbf{b}$ is the zero vector

b $\mathbf{a}-\mathbf{b}=\mathbf{j}$

c $2\mathbf{a}+3\mathbf{b}$ is the zero vector

d $\mathbf{a}+2\mathbf{b}=3\mathbf{i}$

5 Decide which of the following vectors are parallel:

a **i** $\begin{pmatrix}-4\\1\end{pmatrix}$ and $\begin{pmatrix}-8\\2\end{pmatrix}$

ii $\begin{pmatrix}3\\-1\end{pmatrix}$ and $\begin{pmatrix}-6\\-2\end{pmatrix}$

b **i** $\begin{pmatrix}2\\5\end{pmatrix}$ and $\begin{pmatrix}4\\9\end{pmatrix}$

ii $\begin{pmatrix}3\\-1\end{pmatrix}$ and $\begin{pmatrix}-6\\-2\end{pmatrix}$

c **i** $3\mathbf{i}-4\mathbf{j}$ and $1.5\mathbf{i}-2\mathbf{j}$

ii $9\mathbf{i}-2\mathbf{j}$ and $4.5\mathbf{i}+2\mathbf{j}$

6 Given that $\mathbf{a}=\begin{pmatrix}-1\\1\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}5\\3\end{pmatrix}$ find vector $\mathbf{x}$ such that $3\mathbf{a}+4\mathbf{x}=\mathbf{b}$.

7 Given that $\mathbf{a}=3\mathbf{i}-2\mathbf{j}, \mathbf{b}=\mathbf{i}-\mathbf{j}$ and $\mathbf{c}=\mathbf{i}$, find the value of the scalar t such that $\mathbf{a}+t\mathbf{b}=\mathbf{c}$.

8 Given that $\mathbf{a}=\begin{pmatrix}2\\0\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}3\\1\end{pmatrix}$, find the value of the scalar p such that $\mathbf{a}+p\mathbf{b}$ is parallel to the vector $\begin{pmatrix}3\\2\end{pmatrix}$.

9 Given that $\mathbf{x}=2\mathbf{i}+3\mathbf{j}$ and $\mathbf{y}=4\mathbf{i}+\mathbf{j}$, find the value of the scalar λ such that $\lambda\mathbf{x}+\mathbf{y}$ is parallel to vector $\mathbf{j}$.

10 Given that $\mathbf{a} = \mathbf{i} - \mathbf{j}, \mathbf{b} = 2q\mathbf{i} + \mathbf{j}$ and that $p\mathbf{a} + \mathbf{b}$ is parallel to vector $\mathbf{i} + \mathbf{j}$, express q in terms of p.

11 Find the value of k such that the vector $\begin{pmatrix} 3k+1 \\ k-3 \end{pmatrix}$ is parallel to $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

12 Find two vectors of magnitude 20 in the same direction as $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$.

13 Find two vectors of magnitude 3 in the same direction as $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

14 Find two unit vectors in the same direction as $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Worksheet

For another example and more practice of questions like this, see Support sheet 12.

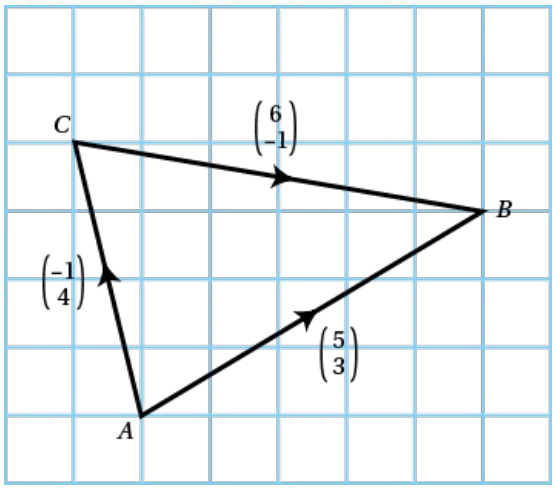
15 **a** Show that vectors $v = \begin{pmatrix} t-1 \\ 2t+1 \end{pmatrix}$ and $u = \begin{pmatrix} t^2-1 \\ 2t^2+3t+1 \end{pmatrix}$ are parallel for all values of t.

b Find the value of t for which the two vectors are equal.

Section 3: Position and displacement vectors

Vectors are used to represent displacements between points. You can think of a vector as describing how to get from one point to another.

For example, to get from A to B in this diagram you need to move 5 units to the right and 3 units up. This can be represented by the **displacement vector** $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

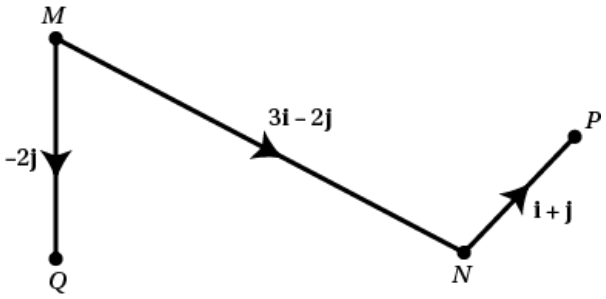


If you now have a third point, C, there are two different ways of getting from A to B: either directly or via C. Both of those journeys represent the same total displacement, so you can write $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$ or, using components, $\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ -1 \end{pmatrix}$.

- If one displacement is followed by another, the total displacement is represented by the sum of the two displacement vectors.
- Multiplying a displacement vector by a scalar represents a displacement in the same direction but different distance.
- Making a displacement negative represents travelling the same distance in the opposite direction.

WORKED EXAMPLE 12.9

The diagram shows points M, N, P, Q such that $\overrightarrow{MN} = 3\mathbf{i} - 2\mathbf{j}, \overrightarrow{NP} = \mathbf{i} + \mathbf{j}$ and $\overrightarrow{MQ} = -2\mathbf{j}$.



a Write the following vectors in component form:

i $\overrightarrow{MP}$

ii $\overrightarrow{PQ}$

b Add a point R to the diagram such that $\overrightarrow{PR} = -2\overrightarrow{PM}$.

You can get from M to P via N.

a i $\overrightarrow{MP} = \overrightarrow{MN} + \overrightarrow{NP}$

$= \left(3\underline{i} - 2\underline{j}\right) + \left(\underline{i} + \underline{j}\right)$

$= 4\underline{i} - \underline{j}$

ii $\overrightarrow{PM} = -\overrightarrow{MP}$

$= -4\underline{i} + \underline{j}$

You can get from P to Q via M. You have already found $\overrightarrow{PM}$ and $\overrightarrow{MQ}$.

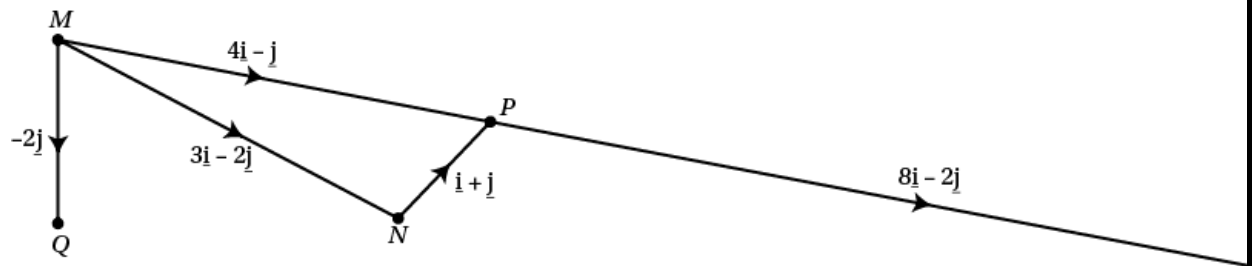
$\overrightarrow{PQ} = \overrightarrow{PM} + \overrightarrow{MQ}$

$= \left(-4\underline{i} + \underline{j}\right) + \left(-2\underline{j}\right)$

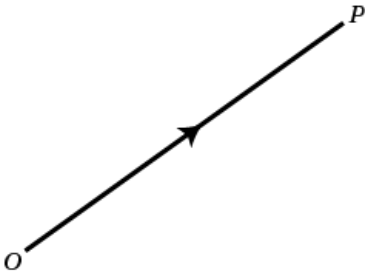
$= -4\underline{i} - \underline{j}$

Getting from P to R is in the opposite direction as getting from M to P, so $\overrightarrow{PR} = -\overrightarrow{MP}$.

b

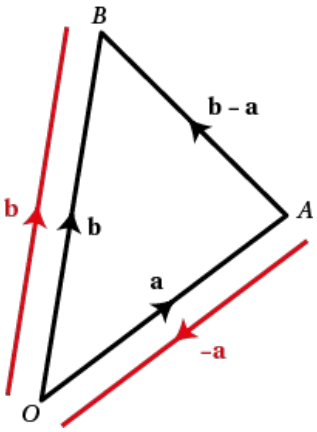


Displacement vectors tell you how to get from one point to another, but nothing about where the points actually are. To describe the position of a point you need a fixed **origin**, defined as $(0, 0)$ and labelled O. Then the the vector $\overrightarrow{OP}$, connecting the origin to a point P is called the **position vector** of P.



It is common to denote the position vector by the same letter as the point: for example, $\overrightarrow{OA} = \mathbf{a}$.

If you know position vectors of two points A and B you can find the displacement $\overrightarrow{AB}$ as shown in the diagram.



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

Key point 12.4

If points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ then $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

Tip

Remember that $\overrightarrow{AB}$ means the vector from A to B.

Tip

You can think of this as being the 'journey' $-\mathbf{a}$ followed by $\mathbf{b}$.

WORKED EXAMPLE 12.10

Points P and Q have position vectors $\overrightarrow{OP} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Find the displacement vector $\overrightarrow{PQ}$.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ Use the result from Key point 12.4.

$= \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

$= \begin{pmatrix} -5 \\ 6 \end{pmatrix}$

Position vectors are closely related to coordinates. If the base vectors $\mathbf{i}$ and $\mathbf{j}$ have directions along the coordinate axes, then the components of the position vector are simply the coordinates of the point.

WORKED EXAMPLE 12.11

Points A and B have coordinates $(3, -1)$ and $(5, 0)$, respectively.

a Write as column vectors:

i the position vectors of A and B

ii the displacement vector $\overrightarrow{AB}$

b Find the position vector of the point C such that $\overrightarrow{CA} = -4\mathbf{i} + 2\mathbf{j}$.

a i $\underline{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ The components of the position vectors are the coordinates of the point.

ii $\overrightarrow{AB} = \underline{\mathrm{b}} - \underline{\mathrm{a}}$ Use $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

$= \begin{pmatrix} 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

b $-4\,\underline{\mathbf{i}} + 2\,\underline{\mathbf{j}} = (3\,\underline{\mathbf{i}} - \underline{\mathbf{j}}) - \underline{\mathbf{c}}$ Use $\overrightarrow{CA} = \mathbf{a} - \mathbf{c}$.

Rearrange to find the position vector of C.

$$\underline{c} = (3\underline{\mathbf{i}} - \underline{\mathbf{j}}) - (-4\underline{\mathbf{i}} + 2\underline{\mathbf{j}})$$
$$= 7\underline{\mathbf{i}} - 3\underline{\mathbf{j}}$$

Distance between two points

Key point 12.5

If points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, then the distance between them, AB, is equal to the magnitude of the vector $\overrightarrow{AB}$:

$AB = \left|\overrightarrow{AB}\right| = |\mathbf{b} - \mathbf{a}|$

WORKED EXAMPLE 12.12

a Find the distance between points A and B with position vectors $\mathbf{a} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$.

b Point C has position vector $\mathbf{c} = \begin{pmatrix} 2 \\ p \end{pmatrix}$. Find the exact values of p such that $AC = 3AB$.

a

$$\underline{b} - \underline{a} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

First find the vector $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$$\therefore AB = \sqrt{1^2 + 2^2}$$
$$= \sqrt{5}$$

Then find the magnitude using Pythagoras' theorem.

b

$$\underline{c} - \underline{a} = \begin{pmatrix} 2 \\ p \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 5 \\ p - 2 \end{pmatrix}$$

First find an expression for the vector $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$ in terms of p.

$$AC = \sqrt{5^2 + (p-2)^2}$$

Then find the magnitude using Key point 12.1.

$$AC = 3AB$$
$$\sqrt{25 + (p-2)^2} = 3\sqrt{5}$$

Form an equation.

$$25 + (p-2)^2 = 45$$

Square both sides to get rid of the root.

$$(p-2)^2 = 20$$
$$p - 2 = \pm\sqrt{20}$$
$$p = 2 \pm 2\sqrt{5}$$

You don't need to expand the brackets. Square root, remembering ±.

WORK IT OUT 12.2

Points P and Q have coordinates $P(-1, 3)$ and $Q(7, -2)$. Find the vector $\overrightarrow{QP}$.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

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$$\overrightarrow{QP} = \mathbf{p} - \mathbf{q}$$
$$= \begin{pmatrix} -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 7 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} -8 \\ 5 \end{pmatrix}$$

Solution 2

$$\overrightarrow{QP} = \mathbf{p} + \mathbf{q}$$
$$= \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 7 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

Solution 3

$$\overrightarrow{QP} = \mathbf{q} - \mathbf{p}$$
$$= \begin{pmatrix} 7 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 8 \\ -5 \end{pmatrix}$$

EXERCISE 12C

1 **a** Write down the following displacement vectors in component form:

i from A to C

ii from C to B

iii from B to C

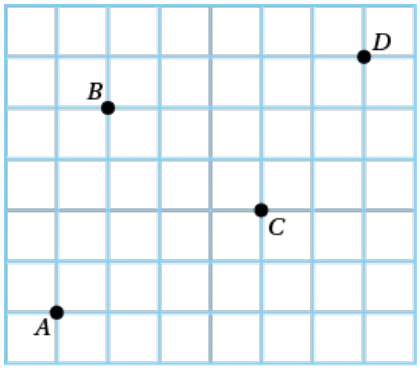
iv from D to A

b Copy the grid and mark on points P, Q and R so that:

i $\overrightarrow{AP} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

ii $\overrightarrow{QC} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

iii $\overrightarrow{RD} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$



2 Four points have coordinates $A(-1, 3), B(3, -3), C(0, -5)$ and $D(-3, -2)$. Write the following vectors in component form:

a **i** the position vector of A

ii the position vector of B

b **i** the displacement from B to D

ii the displacement from C to B

c **i** $\overrightarrow{AB}$

ii $\overrightarrow{DA}$

3 Points A, B, C and D are as in question 2. Find the coordinates of the points satisfying the following conditions:

a **i** point P such that $\overrightarrow{AP} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

ii point Q such that $\overrightarrow{CQ} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

b **i** point R such that $\overrightarrow{RB} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

ii point S such that $\overrightarrow{SD} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$

c **i** point X such that $\overrightarrow{AX} = 3\overrightarrow{AB}$

ii point Y such that $\overrightarrow{BY} = 2\overrightarrow{BD}$

d **i** point Z such that $\overrightarrow{CZ} = -2\overrightarrow{AB}$

ii point W such that $\overrightarrow{DW} = -4\overrightarrow{CA}$

4 Points A, B, C and D have position vectors $\mathbf{a} = 3\mathbf{i} - 7j$, $\mathbf{b} = -3\mathbf{i} + 6\mathbf{j}, \mathbf{c} = \mathbf{i} - 5\mathbf{j}, \mathbf{d} = -4\mathbf{j}$. Find the following distances:

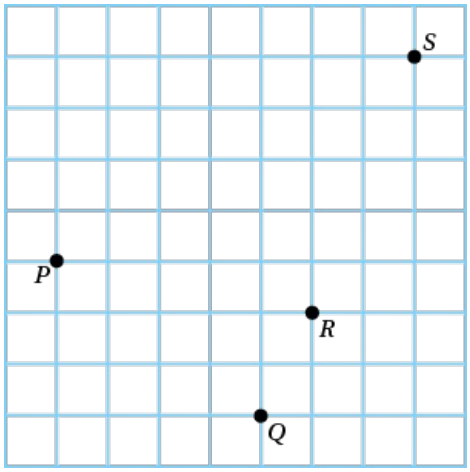
a **i** AB

ii CD

b **i** DA

ii BC

5 The diagram shows four points P, Q, R and S such that $\overrightarrow{PQ} = 4\mathbf{i} - 3\mathbf{j}, \overrightarrow{QR} = \mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{RS} = 2\mathbf{i} + 5\mathbf{j}$.



Express the following vectors in component form:

a $\overrightarrow{PR}$

b $\overrightarrow{SP}$

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6 Points A, B, C and D have coordinates $A(3, -2), B(-4, 1), C(1, -5)$, and $D(0, 7)$.

a Write $\overrightarrow{AC}$ and $\overrightarrow{DB}$ as column vectors.

b Find the coordinates of the point E such that $\overrightarrow{AE} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$.

7 Points A, B and C have coordinates $A(1, 3), B(5, -1)$, and $C(2, -8)$.

Point D is such that $\overrightarrow{AD} = \overrightarrow{BC} + (2x)\overrightarrow{AB} + (3y)\overrightarrow{AC} = \overrightarrow{AB} + (2x)\overrightarrow{AC} + (3y)\overrightarrow{BC}$.

Find the coordinates of D.

8 $\overrightarrow{AB} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ and $\overrightarrow{AC} = \frac{4}{3}\overrightarrow{AB}$.

a Write down $\overrightarrow{AC}$.

Point P is such that $AP = 2BP$.

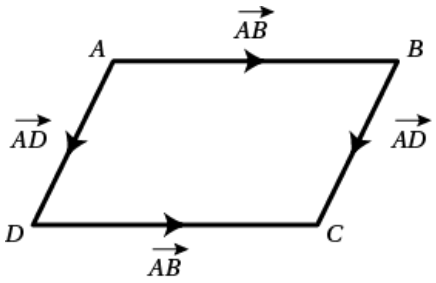
b If $\overrightarrow{AP} = \begin{pmatrix} r \\ q \end{pmatrix}$, write down vectors $\overrightarrow{BP}$ and $\overrightarrow{CP}$ in terms of r and q.

c Show that CP is independent of the values of r and q and evaluate it.

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Section 4: Using vectors to solve geometrical problems

In this section you will learn how to describe and prove various properties of geometrical figures.



Special shapes

Consider four points A, B, C and D such that $\overrightarrow{AB} = \overrightarrow{DC}$. This means that the opposite sides AB and DC are parallel and have equal length, so the shape is a parallelogram. It follows that the other two sides are also equal and parallel, so $\overrightarrow{BC} = \overrightarrow{AD}$.

One special type of parallelogram is a rhombus, which has all four sides of equal length. This means that vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have equal magnitudes. Note that they are not equal vectors because they don't have the same direction.

Gateway to A Level

For a reminder of the properties of special quadrilaterals see Gateway to A Level section N.

Fast forward

Other special types of parallelogram are a rectangle and a square; they have right angles. If you study Further Mathematics in Pure Core Student Book 1 you will learn how to find the angle between two vectors.

Key point 12.6

If $\overrightarrow{AB} = \overrightarrow{DC}$ then $ABCD$ is a parallelogram.

If in addition $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right|$ then $ABCD$ is a rhombus.

WORKED EXAMPLE 12.13

Four points have coordinates $A(2, -1), B(k, k+1), C(2k-3, 2k+2)$ and $D(k-1, k)$.

a Show that $ABCD$ is a parallelogram for all values of k.

b Find the value of k for which $ABCD$ is a rhombus.

a

$$\overrightarrow{AB} = \begin{pmatrix} k \\ k+1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} k-2 \\ k+2 \end{pmatrix}$$

$$\overrightarrow{DC} = \begin{pmatrix} 2k-3 \\ 2k+2 \end{pmatrix} - \begin{pmatrix} k-1 \\ k \end{pmatrix} = \begin{pmatrix} k-2 \\ k+2 \end{pmatrix}$$

$\overrightarrow{AB} = \overrightarrow{DC}$ so ABCD is a parallelogram.

For a parallelogram you need to show that $\overrightarrow{AB} = \overrightarrow{DC}$.

b $|\overrightarrow{AB}| = \sqrt{(k-2)^2 + (k+2)^2}$

$\overrightarrow{BC} = \begin{pmatrix} 2k-3 \\ 2k+2 \end{pmatrix} - \begin{pmatrix} k \\ k+1 \end{pmatrix}$

$= \begin{pmatrix} k-3 \\ k+1 \end{pmatrix}$

$|\overrightarrow{BC}| = \sqrt{(k-3)^2 + (k+1)^2}$

$|\overrightarrow{AB}| = |\overrightarrow{BC}|$

For a rhombus $|\overrightarrow{AB}| = |\overrightarrow{BC}|$ so find expressions for these two distances...

... and then make them equal.

$\Rightarrow (k-2)^2 + (k+2)^2 = (k-3)^2 + (k+1)^2$

$2k^2 + 8 = 2k^2 - 4k + 10$

$4k = 2$

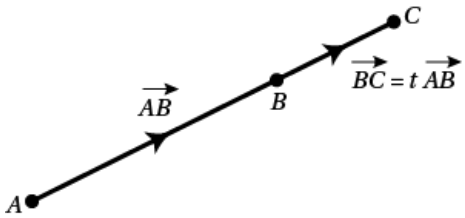
$k = \frac{1}{2}$

Square both sides to get rid of the roots.

Straight lines

If points A, B and C lie in a straight line then vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel, so $\overrightarrow{BC} = t\overrightarrow{AB}$ for some scalar t. This scalar gives the ratio of the lengths BC and AB.

Points that lie on a straight line are said to be **collinear**.



WORKED EXAMPLE 12.14

Three points have coordinates $A(-3, 2), B(6, 5)$ and $C(12, 7)$.

a Show that A, B and C lie on a straight line.

b Find the ratio $AB : BC$.

a $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 5 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 9 \\ 3 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 12 \\ 7 \end{pmatrix} - \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

You need to show that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel. First find these two vectors.

If points A, B and C lie on a straight line: $\overrightarrow{BC} = t\overrightarrow{AB}$

$\begin{cases} 6 = 9t \Rightarrow t = \frac{2}{3} \\ 2 = 3t \Rightarrow t = \frac{2}{3} \end{cases}$

$\therefore \overrightarrow{BC} = \frac{2}{3}\overrightarrow{AB}$

You need to get the same value of t for both components.

Since AB and BC are parallel and contain a common point B, A, B and C lie on a straight line.

Conclude that A, B and C lie on a straight line.

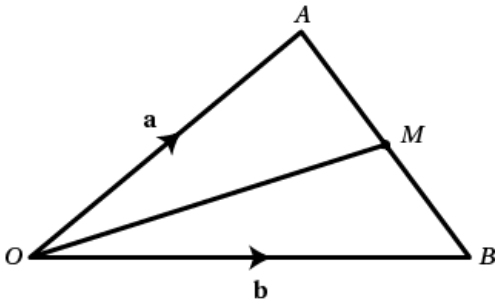
b $BC = \frac{2}{3}AB$

$\therefore AB : BC = 3 : 2$

Since $\overrightarrow{BC} = \frac{2}{3}\overrightarrow{AB}$, the length $BC = \frac{2}{3}AB$.

Midpoints

Consider points A and B with position vectors $\mathbf{a}$ and $\mathbf{b}$ and let M be the midpoint of AB. Then you can express the position vector of M in terms of $\mathbf{a}$ and $\mathbf{b}$.



Key point 12.7

The midpoint of the line segment connecting points with position vectors $\mathbf{a}$ and $\mathbf{b}$ has position vector $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

PROOF 7

$\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

You can get from O to M by going from O to A and then half-way from A to B.

$= \underline{a} + \frac{1}{2}(\underline{b} - \underline{a})$

$= \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b}$

$= \frac{1}{2}(\underline{a} + \underline{b})$

You know that $\overrightarrow{OA} = a$ and $\overrightarrow{AB} = b - a$.

In Worked example 12.15 we use vectors to prove that the diagonals of a parallelogram bisect each other. In Mixed practice 12 question 13 you can prove this result in general.

WORKED EXAMPLE 12.15

Points A, B and C have position vectors $\mathbf{a} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

a Find the position vector of the point D such that $ABCD$ is a parallelogram.

M is the midpoint of the diagonal AC.

b Find the position vector of M.

c Show that M is also the midpoint of BD.

a $\overrightarrow{AB} = \overrightarrow{DC}$

For a parallelogram, $\overrightarrow{AB} = \overrightarrow{DC}$.

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix} - \underline{d}$$

Use $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and $\overrightarrow{DC} = \mathbf{c} - \mathbf{d}$.

$$\begin{pmatrix} 1 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix} - \underline{d}$$

Rearrange to find **d.**

$$\underline{d} = \begin{pmatrix} -3 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ -8 \end{pmatrix}$$

b $\underline{m} = \frac{1}{2}(\underline{a} + \underline{c})$

Using the result for the midpoint from Key point 12.7.

$$= \frac{1}{2}\left(\begin{pmatrix} 3 \\ -5 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix} 0 \\ -6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

c $\overrightarrow{BM} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

If M is half-way between B and D then $\overrightarrow{BM} = \overrightarrow{MD}$.

$$= \begin{pmatrix} -4 \\ -5 \end{pmatrix}$$

You could also have used the condition that if M is the midpoint of BD then $\mathbf{m} = \frac{1}{2}(\mathbf{b} + \mathbf{d})$.

$$\overrightarrow{MD} = \begin{pmatrix} -4 \\ -8 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ -5 \end{pmatrix}$$

$\overrightarrow{BM} = \overrightarrow{MD}$ so M is the midpoint of BD.

Vectors are very useful for proving that two lines are parallel, even if they don't have the same length. Worked example 12.16 combines midpoints with parallel lines.

Tip

Remember that two vectors are parallel if one is a scalar multiple of the other.

WORKED EXAMPLE 12.16

The vertices of triangle *ABC* have position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. M and N are the midpoints of sides AB and AC.

a Express the position vectors of M and N in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

b Prove that MN is parallel to and half the length of BC.

a M is the midpoint of AB so

Find the position vectors of midpoints using the result from Key point 12.7.

$\underline{m} = \frac{1}{2}(\underline{a} + \underline{b})$

N is the midpoint of AC so

$\underline{n} = \frac{1}{2}(\underline{a} + \underline{c})$

b $\overrightarrow{BC} = \underline{c} - \underline{b}$

$\overrightarrow{MN} = \underline{n} - \underline{m}$

$= \frac{1}{2}(\underline{a} + \underline{c}) - \frac{1}{2}(\underline{a} + \underline{b})$

$= \frac{1}{2}\underline{c} - \frac{1}{2}\underline{b}$

$\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$

So MN is parallel to BC and half its length.

You want to compare vectors $\overrightarrow{MN}$ and $\overrightarrow{BC}$.

Use the result from part **a** on $\overrightarrow{MN}$.

Since $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$ the two vectors are parallel.

EXERCISE 12D

1 The diagram shows a parallelogram $ABCD$ with $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.

M is the midpoint of BC and N is the midpoint of CD. Express the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$:

a i $\overrightarrow{BC}$

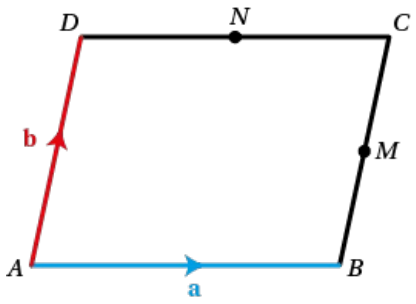
ii $\overrightarrow{AC}$

b i $\overrightarrow{CD}$

ii $\overrightarrow{ND}$

c i $\overrightarrow{AM}$

ii $\overrightarrow{MN}$



2 In the parallelogram $ABCD$, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$. M is the midpoint of BC, Q is the point on the extended line AB such that $BQ = \frac{1}{2}AB$ and P is the point on the extended line BC such that $BC : CP = 3 : 1$, as shown on the diagram. Express the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$:

a i $\overrightarrow{AP}$

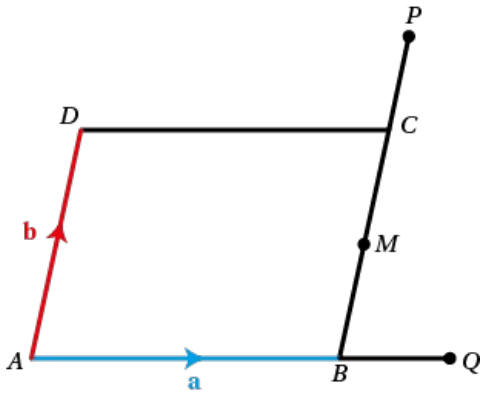
ii $\overrightarrow{AM}$

b i $\overrightarrow{QD}$

ii $\overrightarrow{MQ}$

c i $\overrightarrow{DQ}$

ii $\overrightarrow{PQ}$



3 For the coordinate sets given, determine whether the three points A, B and C are collinear. If they are, find the ratio $AB : BC$.

a i $A(3, 8), B(-1, 2), C(51, 80)$

ii $A(4, -1), B(6, 3), C(11, 13)$

b i $A(2, 1), B(-11, 14), C(4, 3)$

ii $A(3, 5), B(1, 7), C(-6, 3)$

c i $A(4, 2a), B(1, 3a + 2), C(10, 4)$

ii $A(4, 2a), B(1, 2a + 1), \mathrm{C}(7, 2a - 1)$

d i $A\left(a^2 + 2a, 3\right), \mathrm{B}\left(a^2 + a - 1, 1\right), C\left(3a^2 + 2a - 2, 4a - 1\right)$

ii $A\left(a^2 - 2a, 5\right), B\left(a^2 + a + 2, 5\right), C\left(a^2 + 2a, -4\right)$

4 The points A and B have position vectors $\mathbf{a} = 7\mathbf{i} + 18\mathbf{j}$ and $\mathbf{b} = -3\mathbf{i} + 6\mathbf{j}$. Find the position vector of the midpoint of AB.

5 $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$ and $\overrightarrow{CA} = \begin{pmatrix} -9 \\ 21 \end{pmatrix}$.

Show that A, B and C are collinear and find the ratio $AB : BC$.

6 Points A, B and C have position vectors $\mathbf{a} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

a Find the position vector of the point D such that $ABCD$ is a parallelogram.

b Determine whether $ABCD$ is a rhombus.

7 Points A, B, C and D have position vectors.

$\mathbf{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

Point E is the midpoint of BC.

a Find the position vector of E.

b Show that $ABED$ is a parallelogram.

8 The vertices of a quadrilateral $PQRS$ have coordinates $P(-2, 1), Q(5, -3), R(6, 0)$ and $S(-1, 5)$. The midpoints of the sides PQ, QR, RS and SP are A, B, C and D.

Prove that $ABCD$ is a parallelogram.

9 Points A and B have position vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Point C lies on the line segment AB so that $AC : BC = 2 : 3$. Find the position vector of C.

10 Points A, B, C and D have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$.

M is the midpoint of AB and N is the midpoint of BC.

P is the midpoint of CD and Q is the midpoint of AD.

By finding the vectors $\overrightarrow{MN}$ and $\overrightarrow{QP}$, prove that $MNPQ$ forms a parallelogram.

11 Points A and B have coordinates $A(10, 1)$ and $B(2, 7)$. Point C lies on the line segment AB such that $AC : BC = x : 1 - x$, where $0 < x < 1$.

a Find the coordinates of C, in terms of x.

b Point D has coordinates $D(3, 2)$ and $CD = \sqrt{26}$. Find x.

12 $ABCD$ is a parallelogram with $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$. Let M be the midpoint of the diagonal AC.

a Express $\overrightarrow{AM}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

b Show that M is also a midpoint of the diagonal BD.

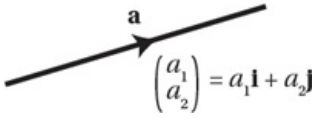
13 Four points have coordinates $A(2, -1), B(k, k + 1), C(2k - 3, 3k + 2)$ and $D(k - 1, 2k)$.

a Show that $ABCD$ is a parallelogram for all values of k.

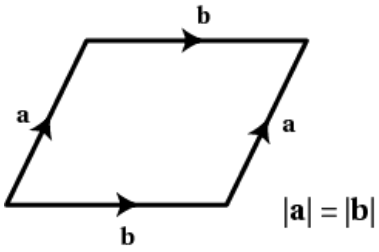
b Show that there is no value of k for which $ABCD$ is a rhombus.

Checklist of learning and understanding

- A vector is a quantity that has both magnitude and direction.
- The magnitude (modulus) of vector $\mathbf{a}$ is written |a|. The direction is the angle it makes with the positive x-axis.
- There are several ways to represent a vector:
  - by drawing a directed arrow: in this case the magnitude of the vector is represented by the length of the arrow
  - by stating its components, either as a column vector or by using base vectors $\mathbf{i}$ and $\mathbf{j}$
  - by stating its magnitude and direction.



- You can perform three operations with vectors: adding, subtracting and multiplying by a scalar.
- If vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel then $\mathbf{b} = t\mathbf{a}$ for some scalar t.
- A unit vector is a vector with magnitude 1.
- Vectors can represent positions of points or displacements between two points.
- The position vector of a point is a vector from the origin to that point.
  - If points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ then the displacement from A to B is $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.
  - The distance between two points is the magnitude of the displacement between them: $AB = \left|\overrightarrow{AB}\right| = |\mathbf{b} - \mathbf{a}|$.
- Vectors can be used to solve problems and prove properties of geometrical shapes. Some of the most useful properties are:
  - If a shape is a parallelogram then the vectors corresponding to the opposite sides are equal.
  - If the shape is a rhombus then the vectors corresponding to all four sides have equal magnitudes.



- The midpoint of the line segment joining points with position vectors $\mathbf{a}$ and $\mathbf{b}$ has position vector $\frac{1}{2}(a+b)$.
- You can use vectors to show that two lines are parallel.

Mixed practice 12

1 Points A and B have position vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = 2\mathbf{j}$. Find the exact distance between A and B.

2 **a** Given the points $P(-5, 2)$ and $Q(1, -3)$, write vector $\overrightarrow{PQ}$ in the form $a\mathbf{i} + b\mathbf{j}$.

b Point R is such that $\overrightarrow{RQ} = \mathbf{i} - 4\mathbf{j}$. Find the coordinates of R.

3 Points A and B have position vectors $\mathbf{a} = \begin{pmatrix} 12 \\ -7 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$.

M is the midpoint of AB.

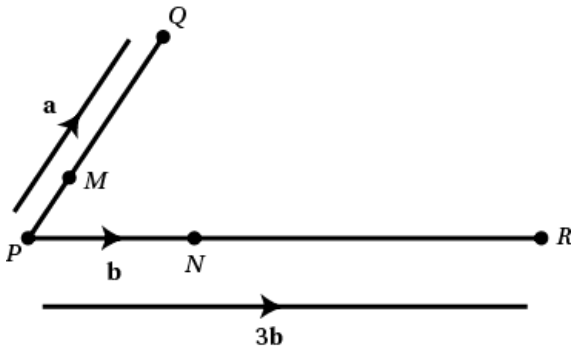
a Find the position vector of M.

b Find the exact distance BM.

4 Points A, B and C have position vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{j}, \mathbf{b} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{c} = 4\mathbf{i} + \mathbf{j}$. Point D is such that $ABCD$ is a parallelogram. Find the position vector of D.

5 The diagram shows points P, Q and R such that $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{PR} = 3\mathbf{b}$.

Points M and N are on PQ and PR such that $PM = \frac{1}{3}PQ$ and $\overrightarrow{PN} = \mathbf{b}$.



Express $\overrightarrow{MN}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and hence prove that MN is parallel to QR.

6 *OAB* is a triangle with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of AB and G is a point on OM such that $OG : GM = 2 : 1$. N is the midpoint of OA. Use vectors to prove that the points B, G and N are collinear.

7 Find the magnitude of the vector $(3\sin\theta)\mathbf{i} + (5\cos\theta)\,\mathbf{j}$ in terms of θ.

8 Points M and N have coordinates $M(-6, 1)$ and $N(3, 5)$. Find a unit vector parallel to $\overrightarrow{MN}$.

9 Points P and Q have coordinates $(1, -8)$ and $(10, -2)$. N is a point on PQ such that $PN : NQ = 1 : 2$.

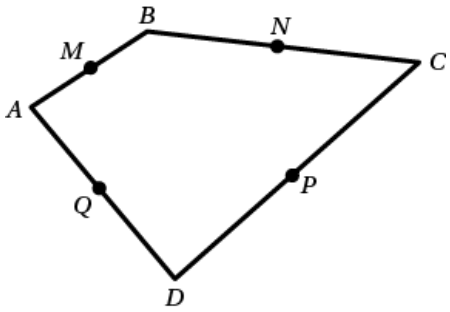
a Find the coordinates of N.

b Calculate the magnitudes of $\overrightarrow{OP}, \overrightarrow{ON}$ and $\overrightarrow{PN}$. Hence show that *ONP* is a right angle.

10 Points A, B, C and D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. M, N, P and Q are midpoints of AB, BC, CD and DA.

a Express vectors $\overrightarrow{MN}$ and $\overrightarrow{PQ}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$.

b What type of quadrilateral is $MNPQ$?



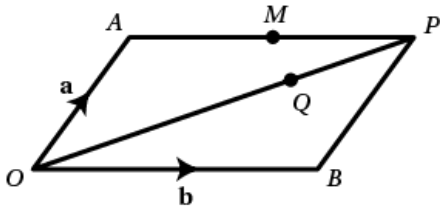
11 Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$. O is the origin and point P is such that $OAPB$ is a parallelogram.

a Write down the position vector of P in terms of $\mathbf{a}$ and $\mathbf{b}$.

b Find the position vector of M, the midpoint of AP.

Point Q lies on OP. Let $\overrightarrow{OQ} = t\overrightarrow{OP}$.

c Express $\overrightarrow{BQ}$ in terms of $t, \mathbf{a}$ and $\mathbf{b}$. Hence find the value of t for which BQM is a straight line.

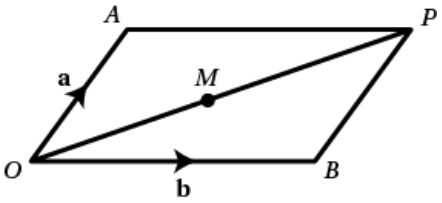


12 Points P and Q have position vectors $p = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $q = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Point H lies on PQ and $\overrightarrow{PH} = t\overrightarrow{PQ}$, with $0 < t < 1$. Let O be the origin.

a Express the vector $\overrightarrow{OH}$ and its length in terms of t.

b Hence find the minimum possible distance of H from the origin, giving your answer in exact form.

13 Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$. O is the origin and point P is such that $OAPB$ is a parallelogram.



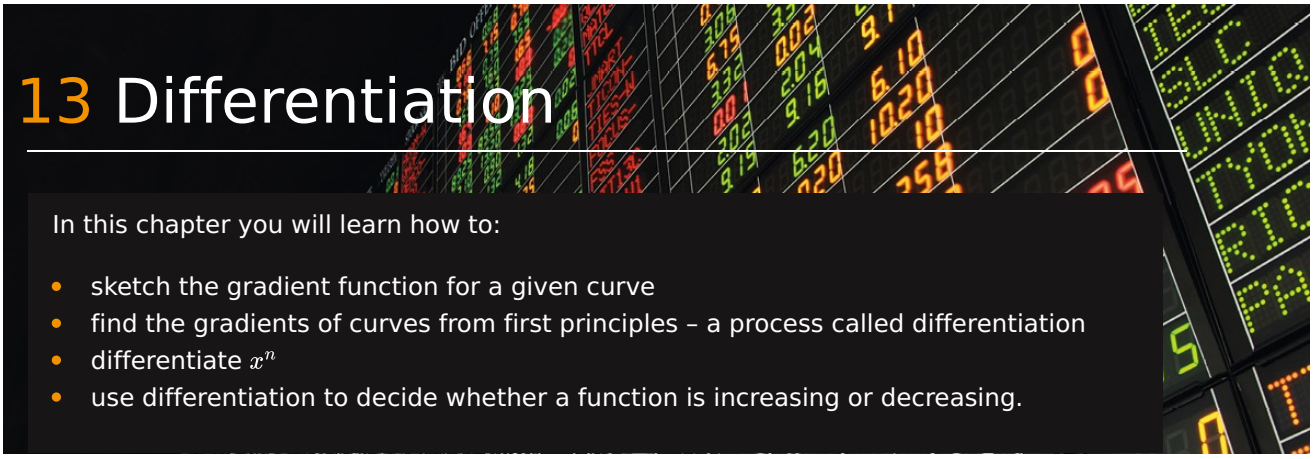
M is the midpoint of OP.

a Show that M lies on AB and determine the ratio $AM : MB$.

b What conclusion can you make about the diagonals of a parallelogram?

Worksheet

For more challenging geometry problems, see Extension sheet 12.



13 Differentiation

In this chapter you will learn how to:

- sketch the gradient function for a given curve
- find the gradients of curves from first principles – a process called differentiation
- differentiate x^n
- use differentiation to decide whether a function is increasing or decreasing.

Before you start...

Chapter 2	You should know how to work with indices.	1 Write the following in the form $nx^a + mx^b$. a $\sqrt[3]{x^2} - \frac{5}{2x}$ b $\frac{2+x}{\sqrt{x}}$
Chapter 3	You should know how to solve linear and quadratic inequalities.	2 Solve these inequalities: a $3x - 4 \geqslant 5x + 2$ b $x^2 - 4x - 12 \geqslant 0$
Chapter 6	You should know how to find the gradient of a straight line.	3 Find the gradient of the line that passes through the point $(-1, 4)$ and $(7, -2)$.
Chapter 9	You should know how to work with the binomial expansion.	4 Expand $(2 + x)^3$.

What is differentiation?

In real life, things change: planets move, babies grow, prices rise and fall. Calculus is the study of change, and one of its most important tools is differentiation − finding the rate at which the y-coordinate of a curve is changing when the x-coordinate changes. For a straight-line graph, this rate of change was given by the gradient of the line. In this chapter you will apply the same idea to curves, where the gradient is different at different points.

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Section 1: Sketching derivatives

You first need to establish exactly what is meant by the gradient of a curve. You know what this means for a straight line, so we can use this idea to make a more general definition.

Key point 13.1

The gradient of a curve at a point P is the gradient of the tangent to the curve at that point.

Rewind

You have already met tangents in Chapters 5 and 6.

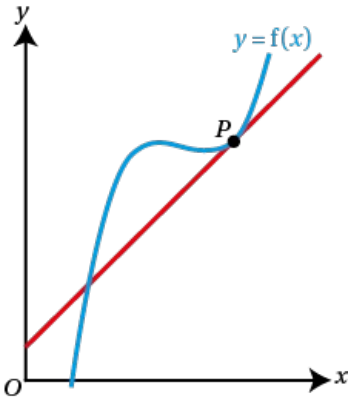
A tangent to a curve at a point is a straight line which touches the curve without crossing it at that point.

Note that when we say that the tangent at P does not cross the curve, we mean this in a 'local' sense: the tangent does not cross the curve close to the point P but it can intersect a different part of the curve (as shown in the diagram).

Fast forward

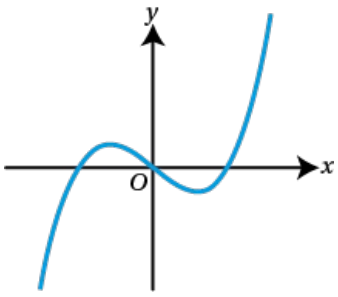
A It is in fact possible for a tangent at a point to cross the curve at that point; this happens at a point of inflection. You will learn about this in Student Book 2, Chapter 12.

The **derivative** (or **gradient function**) of a function $\mathrm{f}(x)$ is another function that gives the gradient of the graph of $y = \mathrm{f}(x)$ at any point. It is often useful to be able to sketch the derivative of a given function.



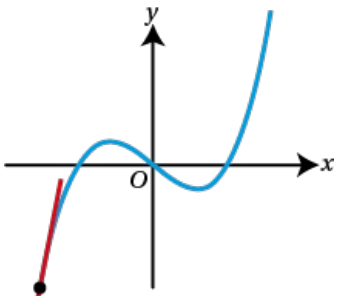
WORKED EXAMPLE 13.1

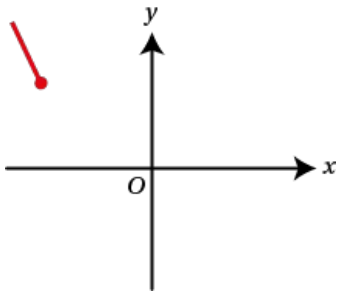
Sketch the derivative of this function:



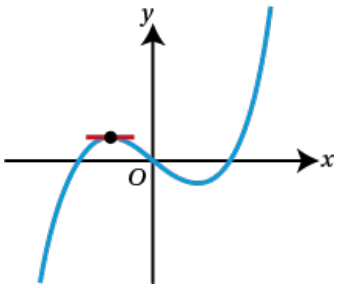
Imagine a point moving along the curve from left to right. Track the tangent of the curve at the moving point and form the graph of its gradient.

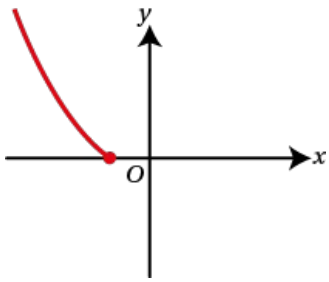
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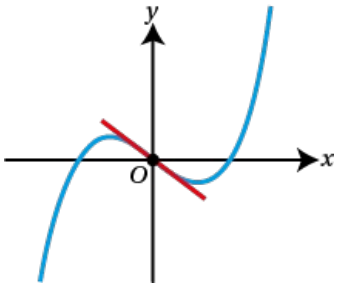


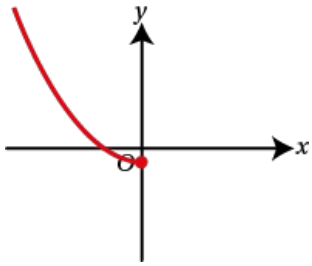
The curve is increasing from left to right, but more and more slowly...
... so the gradient is positive and falling.



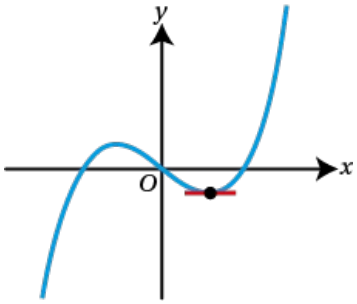


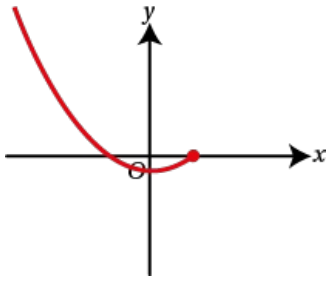
The tangent is horizontal...
... so the gradient is zero.



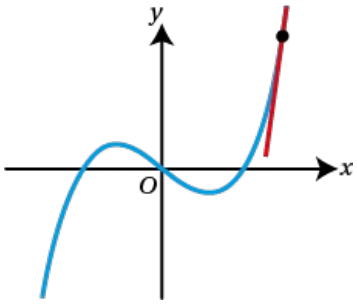


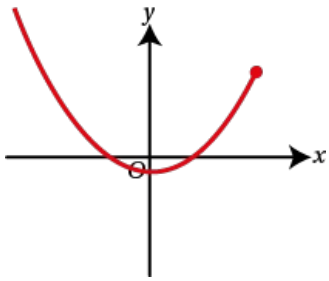
The curve is now decreasing...
... so the gradient is negative.





The tangent is horizontal again...
... so the gradient is zero.





The curve is increasing, and does so faster and faster...
... so the gradient is positive and getting larger.

The relationship between the graph of a function and its derivative can be summarised as follows:

Key point 13.2

- When the graph is **increasing** the gradient is positive.
- When the graph is **decreasing** the gradient is negative.
- When the tangent is horizontal the gradient is zero. A point on the graph where this happens is called a **stationary point**.

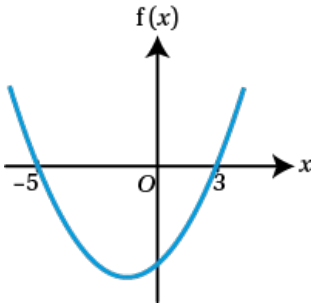
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Fast forward

You will examine stationary points in detail in Chapter 14.

WORKED EXAMPLE 13.2

The diagram shows the graph of the gradient function $f'(x)$.



a Is the function $f(x)$ increasing or decreasing at the point $x = 2$?

b The graph of $y = f(x)$ has two stationary points. Write down their x-coordinates.

a The function $f(x)$ is decreasing.

The gradient is negative at $x = 2$.

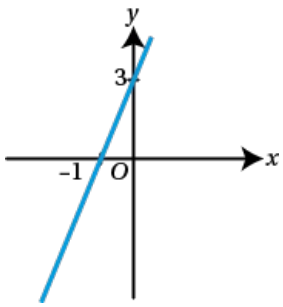
b $x = -5$ and $x = 3$.

Stationary points occur where the gradient is zero, which is where the graph of $y = f'(x)$ crosses the x-axis.

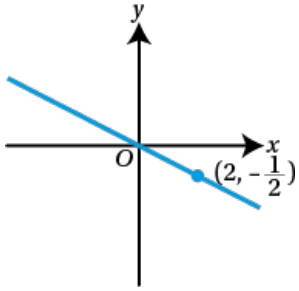
EXERCISE 13A

1 Sketch the derivatives of the following functions, showing any intercepts with the x-axis.

a i

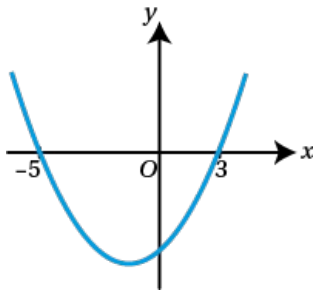


ii

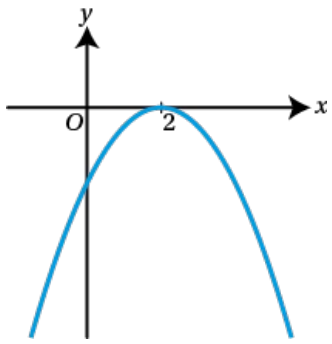


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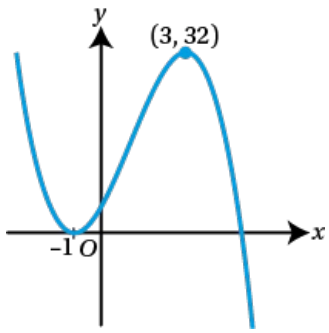
b i



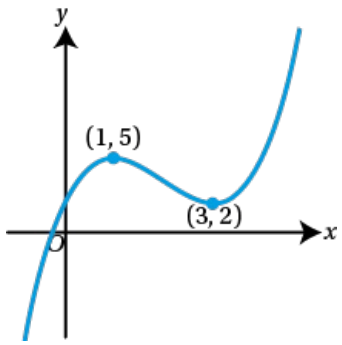
ii



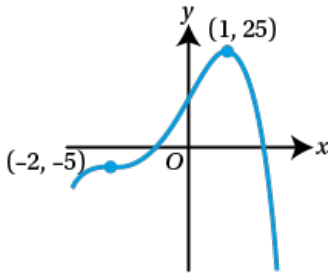
c i



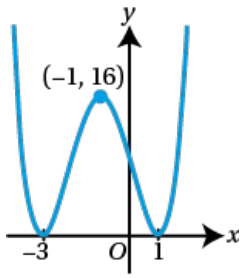
ii



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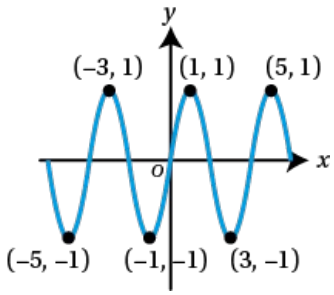


ii

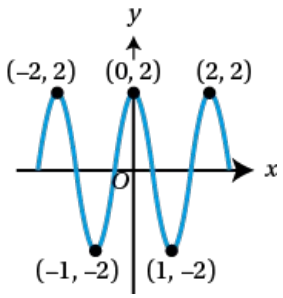


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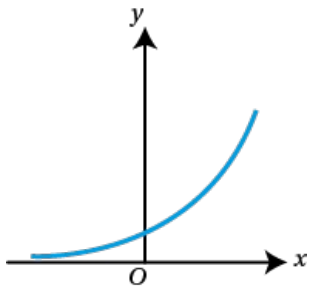
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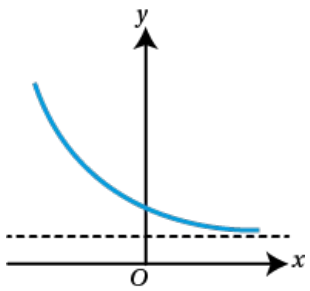
ii



f i

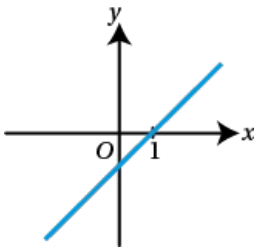


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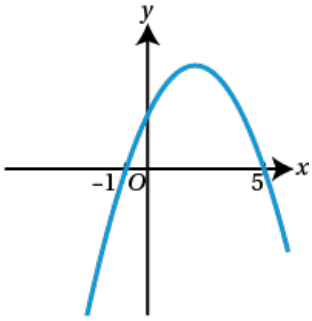


2 Each of the following represents a graph of a function's derivative. Sketch a possible graph for the original function, indicating any stationary points.

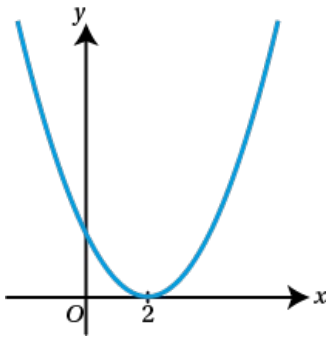
a



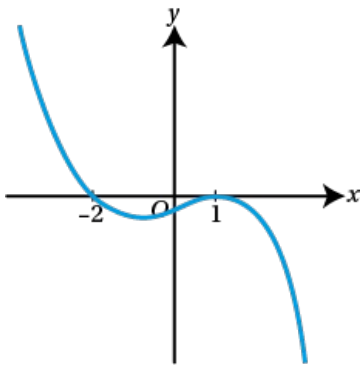
b



c



d



3 Decide whether each of the following statements is true or false. For the statements that are false, give a counter example.

a At a point where the derivative is positive the original function is positive.

b If the original function is negative then the derivative is also negative.

c The derivative crossing the axis corresponds to a stationary point on the graph.

d When the derivative is zero the graph is at a local maximum or minimum point.

e If the derivative function is always positive then part of the original function is above the x-axis.

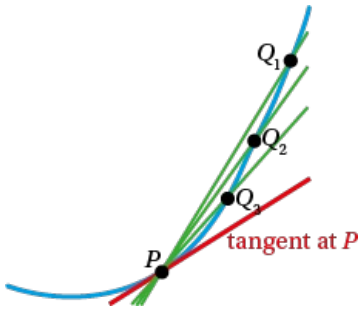
f At the lowest value of the original function the derivative is zero.

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Section 2: Differentiation from first principles

The line segment between two points on a curve is called a **chord**.

The diagram shows the chord PQ. You can see that the closer the point Q is to P, the closer the gradient of the chord is to the gradient of the tangent at P.



You can use this idea to find the gradient of a function at a given point, P.

For example, to find the gradient to $y = x^2$ at the point P where $x = 3$, consider a chord from P to the point Q with a slightly larger x-coordinate, $x = 3 + h$.

Rewind

Remember that in Section 1, the gradient of a curve at a given point was defined as the gradient of the tangent to the curve at that point.

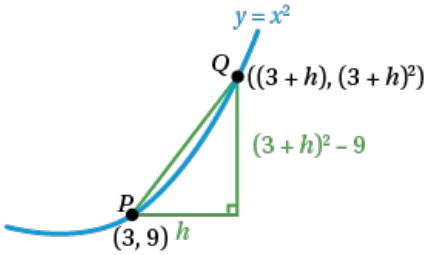
The gradient of the chord, m, is:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{(3+h)^2 - 9}{(3+h) - 3} \\ &= \frac{9 + 6h + h^2 - 9}{h} \\ &= \frac{6h + h^2}{h} \\ &= 6 + h \end{aligned}$$

If the point Q now approaches P, then h tends to 0 and so m tends to 6

Tip

The letter h just denotes a small distance.



Therefore, you can say that when $x = 3$, the gradient of $y = x^2$ is 6.

To denote this idea of the distance h approaching zero, use $\lim_{h \to 0}$ which reads as 'the limit as h tends to 0'.

The process of finding $\lim_{h\to 0}$ of the gradient of the chord is called **differentiation from first principles**.

Did you know?

The credit for developing calculus in the 17th century is usually given to Gottfried Leibniz and Isaac Newton. However, there is evidence that the idea of considering small changes in the value of a function had been used many centuries earlier.

This method can be used to find the gradient at a general point on any function $\mathrm{f}(x)$:

Key point 13.3

$$\mathrm{f}'(x) = \lim_{h\to 0}\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}$$

The expression $\mathrm{f}'(x)$ is called the **derivative** of $\mathrm{f}(x)$. It can also be denoted as f', y' or $\frac{\mathrm{d}y}{\mathrm{d}x}$ where $y = \mathrm{f}(x)$.

So, in this example, we can write $\mathrm{f}'(3) = 6$, or $\frac{\mathrm{d}y}{\mathrm{d}x} = 6$ when $x = 3$.

The process of finding the derivative is called **differentiation**.

You can use this definition to find the derivative of simple polynomial functions.

Tip

'Differentiation from first principles' means finding the derivative using this definition, rather than any of the rules you will meet in the later sections.

WORKED EXAMPLE 13.3

Prove, using differentiation from first principles, that for $\mathrm{f}(x) = x^2 + x, \mathrm{f}'(x) = 2x + 1$.

$$\mathrm{f}'(x) = \lim_{h\to 0}\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}$$

Use the formula with $\mathrm{f}(x) = x^2 + x$, so that $\mathrm{f}(x+h) = (x+h)^2 + (x+h)$.

$$= \lim_{h\to 0}\frac{(x+h)^2 + (x+h) - (x^2 + x)}{h}$$

$$= \lim_{h\to 0}\frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h}$$

Expand $(x+h)^2$ and simplify the expression.

$$= \lim_{h\to 0}\frac{2xh + h^2 + h}{h}$$

$$= \lim_{h\to 0}(2x + h + 1)$$

Then divide top and bottom by h.

$$= 2x + 1$$

Finally, let $h \to 0$.

EXERCISE 13B

1 Prove from first principles that the derivative of the function $f(x) = 8$ is zero.

2 Let $f(x) = -4x$.

Use differentiation from first principles to find $f'(x)$.

3 Let $y = 3x^2$.

Find $\frac{dy}{dx}$ from first principles.

4 Using differentiation from first principles, prove that the derivative of $x^2 + 1$ is $2x$.

5 Differentiate $f(x) = x^2 - 6x$ from first principles.

6 Let $y = x^2 - 3x + 4$.

Using differentiation from first principles, find y'.

7 **a** Expand $(x + h)^3$.

b Hence find from first principles the derivative of $y = x^3$.

8 $f(x) = 3x^3 + 2$.

Find $f'(x)$ from first principles.

9 **a** Expand $(x + h)^4$.

b Hence prove from first principles that if $f(x) = x^4$, then $f'(x) = 4x^3$.

Worksheet

For more challenging questions on differentiation from first principles, see Extension sheet 13.

10 A curve has equation $y = 4x^3$. Points A and B lie on the curve and have x-coordinates a and $a + h$, respectively.

a Find and simplify an expression, in terms of a and h, for the gradient of the chord AB.

b Find, in terms of a, the limit of this gradient when h tends to zero.

c What does your answer to part **b** represent?

11 If k is a constant, prove that the derivative of $kf(x)$ is $kf'(x)$.

12 If $y = f(x) + g(x)$, prove that $y' = f'(x) + g'(x)$.

Section 3: Rules of differentiation

In the previous exercise (questions 7 and 9) you found that the derivative of x^3 is $3x^2$, and that the derivative of x^4 is $4x^3$. This suggests a general rule for differentiating functions of the form $y = x^n$.

Key point 13.4

If $y = x^n$ then: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$

The result in Key point 13.2 is proved here for positive integers using differentiation from first principles, but the result is true (and you will need to use it) for **all** rational powers – positive and negative.

Fast forward

A In Student Book 2 you will use a method called 'implicit differentiation' to extend this proof to all rational powers.

PROOF 8

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

Use the formula with $\mathrm{f}(x) = x^n$.

$$= \lim_{h\to 0}\frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + h^n - x^n}{h}$$

Expand $(x+h)^n$ using the binomial expansion.

$$= \lim_{h\to 0}\frac{\binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + h^n}{h}$$

Simplify...

$$= \lim_{h\to 0}\left[\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h + \ldots + h^{n-1}\right]$$

and then divide top and bottom by h.

$$= \binom{n}{1}x^{n-1}$$

Let $h \to 0$.

$$= nx^{n-1}$$

$\binom{n}{1} = n$

WORKED EXAMPLE 13.4

Let $y = x^6$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^{6-1}$$

Use $\dfrac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$.

$$= 6x^5$$

WORKED EXAMPLE 13.5

For $y = x^{-5}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

$$\frac{dx}{dx} = -5x^{-5-1}$$
$$= -5x^{-6}$$

Use $\frac{dy}{dx} = nx^{n-1}$.

WORKED EXAMPLE 13.6

Differentiate $f(x) = x^{\frac{1}{4}}$.

$$f'(x) = \frac{1}{4}x^{\frac{1}{4}-1}$$
$$= \frac{1}{4}x^{-\frac{3}{4}}$$

Use $f'(x) = nx^{n-1}$.

WORKED EXAMPLE 13.7

Find the derivative of $f(x) = x$.

$$f(x) = 1x^{1-1}$$
$$= x^0$$
$$= 1$$

Use $f'(x) = nx^{n-1}$.

Remember that $x^0 = 1$.

The results in Exercise 13B also suggest the properties of differentiation (they were in fact proved in questions 11 and 12).

Tip

Key point 13.5 just says that you can differentiate a function and then multiply by a constant, and that you can differentiate terms of a sum separately.

Key point 13.5

- If $y = kf(x)$, where k is a constant, then $y' = kf'(x)$.
- If $y = f(x) + g(x)$, then $y' = f'(x) + g'(x)$.

Explore

Some functions don't have derivatives at all points. For example, try finding the derivative of $y = \sqrt{x}$ at $x = 0$. Can you explain why this is the case by looking at the graph?

WORKED EXAMPLE 13.8

Find the derivative of $f(x) = 5x^3$.

$$f'(x) = 5 \times 3x^2$$
$$= 15x^2$$

Differentiate x^3 and then multiply by 5.

WORKED EXAMPLE 13.9

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = 8$.

$y = 8x^0$ — Using the fact that $x^0 = 1$ you can write $y = 8$ as $y = 8x^0$.

$\frac{dy}{dx} = 8 \times \mathrm{O}x^{\mathrm{O}-1}$ — Now you can apply $\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$.

$= \mathrm{O}$ — As 0 is now a factor, the whole expression is zero.

From Worked example 13.9 you can see that the derivative of the function $\mathrm{f}(x) = c$ for any constant c will always be zero. This corresponds to the fact that the gradient of the horizontal line $y = c$ is zero everywhere.

Tip

You do not need to write out $c = cx^0$ every time you differentiate $y = c$. You just need to know that the derivative will always be zero.

WORKED EXAMPLE 13.10

A curve has equation $y = x^4 - 4x^{\frac{1}{2}} + 5x - 3$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 4 \times \frac{1}{2}x^{-\frac{1}{2}} + 5$

$= 4x^3 - 2x^{-\frac{1}{2}} + 5$

Differentiate each term separately. Remember that the derivative of a constant is zero, so the -3 vanishes.

EXERCISE 13C

1 Differentiate the following:

a i $y = x^4$

ii $y = x^6$

b i $y = 3x^7$

ii $y = -4x^5$

c i $y = -x$

ii $y = 3x$

d i $y = 10$

ii $y = -3$

e i $y = \frac{1}{3}x^6$

ii $y = -\frac{3}{4}x^2$

f i $y = 4x^3 - 5x^2 + 2x - 8$

ii $y = 2x^4 + 3x^3 - x$

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g **i** $y = 7x - \dfrac{x^3}{2}$

ii $y = 2 - 5x^4 + \dfrac{x^5}{5}$

2 Find $\mathrm{f}'(x)$ for the following functions:

a **i** $\mathrm{f}(x) = x^{\frac{3}{2}}$

ii $\mathrm{f}(x) = x^{\frac{2}{3}}$

b **i** $\mathrm{f}(x) = 8x^{\frac{1}{2}}$

ii $\mathrm{f}(x) = 6x^{\frac{4}{3}}$

c **i** $\mathrm{f}(x) = \dfrac{4}{9}x^{\frac{3}{4}}$

ii $\mathrm{f}(x) = \dfrac{3}{5}x^{\frac{5}{6}}$

d **i** $\mathrm{f}(x) = 3x^4 - 15x^{\frac{2}{5}} - 2$

ii $\mathrm{f}(x) = x^3 - \dfrac{3}{5}x^{\frac{5}{3}} + \dfrac{4}{3}x^{\frac{1}{2}}$

e **i** $\mathrm{f}(x) = x^{-1}$

ii $\mathrm{f}(x) = x^{-3}$

f **i** $\mathrm{f}(x) = x^{-\frac{1}{2}}$

ii $\mathrm{f}(x) = x^{-\frac{2}{3}}$

g **i** $\mathrm{f}(x) = -6x^{-\frac{4}{3}}$

ii $\mathrm{f}(x) = -8x^{-\frac{3}{4}}$

h **i** $\mathrm{f}(x) = 5x - \dfrac{8x^{-\frac{5}{2}}}{15}$

ii $\mathrm{f}(x) = -\dfrac{7x^{-\frac{3}{7}}}{3} + \dfrac{4x^{-6}}{3}$

3 Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for the following:

a **i** $y = 4x^{2.7}$

ii $y = 3x^{3.5}$

b **i** $y = 5x^{0.6}$

ii $y = -3x^{0.4}$

c **i** $y = -2x^{-0.7}$

ii $y = 4x^{-1.5}$

4 A curve has equation $y = x^2 - 3x^{\frac{1}{2}} + 5$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

5 Given $\mathrm{f}(x) = 4x^3 - 3x^2 + 2x^{-\frac{3}{2}}$, find $\mathrm{f}'(x)$.

6 Find the derivative of the function $\mathrm{f}(x) = 12x^{-\frac{1}{3}} + \dfrac{5x^{\frac{2}{5}}}{6}$.

Section 4: Simplifying into terms of the form ax^n

Notice that there is no rule in Key point 13.5 for differentiating products of functions, $y = \mathrm{f}(x)\mathrm{g}(x)$, or quotients of functions, $y = \dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}$.

Before these can be differentiated, they need to be converted into terms of the form ax^n. This is often done by using the laws of indices.

Fast forward

A Some expressions cannot be simplified using laws of indices. In Student Book 2 you will learn different rules for differentiating products and quotients.

WORKED EXAMPLE 13.11

$y = (x + 2)(x - 5)$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

$y = (x + 2)(x - 5)$
$= x^2 - 3x - 10$ — Expand the brackets.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x - 3$ — Then differentiate.

Rewind

If you need to review the laws of indices, see Chapter 2.

WORKED EXAMPLE 13.12

A curve has equation $y = \dfrac{1}{x^3}$. Find y'.

$y = \dfrac{1}{x^3}$
$= x^{-3}$ — First rewrite the function in the form x^n using the laws of indices.

$y' = -3x^{-4}$ — Then differentiate.

WORKED EXAMPLE 13.13

Find the derivative of $\mathrm{f}(x) = x^2\sqrt{x}$.

$\mathrm{f}(x) = x^2\sqrt{x}$
$= x^2 x^{\frac{1}{2}}$
$= x^{2+\frac{1}{2}}$
$= x^{\frac{5}{2}}$ — First rewrite the function in the form x^n using the laws of indices.

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$$f'(x) = \frac{5}{2}x^{\frac{5}{2}-1}$$
$$= \frac{5}{2}x^{\frac{3}{2}}$$

Then differentiate.

WORKED EXAMPLE 13.14

Differentiate $y = \dfrac{2x-6}{\sqrt{x}}$.

$$y = \frac{2x-6}{\sqrt{x}}$$
$$= \frac{2x-6}{x^{\frac{1}{2}}}$$
$$= 2x^{1-\frac{1}{2}} - 6x^{-\frac{1}{2}}$$
$$= 2x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}$$

You need to write this as a sum of terms of the form x^n.

$$y' = 2 \times \frac{1}{2}x^{\frac{1}{2}-1} - 6\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$$
$$= x^{-\frac{1}{2}} + 3x^{-\frac{3}{2}}$$

Now differentiate each term separately.

WORK IT OUT 13.1

Three students' attempts to differentiate $f(x) = \dfrac{x^2-3x}{x^3}$ are shown.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$$f(x) = \frac{x^2-3x}{x^3}$$
$$f'(x) = \frac{2x-3}{3x^2}$$

Solution 2

$$f(x) = \frac{x^2-3x}{x^3}$$
$$= x^{-1} - 3x^{-2}$$
$$f'(x) = -x^{-2} + 6x^{-3}$$

Solution 3

$$f(x) = \frac{x^2-3x}{x^3}$$
$$= x^{-3}\left(x^2-3x\right)$$
$$f'(x) = -3x^{-4}\left(2x-3\right)$$

EXERCISE 13D

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1 Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following:

a i $y = \sqrt[3]{x}$

ii $y = \sqrt[5]{x}$

b i $y = 8\sqrt[4]{x}$

ii $y = \frac{\sqrt{x}}{3}$

c i $y = -\frac{1}{x}$

ii $y = \frac{1}{x^4}$

d i $y = \frac{3}{x^2}$

ii $y = -\frac{2}{5x^{10}}$

e i $y = \frac{1}{\sqrt{x}}$

ii $y = \frac{1}{\sqrt[3]{x}}$

f i $y = -\frac{10}{\sqrt[5]{x}}$

ii $y = \frac{8}{3\sqrt[4]{x}}$

2 Find $\mathrm{f}'(x)$ for the following:

a i $\mathrm{f}(x) = (2x - 3)(x + 4)$

ii $\mathrm{f}(x) = 3x(x - 5)$

b i $\mathrm{f}(x) = \sqrt{x}\,(4x + 3)$

ii $\mathrm{f}(x) = \sqrt[3]{x}\,(x - 1)$

c i $\mathrm{f}(x) = (\sqrt{x} + 2x)^2$

ii $\mathrm{f}(x) = (\sqrt[4]{x} - 4)^2$

d i $\mathrm{f}(x) = \left(x + \frac{1}{x}\right)^2$

ii $\mathrm{f}(x) = \left(x + \frac{2}{x}\right)\left(x - \frac{2}{x}\right)$

3 Differentiate the following:

a i $\mathrm{f}(x) = \frac{3x - 2}{x}$

ii $\mathrm{f}(x) = \frac{1 + 4x^2}{2x}$

b i $\mathrm{f}(x) = \frac{\sqrt{x} - 3}{x^2}$

ii $\mathrm{f}(x) = \frac{x^2 + 4}{\sqrt{x}}$

4 Differentiate $y = x^2(3x - 4)$.

5 A curve has equation $y = 2\sqrt{x}\,(x^3 + 4)$. Find y'.

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6 $y = \sqrt[5]{x^4}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

7 Find the derivative of the function $\mathrm{f}(x) = \frac{8}{3\sqrt[4]{x^3}}$.

8 A curve has equation $y = \frac{3x^5 - 2x}{x^2}$.

a Express $y = \frac{3x^5 - 2x}{x^2}$ in the form $y = ax^p + bx^q$.

b Hence find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

9 $\mathrm{f}(x) = \frac{(x+1)(x+9)}{x}$

Show that $\mathrm{f}'(x) = \frac{(x-3)(x+3)}{x^2}$.

10 $\mathrm{f}(x) = \frac{9x^2 + 3}{2\sqrt[3]{x}}$. Find $\mathrm{f}'(x)$.

11 Find the derivative of the curve $y = \frac{(2\sqrt{x} - 3)^2}{\sqrt{x^3}}$.

Section 5: Interpreting derivatives and second derivatives

The derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ has two related interpretations:

- It is the gradient of the graph of y against x.
- It measures how fast y changes when x is changed – this is called the **rate of change** of y with respect to x.

To calculate the gradient (or the rate of change) at any particular point, you simply substitute the value of x into the equation for the derivative.

Tip

Your calculator may be able to find the gradient at a given point, but it cannot give you the expression for the derivative.

WORKED EXAMPLE 13.15

Find the gradient of the graph $y = 4x^3$ at the point where $x = 2$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2$ — The gradient is given by the derivative, so find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

When $x = 2$, — Substitute the given value for x.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 12 \times 2^2 = 48$

So the gradient is 48.

If you know the gradient of a graph at a particular point, you can find the value of x at that point. This involves solving an equation.

WORKED EXAMPLE 13.16

Find the values of x for which the graph of $y = x^3 - 7x + 1$ has gradient 5.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 7$ — The gradient is given by the derivative.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 5$ — You know that $\frac{\mathrm{d}y}{\mathrm{d}x} = 5$ so you can form an equation.

$3x^2 - 7 = 5$

$3x^2 = 12$

$x^2 = 4$

$x = 2$ or -2

Increasing and decreasing functions

The sign of the gradient at a point tells you whether the function is increasing or decreasing at that point.

Key point 13.6

- If $\frac{dy}{dx}$ is positive the function is increasing – as x gets larger, so does y.
- If $\frac{dy}{dx}$ is negative the function is decreasing – as x gets larger, y gets smaller.

A function can increase on some intervals and decrease on others.

Fast forward

In Chapter 14 you will find out what happens when $\frac{dy}{dx} = 0$.

WORKED EXAMPLE 13.17

Find the range of values of x for which the function $f(x) = 2x^3 - 6x$ is decreasing.

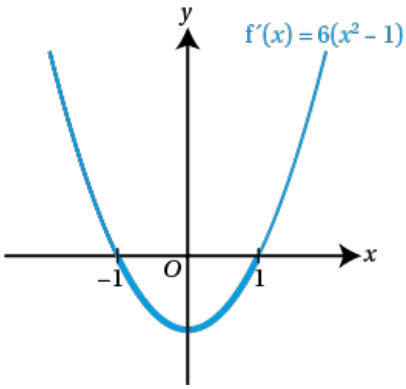
$f'(x) = 6x^2 - 6$ — First find the derivative.

$f'(x) < 0$ — A decreasing function has negative gradient.

$6x^2 - 6 < 0$

$6(x^2 - 1) < 0$ — This is a quadratic inequality. To solve it, first sketch the graph.

$6(x - 1)(x + 1) < 0$



$\therefore -1 < x < 1$ — The graph is below the x-axis between the two intercepts.

Some functions increase (or decrease) for all values of x.

Rewind

See Chapter 3 for a reminder of quadratic inequalities.

WORKED EXAMPLE 13.18

Show that the function $f(x) = 3x^3 + 5x$ is increasing for all x.

$f'(x) = 9x^2 + 5$ — The derivative tells us whether a function is increasing or decreasing.

Since $x^2 \geqslant 0$, $f'(x) > 0$ for all x. — Squares can never be negative.

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Hence $\mathrm{f}(x)$ is always increasing.

Positive derivative means that the function is increasing.

WORK IT OUT 13.2

Is the function $\mathrm{f}(x) = 5x - 3x^2$ increasing or decreasing at $x = -2$?

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$$\begin{aligned}\mathrm{f}(-2) &= 5(-2) - 3(-2^2)\\ &= -10 + 12\\ &= 2 > 0\end{aligned}$$

So it is increasing.

Solution 2

$$\begin{aligned}\mathrm{f}'(x) &= 5 - 6x\\ \mathrm{f}'(-2) &= 5 + 12 = 17 > 0\end{aligned}$$

So it is increasing.

Solution 3

$\mathrm{f}'(x) = 5 - 6x$

When $x = -2$:

$5 - 12 = -7 < 0$

So it is decreasing.

There is nothing special about the variables y and x, and any letters could be used. To emphasise which variables you are using, you call $\frac{\mathrm{d}y}{\mathrm{d}x}$ the derivative of y with respect to x.

WORKED EXAMPLE 13.19

Given that $a = \sqrt{S}$, find the rate of change of a with respect to S when $S = 9$.

The rate of change is given by the derivative.

$$a = S^{\frac{1}{2}}$$

$$\begin{aligned}\frac{da}{dS} &= \frac{1}{2}S^{-\frac{1}{2}}\\ &= \frac{1}{2\sqrt{S}}\end{aligned}$$

When $S = 9$:

Substitute the given value for S.

$$\frac{da}{dS} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

Higher derivatives

Since the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ is itself a function of x, you can differentiate it with respect to x. The result is called the **second derivative**.

The second derivative is the derivative of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and is given the symbol $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ or $\mathrm{f}''(x)$. It measures the rate of change of the gradient.

You can differentiate again to find the third derivative $\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3} \text{ or } \mathrm{f}'''(x)\right)$, fourth derivative $\left(\frac{\mathrm{d}^4y}{\mathrm{d}x^4} \text{ or } \mathrm{f}^{(4)}(x)\right)$, and so on.

WORKED EXAMPLE 13.20

Let $\mathrm{f}(x) = 5x^3 - 4x$.

a Find $\mathrm{f}''(x)$.

b Find the rate of change of the gradient of the graph of $y = \mathrm{f}(x)$ at the point where $x = -1$.

c Show that, when $x = 0.1$, the graph of $y = \mathrm{f}(x)$ is decreasing, but its gradient is increasing. Hence sketch the graph of $y = \mathrm{f}(x)$ near $x = 0.1$.

a $\mathrm{f}'(x) = 15x^2 - 4$

$\mathrm{f}''(x) = 30x$

Differentiate $\mathrm{f}(x)$ and then differentiate the result.

b $\mathrm{f}''(-1) = -30$

The rate of change of the gradient means the second derivative.

c $\mathrm{f}'(0.1) = 15 \times 0.1^2 - 4$

$= -3.85 < 0$

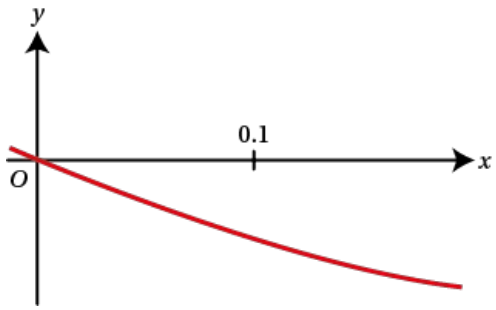
So the function is decreasing.

A decreasing function has a negative gradient.

$\mathrm{f}''(0.1) = 30 \times 0.1 = 3 > 0$

So the gradient is increasing.

The rate of change of gradient is $\mathrm{f}''(0.1)$. If this is positive, the gradient is increasing.



$\mathrm{f}(x)$ is decreasing, which means that the graph is going down (looking from left to right).

The gradient is increasing, but it is negative. This means that it is getting less negative, and so the graph is becoming less steep.

EXERCISE 13E

1 Write the following rates of change as derivatives.

a The rate of change of z as t changes.

b The rate of change of Q with respect to P.

c How fast R changes when m is changed.

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d How quickly the volume of a balloon (V) changes over time (t).

e The rate of increase of the cost of apples (y) as the mass of the apple (x) increases.

f The rate of change of the rate of change of z as y changes.

g The second derivative of H with respect to m.

2 **a i** If $\mathrm{f} = 5x^{\frac{1}{3}}$ what is the derivative of f with respect to x?

ii If $p = 3q^5$ what is the derivative of p with respect to q?

b i Differentiate $d = 3t + 7t^{-1}$ with respect to t.

ii Differentiate $r = c + \dfrac{1}{c}$ with respect to c.

c i Find the second derivative of $y = 9x^2 + x^3$ with respect to x.

ii Find the second derivative of $z = \dfrac{3}{t}$ with respect to t.

3 **a i** If $y = 5x^2$ find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 3$.

ii If $y = x^3 + \dfrac{1}{x}$ find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 1.5$.

b i If $A = 7b + 3$ find $\dfrac{\mathrm{d}A}{\mathrm{d}b}$ when $b = -1$.

ii If $\Phi = \theta^2 + \theta^{-3}$ find $\dfrac{\mathrm{d}\Phi}{\mathrm{d}\theta}$ when $\theta = 0.1$.

c i Find the gradient of the graph of $A = x^3$ when $x = 2$.

ii Find the gradient of the tangent to the graph of $z = 2a + a^2$ when $a = -6$.

d i How quickly does $\mathrm{f} = 4T^2$ change as T changes when $T = 3$?

ii How quickly does $\mathrm{g} = y^4$ change as y changes when $y = 2$?

e i What is the rate of increase of W with respect to p when p is -3 if $W = -p^2$?

ii What is the rate of change of L with respect to c when $c = 6$ if $L = 7\sqrt{c} - 8$?

Tip

Φ is the Greek letter phi. This is the capital form of the letter.

4 **a i** If $y = ax^2 + (1 - a)x$ where a is a constant, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

ii If $y = x^3 + b^2$ where b is a constant, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

b i If $Q = \sqrt{ab} + \sqrt{b}$ where b is a constant, find $\dfrac{\mathrm{d}Q}{\mathrm{d}a}$.

ii If $D = 3(av)^2$ where a is a constant, find $\dfrac{\mathrm{d}D}{\mathrm{d}v}$.

5 **a i** If $y = x^3 - 5x$ find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x = 9$.

ii If $y = 8 + 2x^4$ find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x = 4$.

b i If $S = 3A^2 + \dfrac{1}{A}$ find $\dfrac{\mathrm{d}^2S}{\mathrm{d}A^2}$ when $A = 1$.

ii If $J = v - \sqrt{v}$ find $\dfrac{\mathrm{d}^2J}{\mathrm{d}v^2}$ when $v = 9$.

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c i Find the second derivative of B with respect to n if $B = 8n$ and $n = 2$.

ii Find the second derivative of g with respect to r if $\mathrm{g} = r^7$ and $r = 1$.

6 **a i** If $y = 3x^3$ find x at the point where $\frac{\mathrm{d}y}{\mathrm{d}x} = 36$.

ii If $y = x^4 + 2x$ find x at the point where $\frac{\mathrm{d}y}{\mathrm{d}x} = 6$.

b i If $y = 2x + \frac{8}{x}$ find y at the point where $\frac{\mathrm{d}y}{\mathrm{d}x} = -30$.

ii If $y = \sqrt{x} + 3$ find y at the point where $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{6}$.

c i If $y = 3x^3$ find x at the point where $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -54$.

ii If $y = \frac{2}{x}$ find x at the point where $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{2}$.

Worksheet

For more practice of increasing and decreasing functions see Support sheet 13.

7 **a i** Find the interval in which $x^2 - x$ is an increasing function.

ii Find the interval in which $x^2 + 2x - 5$ is a decreasing function.

b i Find the range of values of x for which $x^3 - 3x$ is an increasing function.

ii Find the range of values of x for which $x^3 + 2x^2 - 5$ is a decreasing function.

c i Find the interval in which the gradient of $y = x^3 - 3x$ is decreasing.

ii Find the interval in which the gradient of $y = 2x^3 - x^2$ is increasing.

8 **a** Find the rate of change of $y = \frac{3}{\sqrt{x}}$ at the point where $x = 9$.

b Is the gradient increasing or decreasing at this point?

9 Find the rate of change of the gradient of $\mathrm{f}(x) = \sqrt{x} + \frac{1}{x}$ when $x = 4$.

10 Find the range of values of x for which the function $y = 3x^3 - 4x + 1$ is decreasing.

11 Find the value of x for which the graph of $\frac{1}{3}x^3 - x^2 - 12x + 1$ has gradient 3.

12 Find the x-coordinates of the points on the curve $y = x^4 - 12x^2 + 3x - 1$ where $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$.

13 Show that $y = x^3 + kx + c$ is always increasing if $k > 0$.

14 Find all points of the graph of $y = x^2 - 2x + 1$ where the gradient equals the y-coordinate.

15 Find the range of x-values for which $\mathrm{f}(x) = x^3 - 6x^2 + 5$ is an increasing function.

16 The function $\mathrm{f}(x) = 12x - 2x^2 - \frac{1}{3}x^3$ is increasing for $a < x < b$. Find the constants a and b.

17 Find the interval in which the gradient of the graph of $y = 7x - x^2 - x^3$ is decreasing.

18 In what interval is the gradient of the graph $y = \frac{1}{4}x^4 + x^3 - \frac{1}{2}x^2 - 3x + 6$ decreasing?

19 Find the range of values of x for which the function $\mathrm{f}(x) = x^3 - 6x^2 + 9x + 2$ is decreasing but its gradient is increasing.

20 Find an alternative expression for $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n)$.

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Checklist of learning and understanding

- The gradient (or the derivative) of a function at the point P is the gradient of the tangent to the function's graph at that point.
- To find the derivative of a function you differentiate.
  - Differentiation from first principles:
    $\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$
  - If $\mathrm{f}(x) = ax^n$, then $\mathrm{f}'(x) = anx^{n-1}$.
  - The derivative of a sum is the sum of the derivatives of each term.
- The derivative at a point tells you the rate of change of the y-coordinate at that point.
- You can use the derivative to tell whether a function is increasing or decreasing:
  - If $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ the function is increasing.
  - If $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ the function is decreasing.
- The second derivative, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, gives you the rate of change of gradient.

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Mixed practice 13

1 Find the gradient function of $f(x) = \dfrac{3x-2}{\sqrt{x}}$.

2 A curve has equation $y = (4x^2 - 1)(3 - x)$. Find $\dfrac{dy}{dx}$.

3 $f(x) = \dfrac{x^2-4}{2x}$. Find $f''(2)$.

4 Given that $f(x) = 3\sqrt{x} - \dfrac{2}{\sqrt{x}}$ find:

a $f'(x)$

b the gradient of the graph of $y = f(x)$ at the point where $x = 4$.

5 $f(x) = x^2 + bx + c$. If $f(1) = 2$ and $f'(2) = 12$ find the values of b and c.

6 **a** Find the gradient of the curve $y = 3\sqrt{x} - 2$ at the point where it crosses the x-axis.

b Is the curve increasing or decreasing at this point? Give a reason for your answer.

7 Find the range of values of x for which the function $y = 3x^2 - 4x$ is increasing.

8 Find the rate of change of gradient of $y = x^2 - 2\sqrt{x}$ at the point where $x = 9$.

9 Given that $y = \dfrac{5}{x^2} - \dfrac{1}{4x} + x$ find:

a $\dfrac{dy}{dx}$

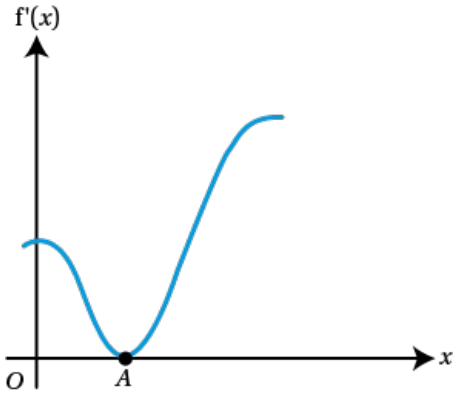
b $\dfrac{d^2y}{dx^2}$

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[Question part reference style adapted]

10 $y = x^2 + ax - 7$ is increasing for $x > 5$. Find a.

11 What is the rate of change of the gradient of $y = x^3 + 4x^2 - 2x + 1$ at $x = \dfrac{1}{2}$?

12 This graph shows the gradient function, $f'(x)$, of a function $f(x)$. Which of the following is definitely true at the point A?



A $f(x)$ has a minimum

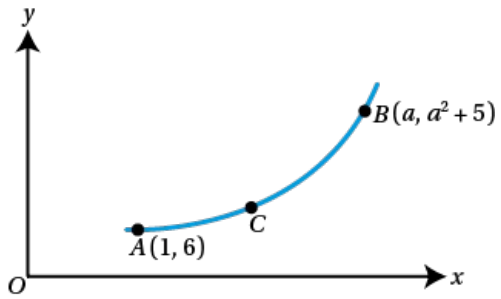
B $f(x)$ has a maximum

C $f(x) = 0$

D $f''(x) = 0$

13 The diagram shows part of the curve $y = x^2 + 5$. The point A has coordinates $(1, 6)$. The point

B has coordinates $\left(a, a^2+5\right)$, where a is a constant greater than 1. The point C is on the curve between A and B.



a Find by differentiation the value of the gradient of the curve at the point A.

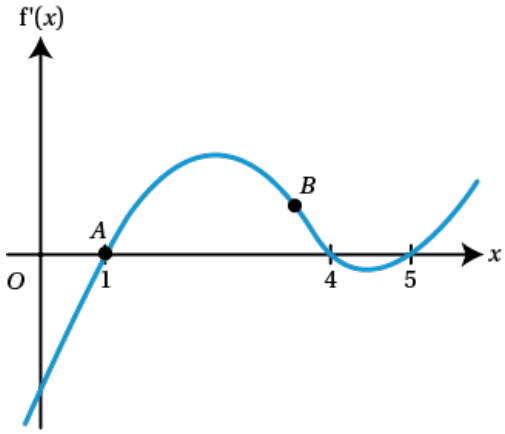
b The line segment joining the points A and B has gradient 2.3. Find the value of a.

c State a possible value for the gradient of the line segment joining the points A and C.

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[Question part reference style adapted]

14 Use differentiation from first principles to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = x^3 - 5x$.

15 The diagram shows the graph of $y = \mathrm{f}'(x)$.



a State the value of the gradient of the graph of $y = \mathrm{f}(x)$ at the point marked A.

b Is the function $\mathrm{f}(x)$ increasing or decreasing at the point marked B?

c Sketch the graph of $y = \mathrm{f}(x)$.

16 Find the coordinates of the point on the curve $y = \sqrt{x} + 3x$ where the gradient is 5.

17 Find the gradient of the graph of $y = \frac{1}{2\sqrt{x}}$ at the point where the y-coordinate is 3.

18 $\mathrm{f}(x) = ax^3 + bx^{-2}$ where a and b are constants. $\mathrm{f}'(1) = 18$ and $\mathrm{f}''(1) = 18$. Find a and b.

19 $\mathrm{f}(x) = \sqrt{x^3} + 15\sqrt{x}$

Find the values of x for which the gradient of $\mathrm{f}(x)$ is 9.

20 Find the range of values of x for which the gradient of the graph $y = x^4 - 2x^2 + 3$ is decreasing.

14 Applications of differentiation

In this chapter you will learn how to:

- find the equations of tangents and normals to curves at given points
- find maximum and minimum points on curves
- solve problems which involve maximising or minimising quantities.

Before you start...

Chapter 13	You should know how to differentiate functions involving x^n.	1 Differentiate the following: a. $\frac{3x^4-2}{5x^2}$ b. $\frac{3}{2\sqrt{x}}$
Chapter 13	You should know how to evaluate second derivatives.	2 Given that $y=\frac{3}{2x^2}$ evaluate $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x=-2$.
Chapter 6	You should know how to find the equation of a straight line.	3 Find the equation of the line through the point $(-2,1)$ with gradient 3.
Chapter 6	You should know how to find the equation of a perpendicular to a line.	4 Find the equation of the perpendicular to the line with gradient $\frac{3}{4}$ that passes through the point $(2,-3)$.

Where can you use differentiation?

You can apply the techniques from the Chapter 13 to solve a variety of problems. You will first look at graphs of functions, learning how to find equations of tangents and normals. These have applications in mechanics when studying collisions, as well in pure mathematics when defining ‘curvature’ of a graph. You will then turn to finding coordinates of maximum and minimum points and solving practical problems where you need to minimise or maximise quantities, such as production costs.

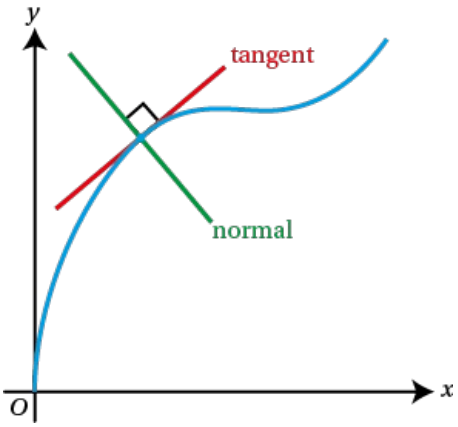
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Section 1: Tangents and normals

The **normal** to a curve at a given point is a straight line that crosses the curve at that point and is perpendicular to the tangent.

You know from the previous chapter that the gradient of the tangent at a point is the value of the curve's gradient at that point; this can be found by substituting the value of x into the equation for the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$.

The gradient of the normal can then be found by using the fact that if two lines with gradients m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$.



Once you have the gradient you can use it, together with the coordinates of the point, to find the equations of the tangent and normal.

Rewind

See Chapter 6, Section 1 for a reminder of how to find the equation of the line with a given point and gradient.

WORKED EXAMPLE 14.1

A curve has equation $y = x^3 - 5x^2 - x^{\frac{3}{2}} + 22$.

Find the equations of:

a the tangent

b the normal

to the curve at the point $(4, -2)$.

In each case give your answer in the form $ax + by + c = 0$ where a, b, c are integers.

a $y' = 3x^2 - 10x - \frac{3}{2}x^{\frac{1}{2}}$

First find y'.

At $x = 4$,

$y' = 3(4)^2 - 10(4) - \frac{3}{2}(4)^{\frac{1}{2}}$

$= 48 - 40 - 3$

$= 5$

Then evaluate the derivative at $x = 4$.
This will give the gradient of the tangent at $x = 4$.

Equation of the tangent:

$y-(-2)=5(x-4)$

$y+2=5x-20$

$5x-y-22=0$

Use $y-y_1=m(x-x_1)$ to find the equation of the straight line with gradient 5, passing through (4, −2).

b Gradient of the normal:

$m_n=\frac{-1}{5}$

Find the gradient of the normal, using $m_1m_2=-1$.

Equation of the normal:

$y-(-2)=\frac{-1}{5}(x-4)$

$5y+10=-x+4$

$x+5y+6=0$

Use the equation of the line again. The coordinates of the point are the same, but now use the gradient of the normal.

The procedure for finding the equations of tangents and normals is summarised in Key point 14.1.

Key point 14.1

For the point on the curve $y=\mathrm{f}(x)$ with $x=a$:

- the gradient of the tangent is $\mathrm{f}'(a)$
- the gradient of the normal is $-\frac{1}{f'(a)}$
- the coordinates of the point are $x_1=a$, $y_1=\mathrm{f}(a)$.

To find the equation of the tangent or the normal, use $y-y_1=m(x-x_1)$ with the appropriate gradient.

Harder questions may give you some information about the tangent and require you to find other information.

WORKED EXAMPLE 14.2

The tangent at point P on the curve $y=x^2+1$ passes through the origin. Find the possible coordinates of P.

Let P have coordinates (p,q)

You want to find the equation of the tangent at P, but you don't know the coordinates.

Then $q=p^2+1$

As P lies on the curve, (p, q) must satisfy $y=x^2+1$.

$\frac{\mathrm{d}y}{\mathrm{d}x}=2x$

When $x=p$:

$\frac{dy}{dx}=2p$

$\therefore m=2p$

The gradient of the tangent is given by $\frac{dy}{dx}$ when $x=p$.

Equation of the tangent:

$y-q=2p(x-p)$

$y-\left(p^2+1\right)=2p(x-p)$

Write the equation of the tangent, remembering it passes through (p, q).

Passes through $(0,0)$:

The tangent passes through the origin, so set $x=0$, $y=0$.

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$0 - (p^2 + 1) = 2p(0 - p)$

$-p^2 - 1 = -2p^2$

$p^2 = 1$

$p = 1 or -1$

When $p = 1$, $q = 2$

When $p = -1$, $q = 2$

So the coordinates of P are $(1, 2)$ or $(-1, 2)$.

Now use $q = p^2 + 1$ to find the corresponding values of q.

Tip

Notice that letters other than x and y are used for the unknown coordinates at the point P, as x and y will appear in the equation of the tangent, $y - y_1 = m(x - x_1)$.

WORK IT OUT 14.1

Find the equation of the normal to the curve $y = \frac{3}{x}$ at the point where $x = 2$.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$\frac{dy}{dx} = -\frac{3}{x^2}$

$x = 2$, $m = -\frac{3}{2^2} = -\frac{3}{4}$, $y = \frac{3}{2}$

equation:

$y - \frac{3}{2} = -\frac{3}{4}(x - 2)$

$3x + 4y = 12$

Solution 2

$\frac{dy}{dx} = -\frac{3}{x^2}$

$x = 2$, $m = \frac{2^2}{3} = \frac{4}{3}$, $y = \frac{3}{2}$

equation:

$y - \frac{3}{2} = \frac{4}{3}(x - 2)$

$8x - 6y = 7$

Solution 3

$\frac{dy}{dx} = -\frac{3}{x^2}$

so normal $m = \frac{x^2}{3}$

$x = 2$, $y = \frac{3}{2}$

equation:

$y - \frac{3}{2} = \frac{x^2}{3}(x - 2)$

$y = \frac{x^3}{3} - \frac{2}{3}x^2 + \frac{3}{2}$

WORKED EXAMPLE 14.3

The point P on the curve $y = \frac{a}{x}$ has x-coordinate 3. The normal to the curve at P is parallel to the line $x - 3y + 6 = 0$. Find the constant a.

Normal at P is parallel to:

$x - 3y + 6 = 0$

$y = \frac{1}{3}x + 2$

Rearrange the equation into the form $y = mx + c$ to find the gradient of the normal.

So, gradient of curve at P is

$m = -\frac{1}{\frac{1}{3}}$

$= -3$

Find the gradient of the tangent at P using $m_1 m_2 = -1$.

$y = ax^{-1}$

$y' = -ax^{-2}$

The gradient of the curve is given by y'.

When $x = 3$,

$y' = -a(3)^{-2}$

$= -\frac{a}{9}$

Evaluate at $x = 3$.

Therefore,

$-\frac{a}{9} = -3$

$a = 27$

The gradient of the tangent and the gradient of the curve at P are the same, so set them equal and solve for a.

EXERCISE 14A

1 Find the equation of the tangent and normal at the following points. In each case, give your answer in the form $ax + by + c = 0$.

a i $y = 2x^2 - 4x$ at $(1, -2)$

ii $y = x^2 - 6x + 9$ at $(1, 4)$

b i $y = \frac{3}{x}$ at (-3, -1)

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ii $y = -\dfrac{6}{x^2}$ at $\left(2, -\dfrac{3}{2}\right)$

c i $y = x + \dfrac{12}{\sqrt{x}}$ at $(4, 10)$

ii $y = 4\sqrt[3]{x} - \dfrac{x}{2}$ at $(8, 4)$

2 Find the equation of the tangent to the curve $y = \dfrac{x^2 + 4}{\sqrt{x}}$ at the point where $x = 4$.

3 Find the equation of the normal to the curve $y = 2x - \dfrac{3}{\sqrt{x}}$ at the point where $x = 4$.

4 The normal to the curve with equation $y = 2x^3 - 4x$ at the point $x = 2$ crosses the x-axis at P. Find the coordinates of P.

5 The tangent to the graph of $y = \dfrac{3}{x^2}$ at $x = 3$ crosses the coordinate axes at the points A and B. Find the area of the triangle *AOB*, where O is the origin.

6 Find the equations of the tangents to the curve $y = 2x^3 - 9x^2 + 12x + 1$ which are parallel to the x-axis.

7 Find the x-coordinates of the points on the curve $y = x^3 - 2x^2$ where the tangent is parallel to the normal at the point $(1, -1)$.

8 The normal to the curve $y = ax^3 + bx^2$ at the point $(1, 2)$ is parallel to the line $x - 2y + 14 = 0$. Find a and b

9 The tangent to the curve $y = p\sqrt{x} + qx$ at the point $\left(1, -\dfrac{7}{6}\right)$ is parallel to the line $2x + 3y - 12 = 0$. Find p and q.

10 Given that $\mathrm{f}(x) = \dfrac{2}{x}$ $y = \mathrm{f}(x)$ find the coordinates of the point where the tangent to $y = \mathrm{f}(x)$ at $x = 1$ intersects the normal to $y = \mathrm{f}(x)$ at $x = -2$.

11 Find the coordinates of the point where the tangent to the curve $y = x^3 - 3x^2$ at $x = 2$ meets the curve again.

12 Find the coordinates of the point on the curve $y = (x - 1)^2$ where the normal passes through the point $(0, 0)$.

13 $y = -\dfrac{1}{2}x + c$ is the equation of a normal to the curve $y = x^2 + 5x + 4$. Find the constant c.

14 A tangent is drawn on the graph $y = \dfrac{k}{x}$ at the point where $x = a (a > 0)$.The tangent intersects the y-axis at P and the x axis at Q. If O is the origin, show that the area of the triangle *OPQ* is independent of a.

15 Show that the tangent to the curve $y = x^3 - x$ at the point with x-coordinate a meets the curve again at a point with x-coordinate $-2a$.

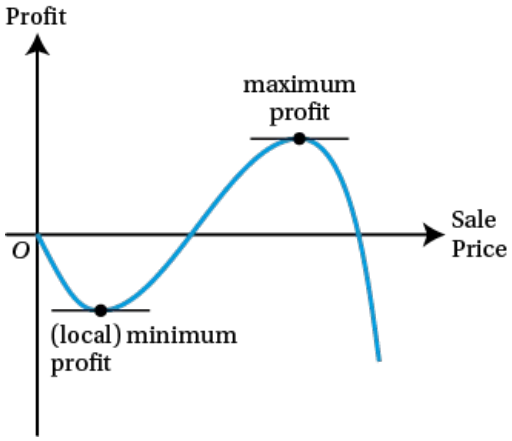
Worksheet

For more questions with tangents and normals, see Extension sheet 14.

Section 2: Stationary points

In real life, you might be interested in maximising profits or minimising the drag on a car. You can use calculus to mathematically describe such problems.

The quantity you wish to maximise or minimise often depends on another variable. For example, the profit might depend on the selling price of the item. You can represent this relationship on a graph.



The gradients at both the local maximum and the local minimum point on this graph are zero.

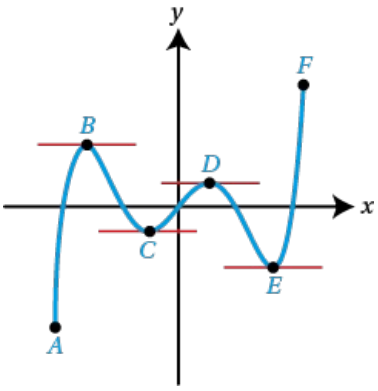
Key point 14.2

To find local maximum and local minimum points, solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

You use the terms **local maximum** and **local minimum** because it is possible that the largest or smallest value of the whole function occurs at the endpoint of the graph, or that there are several points that have gradient zero. The points that you have found are just the largest or smallest values of y in their local area.

For this function, $y = \mathrm{f}(x)$, defined for $x_A \leqslant x \leqslant x_F$:

- B and D are local maximum points
- C and E are local minimum points
- A is the point where y is smallest
- F is the point where y is largest.



Points such as B, C, D and E, which have a gradient of zero, are called **stationary points**.

WORKED EXAMPLE 14.4

Find the coordinates of the stationary points of $y = 2x^3 - 15x^2 + 24x + 8$.

$\frac{dy}{dx} = 6x^2 - 30x + 24$ — First differentiate

For stationary points $\frac{dy}{dx} = 0$: — Then solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$6x^2 - 30x + 24 = 0$

$x^2 - 5x + 4 = 0$

$(x - 4)(x - 1) = 0$

$x = 1 \quad \text{or} \quad x = 4$

When $x = 1$: — Remember to find the y-coordinate for each point.

$y = 2(1)^3 - 15(1)^2 + 24(1) - 8$

= 19

When $x = 4$:

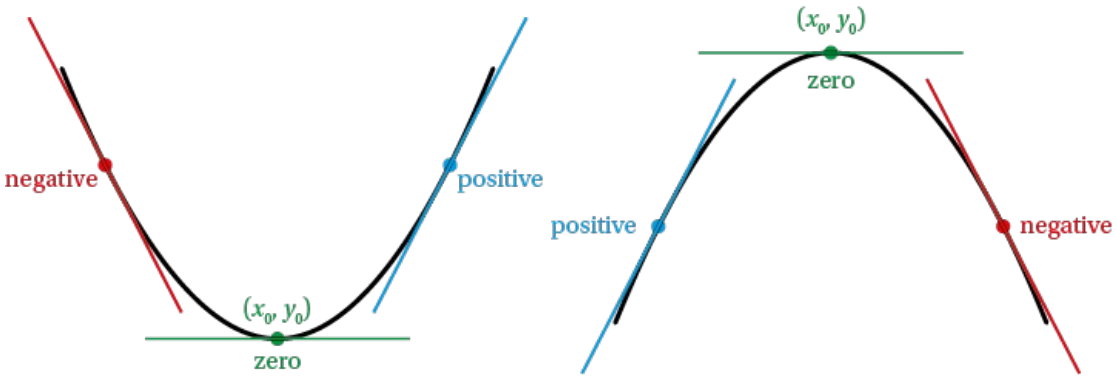
$y = 2(4)^3 - 15(4)^2 + 24(4) + 8$

$= -8$

Therefore, stationary points are: — State the coordinates.

(1,19) and $(4, -8)$

The calculation in Worked example 14.4 does not tell you whether the stationary points you found are maximum or minimum points. To decide this, we need to look at how the gradient changes on either side of the stationary point.



It can be seen from these diagrams that for a minimum point the gradient goes from negative to positive. This means that the gradient is increasing, so the rate of change of the gradient, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, is positive. For a maximum point, the gradient goes from positive to negative, so $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is negative. This test does not tell us what happens to the gradient if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is zero.

Key point 14.3

Given a stationary point (x_0, y_0) of a function $y = \mathrm{f}(x)$:

- if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ at x_0 then (x_0, y_0) is a local **maximum**
- if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ at x_0 then (x_0, y_0) is a local **minimum**
- if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at x_0 then no conclusion can be drawn.

If a question asks you to ‘classify’ or ‘determine the nature of’ the stationary points on a curve, you need

to decide whether each point is a local maximum or a local minimum.

Tip

It is common to simply say ‘maximum’ instead of ‘local maximum’ when the intended meaning is clear.

WORKED EXAMPLE 14.5

Find and classify the stationary points of the curve $y = 2x^3 - 15x^2 + 24x + 8$ from Worked example 14.4.

Stationary points: (1,19) and (4,-8)	You have already found the stationary points.
$\frac{d^2y}{dx^2} = 12x - 30$	Find the second derivative.
At $x = 1$: $\frac{d^2y}{dx^2} = 12(1) - 30$	Evaluate the second derivative at the stationary point with $x = 1$.
$= -18 < 0$ $\therefore$(1,19) is a maximum	$\frac{d^2y}{dx^2} < 0$ so the point is a local maximum.
At $x = 4$: $\frac{d^2y}{dx^2} = 12(4) - 30$	Evaluate the second derivative at the stationary point with $x = 4$.
= 18 > 0 $\therefore(4, -8)$ is a minimum	$\frac{d^2y}{dx^2} < 0$ so the point is a local minimum.

Tip

Remember to solve $\frac{dy}{dx} = 0$ first, then substitute the x-value into $\frac{d^2y}{dx^2}$ to test whether the points are maximum or minimum. Do not try to solve the equation $\frac{d^2y}{dx^2} = 0$.

Fast forward

A You will see in Student Book 2 that there is another type of stationary point other than a maximum or minimum.

One use of stationary points is to find the largest and smallest values of a function for a given set of x-values.

Tip

Remember that the largest/smallest value of a function in a given interval could occur at the endpoint(s) of the interval instead of at the local maximum/minimum.

Fast forward

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A You will see in Student Book 2 that this is related to the *range* of a function.

WORKED EXAMPLE 14.6

Find the largest and smallest values of $f(x) = 2x^3 - 6x + 3$ for $-2 \leqslant x \leqslant 3$.

Endpoints:

$f(-2) = 2(-2)^3 - 6(-2) + 3$
$= -1$
$f(3) = 2(3)^3 - 6(3) + 3$
$= 39$

Evaluate the function at the endpoints; it may have a greater value at an endpoint than it does at any local maximum and/or be less than any local minimum.

Stationary points:

$f'(x) = 0$

$6x^2 - 6 = 0$

$6(x - 1)(x + 1) = 0$

$x = 1$ or -1

Stationary points give local minimum and maximum values. Solve $f'(x) = 0$ to find any.

$f(1) = 2(1)^3 - 6(1) + 3$
$= -1$
$f(-1) = 2(-1)^3 - 6(-1) + 3$
$= 7$

Find the value of the function at the stationary points. Second derivative analysis is not needed; the maximum and minimum values of the function must be either at endpoints or at stationary points.

So, the largest value of $f(x)$ is 39.

The smallest value of $f(x)$ is -1

$-1 \leqslant f(x) \leqslant 39$

$f(-2)$ is the same as at the local minimum, but $f(3)$ is larger than at the local maximum.

EXERCISE 14B

IRANINOVIN

1 Find and classify the stationary points on the following curves.

a i $y = x^3 - 5x^2$

ii $y = 1 + 24x^2 - x^3$

b i $y = x^4 - 8x^2$

ii $y = x^4 - 2x^2 + 3$

c i $y = 2x - \frac{1}{x}$

ii $y = 3x^2 + \frac{9}{x}$

d i $y = 2\sqrt{x} - 3x$

ii $y = \frac{1}{\sqrt{x}} + 3x$

Worksheet

See Support sheet 14 for more practice.

2 Find and classify the stationary points on the curve $y = x^3 + 3x^2 - 24x + 12$.

3 Prove that the curve $y = x^3 - 3x^2 + 4x - 1$ has no stationary points.

4 $\mathrm{f}(x) = \frac{9x^2 + 1}{x}$

a Find the x-coordinates of the stationary points on the curve $y = \mathrm{f}(x)$.

b Determine whether each is a maximum or minimum point.

5 Find the coordinates of the stationary point on the curve $y = x - \sqrt{x}$ and determine its nature.

6 **a** Find the coordinates of the stationary points on the curve with equation $y = x^2\,(6\sqrt[3]{x} - 7)$.

b Establish whether each is a maximum or minimum point.

7 The curve $y = x^3 + 10x^2 + kx - 2$ has a stationary point at $x = -8$.

a Find the x-coordinate of the other stationary point.

b Determine the nature of both stationary points.

8 The curve $y = ax^2 + bx - 2$ has a minimum point at $x = -2$ and passes through the point $(1, 13)$. Find a and b.

9 $\mathrm{f}(x) = 4 - 9x + 6x^2 - x^3, -2 \leqslant x \leqslant 4$

Find the largest and smallest values of $\mathrm{f}(x)$.

10 $\mathrm{f}(x) = -\frac{1}{4}x^4 - x^3 + 2x^2 - 10$

Show that $\mathrm{f}(x) \leqslant 22$ for all x.

11 **a** Find the stationary points of the curve $y = 3x^4 - 16x^3 + 18x^2 + 6$.

b Find the set of values of c for which the equation $3x^4 - 16x^3 + 18x^2 + 6 = c$ has four real roots.

12 Find and classify in terms of k the stationary points on the curve $y = kx^3 + 6x^2$.

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Section 3: Optimisation

You can now apply results from the previous section to solve practical problems which involve maximising or minimising quantities.

WORKED EXAMPLE 14.7

The distance travelled, in metres, by a paper aeroplane of mass w grams where $w > 1$ is

$$s = \left(1 - \frac{1}{\sqrt{w}}\right)\left(\frac{5}{w}\right)$$

a Find the mass the aeroplane needs to be to travel its maximum distance.

b Find this maximum distance.

a $s = \left(1 - \frac{1}{\sqrt{w}}\right)\left(\frac{5}{w}\right)$

Expand the brackets and write terms in the form axn using laws of indices.

$= \frac{5}{w} - \frac{5}{w\sqrt{w}}$

$= 5w^{-1} - 5w^{-\frac{3}{2}}$

$\frac{ds}{dw} = -5w^{-2} + \frac{15}{2}w^{-\frac{5}{2}}$

Then find the maximum by solving $\frac{ds}{dw} = 0$.

Stationary points:

$\frac{ds}{dw} = 0$

$-5w^{-2} + \frac{15}{2}w^{-\frac{5}{2}} = 0$

$\frac{15}{2}w^{-\frac{5}{2}} = 5w^{-2}$

$3w^{-\frac{5}{2}} = 2w^{-2}$

Multiply through by 2 and divide by 5.

$\frac{3}{w^{\frac{5}{2}}} = \frac{2}{w^2}$

Rewrite without the negative indices.

$3 = \frac{2w^{\frac{5}{2}}}{w^2}$

Multiply through by $w^{\frac{5}{2}}$.

$3 = 2w^{\frac{1}{2}}$

Simplify.

$\sqrt{w} = \frac{3}{2}$

Solve for w.

$w = \frac{9}{4}$ g

Nature of stationary point:

You know that there is a stationary point at this value of w but you need to check that this is a maximum.

$\frac{d^2s}{dw^2} = 10w^{-3} - \frac{75}{4}w^{-\frac{7}{2}}$

When $w = \frac{9}{4}$:

$\frac{d^2s}{dw^2} = -0.22 < 0$

$\therefore$ maximum

b Maximum value of s:

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$$s = \left(1 - \frac{1}{\sqrt{\frac{9}{4}}}\right)\left(\frac{\frac{5}{9}}{\frac{9}{4}}\right)$$
$$= \left(1 - \frac{2}{3}\right)\left(\frac{20}{9}\right)$$
$$= \frac{20}{27}\,m$$

Find the maximum value of the distance by evaluating the original expression for s at $w = \frac{9}{4}$.

In some examples, a function appears to depend on two different variables. However, these two variables will always be related by a **constraint**, a condition that allows one of the variables to be eliminated. You can then follow the normal procedure for finding maxima or minima.

The two common types of constraints are:

- A shape has a fixed perimeter, area or volume – this gives an equation relating different variables (height, length, radius...).
- A point lies on a given curve – this gives a relationship between x and y.

Focus on ...

Focus on ... Problem solving 2 looks at choosing the right variables in this sort of optimisation problem.

WORKED EXAMPLE 14.8

A wire of length $12\,\text{cm}$ is bent to form a rectangle.

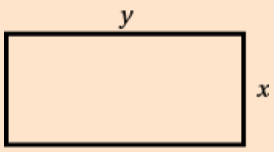
a Show that the area, A, is given by $A = 6x - x^2$ where x is the width of the rectangle.

b Find the maximum possible area.

a Let the length of the rectangle be y.

Then $A = xy$

Introduce a variable, y, for the length in order to write an expression for the area (which is to be maximised).



$P = 2x + 2y$

Since perimeter $P = 12$,

$12 = 2x + 2y$

$6 = x + y$

Use the information about the perimeter to form a second equation (the constraint).

$y = 6 - x$

Make y the subject.

So

$A = xy$

$= x(6 - x)$

$= 6x - x^2$

Substitute into the expression for A in order to eliminate y and express A as a function of x only.

b Stationary points:

$\frac{dA}{dx} = 6 - 2x = 0$

$x = 3$

Find the value of x at which the maximum of A occurs by solving $\frac{dA}{dx} = 0$.

Nature of stationary point:

$\frac{d^2A}{dx^2} = -2 < 0$

$\therefore$ maximum

Check that the stationary point is a maximum.

Maximum value of A is

$A = x(6 - x)$

$= 3(6–3)$

$= 9 \text{ cm}^2$

Find the maximum value of A by substituting $x = 3$ into $A = x(6 - x)$.

Explore

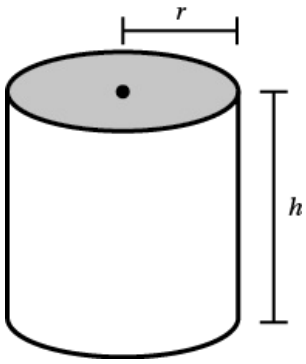
Worked example 14.8 is an example of the isoperimetric problem – to find the shape with the fixed perimeter and maximum area. Find out more about it – it has some very interesting applications.

Tip

Don't forget to substitute the value of x that maximises/minimises the quantity back into the original expression to find the maximum/minimum value of that quantity.

WORKED EXAMPLE 14.9

An open-top cylindrical can has base radius r and height h.



The external surface area $A = 243\pi cm^2$.

a Show that the volume, V, is given by $V = \frac{\pi}{2}(243r - r^3)$.

b Hence find the maximum capacity of the can.

a $V = \pi r^2 h$

Write down an expression for the quantity to be maximised.

$A = \pi r^2 + 2\pi rh$

The surface area is made up of the area of the base (pr2) and the curved surface area of the cylinder (2prh).

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Since $A = 243\pi$,

$243\pi = \pi r^2 + 2\pi rh$

$243 = r^2 + 2rh$

Use the information about the surface area to form a second equation (the constraint).

$h = \dfrac{243 - r^2}{2r}$

Make h the subject.

So

$V = \pi r^2 \left(\dfrac{243 - r^2}{2r}\right)$

$= \dfrac{\pi r^2}{2r}(243 - r^2)$

$= \dfrac{\pi r}{2}(243 - r^2)$

$= \dfrac{\pi}{2}(243r - r^3)$

Substitute into the expression for V in order to eliminate h and express V as a function of r only.

b Stationary points:

$\dfrac{\mathrm{d}V}{\mathrm{d}r} = 0$

$\dfrac{\pi}{2}(243 - 3r^2) = 0$

$243 - 3r^2 = 0$

$r^2 = 81$

$r = 9$ (r > 0)

Find the value of r at which the maximum of V occurs by solving $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 0$.

Nature of stationary point:

$\dfrac{\mathrm{d}^2V}{\mathrm{d}r^2} = \dfrac{\pi}{2}(-6r)$

$= -3\pi r$

$= -3\pi(9)$

$= -27\pi < 0$

$\therefore$ maximum

Check that the stationary point is a maximum.

Maximum value of V is

$V = \dfrac{\pi}{2}(243r - r^3)$

$= \pi(243(9) - (9)^3)$

$= 729\pi\, cm^3$

Don't forget to find the maximum value of V by substituting $r = 9$ into $V = \dfrac{\pi}{2}\left(243r - r^3\right)$.

Gateway to A Level

If you need a reminder about volume and area, see Gateway to A Level section R.

Gateway to A Level

For a reminder of changing the subject of the formula, and substituting one formula into another, see Gateway to A Level section S.

WORKED EXAMPLE 14.10

Let A be a point on the curve $y = x^2$ with x-coordinate a.

Let L be the distance from A to the point $P\left(2, \frac{1}{2}\right)$.

a Write down an expression for L^2 in terms of a.

b Find the minimum possible value of L^2.

c Hence write down the coordinate of the point on the curve $y = x^2$ that is closest to the point $P\left(2, \frac{1}{2}\right)$.

a The coordinates of A are (a, a^2)

For any point of the curve, the y-coordinate equals x^2.

The distance between A and P is

Use $L^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

$$L^2 = (a-2)^2 + \left(a^2 - \frac{1}{2}\right)^2$$

b
$$\begin{aligned} L^2 &= a^2 - 4a + 4 + a^4 - a^2 + \frac{1}{4} \\ &= a^4 - 4a + \frac{17}{4} \end{aligned}$$

Before differentiating to find stationary points, you need to expand the brackets.

$$\frac{\mathrm{d}\left(L^2\right)}{\mathrm{d}a} = 4a^3 - 4$$

Stationary points:

$$\begin{aligned} \frac{\mathrm{d}\left(L^2\right)}{\mathrm{d}a} &= 0 \\ 4a^3 - 4 &= 0 \\ 4\left(a^3 - 1\right) &= 0 \\ a &= 1 \end{aligned}$$

Nature of stationary point:

Check that the stationary point is a minimum.

$$\begin{aligned} \frac{\mathrm{d}^2\left(L^2\right)}{\mathrm{d}a^2} &= 12a \\ &= 12(1) \\ &= 12 > 0 \end{aligned}$$

$\therefore$ minimum

The minimum value of L^2 is:

The question was to find the minimum value of L^2.

$$\begin{aligned} L^2 &= 1^4 - 4 + \frac{17}{4} \\ &= \frac{5}{4} \end{aligned}$$

c From part **b**, the minimum value of L is when $a = 1$, so the point is $A(a, a^2) = (1, 1)$

The minimum of L occurs at the same value of a as the minimum of L^2.

Note that the minimum distance from P to the curve is $\frac{\sqrt{5}}{2}$.

Rewind

You will need the formula for the distance between two points from Chapter 6.

EXERCISE 14C

IRANINOVIN

1 **a** **i** Find the maximum value of *xy* given that $x + 2y = 4$.

ii Find the maximum possible value of *xy* given that $3x + y = 7$.

b **i** Find the minimum possible value of $a + b$ given that $ab = 3$ and $a, b > 0$.

ii Find the minimum possible value of $2a + b$ given that $ab = 4$ and $a, b > 0$.

c **i** Find the maximum possible value of $4r^2h$ if $2r^2 + rh = 3$ and $r, h > 0$.

ii Find the maximum possible value of rh^2 if $4r^2 + 3h^2 = 12$ and $r, h > 0$.

2 A rectangle has width x metres and length 30 - x metres.

a Find the maximum area of the rectangle.

b Show that as x changes the perimeter stays constant and find the value of this perimeter.

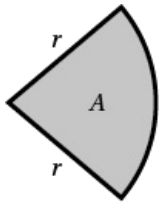
3 The sector of a circle with radius r has perimeter 40 cm.

The sector has area A.

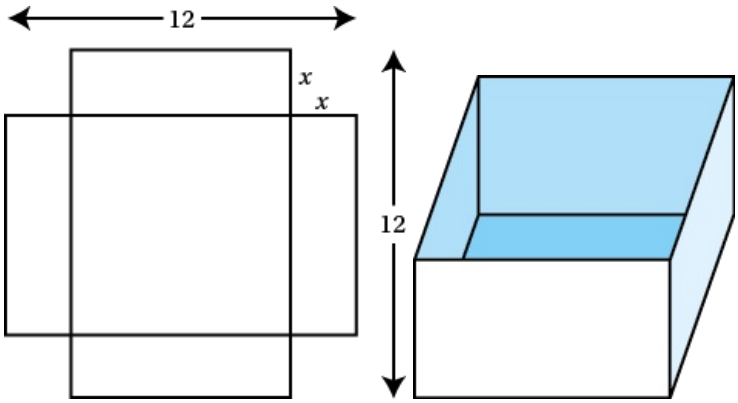
a Show that $A = 20r - r^2$.

b **i** Find the value of r for which A is a maximum.

ii Show that this does give a maximum.



4 A square sheet of card of side 12 cm has four squares of side x cm cut from the corners. The sides are then folded to make a small open box.



a Show that the volume, V, is given by $V = x(12 - 2x)^2$.

b Find the value of x for which the volume is maximum possible, and prove that it is a maximum.

5 Prove that the minimum possible value of the sum of a positive real number and its reciprocal is 2.

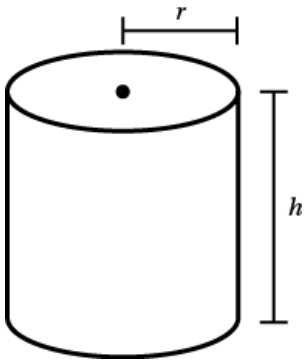
6 A solid cylinder has radius r and height h.

The total surface area of the cylinder is 450 cm^2.

a Find an expression for the volume of the cylinder in terms of r only.

b Hence find the maximum possible volume, justifying that the value found is a maximum.

IRANINOVIN

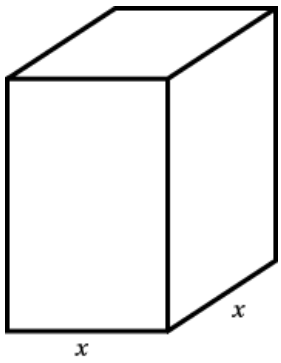


7 A closed carton is in the shape of a cuboid. The base is a square of side x.

The total surface area is 486 cm^2.

a Find an expression for the volume of the carton in terms of x only.

b Hence find the maximum possible volume, justifying that the value found is a maximum.



8 A certain type of chocolate is sold in boxes that are in the shape of a triangular prism. The cross-section is an equilateral triangle of side x cm. The length is y cm.

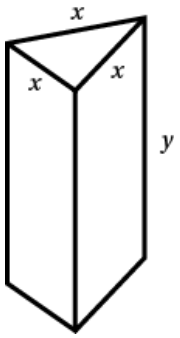
The volume of the box needs to be 128 cm^3.

The manufacturer wishes to minimise the surface area.

a Show that $A = \sqrt{3}\left(\frac{512}{x} + \frac{x^2}{2}\right)$.

b Find the minimum value of A.

c Prove that the value found is a minimum.

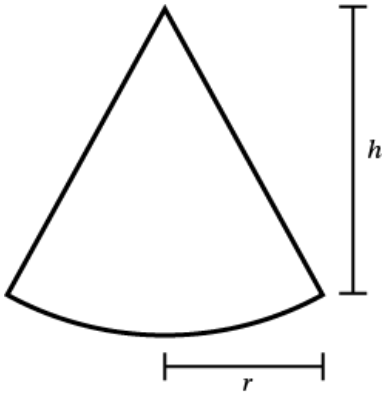


9 A cone of radius r and height h has volume 81π.

a Show that the curved surface area of the cone is given by $S = \pi\sqrt{r^4 + \frac{243^2}{r^2}}$.

It is required to make the cone so that the curved surface area is the minimum possible.

b By considering stationary points of S^2, or otherwise, find the radius and the height of the cone.



10 The sum of two numbers x and y is 6, and $x, y \geqslant 0$. Find the two numbers if the sum of their squares is the:

a minimum possible

b maximum possible.

11 The time in minutes taken to melt 100 g of butter (t) depends upon the percentage of the butter that is made of saturated fats (p), as shown in the following function:

$$t = \frac{p^2}{10\,000} + \frac{p}{100} + 2$$

Find the maximum and minimum times to melt 100 g of butter.

12 A 20 cm piece of wire is bent to form an isosceles triangle with base b.

a Show that the area of the triangle is given by $A = \frac{1}{2}\sqrt{100b^2 - 10b^3}$.

b Show that the area of the triangle is the largest possible when the triangle is equilateral.

13 The sum of the squares of two positive numbers is a. Prove that their product is the maximum possible when the two numbers are equal.

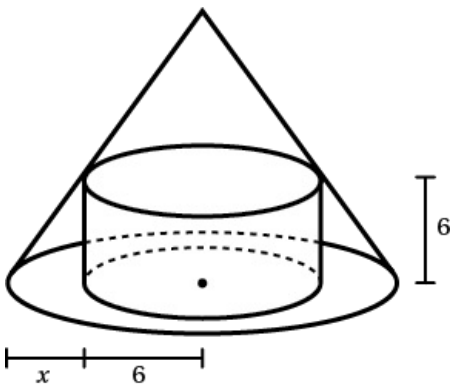
14 Find the coordinates of the point on the curve $y = x^2, x \geqslant 0$, closest to the point $(0, 4)$.

15 A cylinder of radius 6 cm and height 6 cm fits perfectly inside a cone, leaving a constant ring of width x around the base of the cylinder.

a Show that the height, h, of the cone is $h = \frac{36}{x} + 6$.

b Find the volume of the cone in terms of x.

c Hence find the minimum value of the volume, justifying that the value you have found is a minimum.



Checklist of learning and understanding

IRANINOVIN

- The **tangent** to a curve at a point (x_0, y_0) has gradient equal to $\frac{\mathrm{d}y}{\mathrm{d}x}$ evaluated at that point.
- The **normal** to the curve at a point (x_0, y_0) is perpendicular to the tangent at that point.
- **Stationary points** of a function are points where the gradient is zero, i.e. $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.
- The second derivative can be used to determine the nature of a stationary point. At a stationary point (x_0, y_0):
  - if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ at x_0 then (x_0, y_0) is a maximum
  - if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ at x_0 then (x_0, y_0) is a minimum
  - if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at x_0 then no conclusion can be drawn.

Mixed practice 14

1 $y = \mathrm{f}(x)$ has a stationary point, P, at $x = 2$. What is the equation of the tangent at P?

Choose from the following options:

A $y = \mathrm{f}'(2)$

B $x = 2$

C $y = \mathrm{f}(2)$

D $x = -\frac{1}{2}$

2 Find the equation of the tangent to the curve $y = x^2 - \frac{3}{x}$ at the point where $x = 2$. Give your answer in the form $ax + by = c$ where a, b and c are integers.

3 Find the equation of the normal to the curve $y = (x - 2)^3$ when $x = 2$.

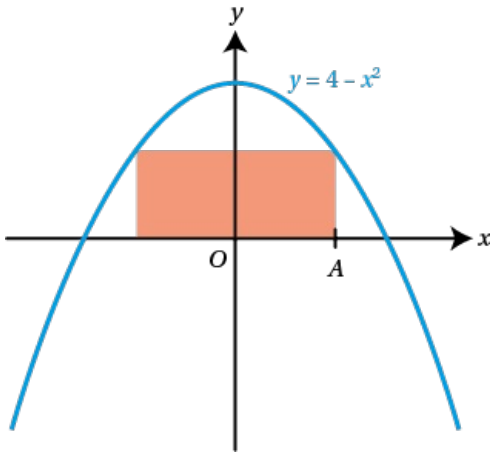
4 Find the x-coordinates of the stationary points on the graph of $y = \frac{x^3}{6} - x^2 + x$ and determine their nature.

5 A curve C has equation $y = \frac{1}{4}x^4 - 2x^3 + 4x^2 - 1$.

a Find the coordinates of the stationary points on C.

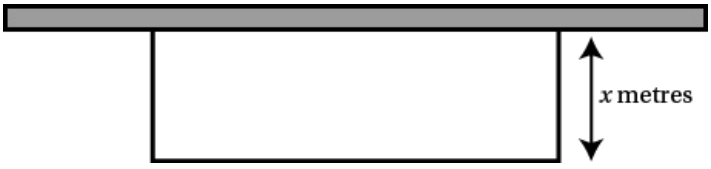
b Determine the nature of each stationary point.

6 A rectangle is drawn inside the region bounded by the curve $y = 4 - x^2$ and the x-axis, so that two of the vertices lie on the axis and the other two on the curve.



Find the coordinates of vertex A so that the area of the rectangle is the maximum possible.

7



The diagram shows a rectangular enclosure, with a wall forming one side. A rope, of length 20 metres, is used to form the remaining three sides. The width of the enclosure is x metres.

a Show that the enclosed area, A m^2, is given by $A = 20x - 2x^2$.

b Use differentiation to find the maximum value of A.

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8 $y = \frac{1}{3}x^3 - ax^2 + 3ax + 1$ where $a \neq 0$ has one stationary point. What is the value of a?

9 The curve $y = ax^3 + bx^2 + 8x - 1$ has stationary points at $x = \frac{1}{3}$ and $x = 4$. Find a and b.

10 $f(x) = x^2 - 4x\sqrt{x} + 4x - 3$

Show that the curve $y = f(x)$ has two stationary points and determine whether each is a maximum or minimum.

11 A function is defined by $f(x) = x^3 - 9x \text{for} -2 \leqslant x \leqslant 5$.

a Find the coordinates of the stationary points on the curve $y = f(x)$.

b Find the minimum and maximum values of $f(x)$.

12 The tangent to the graph of $y = \frac{1}{x^2}$ at the point where $x = 3$ crosses the coordinate axes at points M and N. Find the exact area of the triangle *MON*.

13 A car tank is being filled with petrol such that the volume in the tank in litres (V) over time in minutes (t) is given by $V = 300(t^2 - t^3) + 4 \text{for} 0 < t < 0.5$.

a How much petrol was initially in the tank?

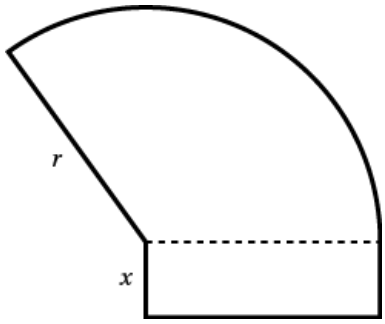
b After 30 seconds the tank was full. What is the capacity of the tank?

c At what time is petrol flowing in at the greatest rate?

14 A gardener is planting a lawn in the shape of a sector of a circle joined to a rectangle. The sector has radius r and angle $\frac{2\pi}{3}$ radians.

He needs the area, A, of the lawn to be 200 m^2.

A fence is to be built around the perimeter of the lawn.



a Show that the length of the fence, P, is given by $P = 2r + \frac{400}{r}$.

b Hence find the minimum length of fence required, justifying that this value is a minimum.

15 **a** Find the coordinates of the stationary point on the curve $y = 3x^2 - \frac{6}{x} - 2$.

b Determine whether the stationary point is a maximum point or a minimum point.

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16 The line $y = 24(x - 1)$ is tangent to the curve $y = ax^3 + bx^2 + 4$ at $x = 2$.

a Use the fact that the tangent meets the curve to show that $2a + b = 5$.

b Use the fact that the tangent has the same gradient as the curve to find another relationship between a and b.

c Hence find the values of a and b.

d The line meets the curve again. Find the coordinates of the other point of intersection.

17 On the curve $y = x^3$ a tangent is drawn from the point $(a,\ a^3)$ and a normal is drawn from the point $(-a, -a^3)$. The tangent and the normal meet on the y-axis. Find the value of a.

18 The curve $y = ax^2 + \dfrac{24}{x}$ has a stationary point at $y = 18$. Find a.

15 Integration

In this chapter you will learn how to:

- reverse the process of differentiation (this process is called integration)
- find the equation of a curve given its derivative and a point on the curve
- find the area between a curve and the x-axis.

Before you start...

Chapter 3	You should know how to solve quadratic and cubic equations by factorising.	1 Solve the following equations: a $3x^2 - 4x = 0$ b $x^3 - 4x^2 = 0$ c $x^3 - 5x = 0$
Chapter 13	You should know how to differentiate expressions of the form ax^n.	2 Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following: a $y = 3x^2 - x$ b $y = x^{\frac{1}{2}} - 3x^{-\frac{2}{3}} + 2$
Chapter 13	You should know how to convert expressions to the form ax^n in order to differentiate.	3 Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following: a $y = \frac{3}{2x^2}$ b $y = 3x\sqrt{x}$

What is integration?

As in many areas of mathematics, as soon as you learn a new process you must then learn how to undo it. Reversing differentiation answers the question: If you know the equation for the gradient of a curve, can you find the equation of the curve itself?

This question is important because, in many applications, the rate of change is easier to measure or model than the quantity itself. For example, acceleration, which is the rate of change of velocity, can be calculated if you know the forces acting on an object. You can then 'undifferentiate' the equation for acceleration to find the equation for velocity.

It turns out that undoing the process of differentiation opens up the possibility of answering another, seemingly unconnected problem: how to find the area under a curve. As you will learn in the mechanics chapters, this enables you to find distance travelled from a velocity−time graph.

Fast forward

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In See Chapters 19 and 20 for more about motion.

Section 1: Rules for integration

If you know the function describing a curve's gradient, you can find the equation of the curve by 'undoing' the differentiation. This process of reversing differentiation is known as **integration**.

For example, if $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ you know that the original function, y, must have contained x^2. However, it could have been $y = x^2 + 2$ or $y = x^2 - \frac{2}{3}$ or in fact $y = x^2 + c$, for any constant c.

All of these functions have $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, as the derivative of any constant is zero.

Without further information, it is impossible to know what the constant was in the original function, so you say:

If $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, then $y = x^2 + c$

or, equivalently, using the integration symbol:

$$\int 2x \, \mathrm{d}x = x^2 + c$$

Here, the d x simply states that the integration is taking place with respect to the variable x in exactly the same way that in $\frac{\mathrm{d}y}{\mathrm{d}x}$ it states that the differentiation is taking place with respect to x. You could equally well write, for example:

$$\int 2t \, \mathrm{d}t = t^2 + c$$

This relationship between differentiation and integration is so important that it is given a special name.

Fast forward

You will see how to find the constant of integration in Section 4.

Key point 15.1

The fundamental theorem of calculus:

$\int \mathrm{f}(x) \, \mathrm{d}x = \mathrm{F}(x) + c$ means that $\mathrm{f}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}(x)$

This just means that integration is the reverse of differentiation.

Tip

This theorem implies that the processes of integration and differentiation are inverses of each other.

Reversing the formula you know for differentiation gives the rule for integration.

Key point 15.2

$\int x^n \, \mathrm{d}x = \frac{1}{n+1}x^{n+1} + c$ for any $n \neq -1$

Tip

It may be helpful to think of this in words: 'add one to the power and divide by the new power'.

Explore

The statement in Key point 15.1 is one way to define integration. It can also be defined using areas, as you will find out in the next chapter. The full statement of the fundamental theorem of calculus says that the two definitions are equivalent; find out how to prove this.

Tip

Note the condition $n \neq -1$, which ensures that you are not dividing by zero.

WORKED EXAMPLE 15.1

If $\frac{dy}{dx} = x^3$ find y.

$y = \int x^3 \, dx$ — Integrate to get y.

$= \frac{1}{3+1}x^{3+1} + c$ — Use $\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c$.

$y = \frac{1}{4}x^4 + c$

Tip

Don't forget the $+c$. It is a part of the answer and you **must** write it every time.

WORKED EXAMPLE 15.2

Find $\int x^{-4} \, dx$.

$\int x^{-4} dx = \frac{1}{-4+1}x^{-4+1} + c$ — Use $\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c$.

$= -\frac{1}{3}x^{-3} + c$

WORKED EXAMPLE 15.3

If $f'(x) = x^{\frac{1}{2}}$ find $f(x)$.

$f(x) = \int x^{\frac{1}{2}} \, dx$ — Integrate $f'(x)$ to get $f(x)$.

$= \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + c$ — Use $\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c$.

Simplify.

$$= \frac{1}{\frac{3}{2}} x^{\frac{3}{2}} + c$$
$$= \frac{2}{3} x^{\frac{3}{2}} + c$$

Some important properties of integration follow directly from the properties on differentiation given in Key point 13.4.

Key point 15.3

- $\int kf(x)\ dx = k\int f(x)\ dx$ where k is a constant.
- $\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx$.

Tip

Key point 15.3 just says that you can integrate a function and then multiply by a constant, and that you can integrate terms of a sum separately.

WORKED EXAMPLE 15.4

Find $\int 3x^5\ dx$.

$$\int 3x^5\ dx = 3 \times \frac{1}{6}x^6 + c$$
$$= \frac{1}{2}x^6 + c$$

Integrate x^5 and then multiply by 3.

WORKED EXAMPLE 15.5

Find $\int 7\,dx$.

$$\int 7dx = \int 7x^0 dx$$

Using the fact that $x^0 = 1$ you can write 7 as $7x^0$.

$$= 7 \times \frac{1}{1}x^1 + c$$

Now integrate x^0.

$$= 7x + c$$

From Worked example 15.5 you can see that $\int k\,dx = kx + c$ for any constant k.

WORKED EXAMPLE 15.6

A curve has equation $\frac{dy}{dx} = 4x^{-\frac{1}{2}} - 3x + 8$. Find an equation for y in terms of x.

$y = \int 4x^{-\frac{1}{2}} - 3x + 8\,dx$

Integrate to find y.

$= 4 \times \frac{1}{\frac{1}{2}} x^{\frac{1}{2}} - 3 \times \frac{1}{2}x^2 + 8x + c$

Integrate each term separately. Remember that the integral of a constant, k, is just kx.

$= 8x^{\frac{1}{2}} - \frac{3}{2}x^2 + 8x + c$

Tip

You do not need to write out $k = kx^\circ$ every time you integrate $y = k$. You just need to know that the integral will always be $kx + c$.

EXERCISE 15A

1 Find y for the following:

a **i** $\frac{dy}{dx} = x^5$

ii $\frac{dy}{dx} = x^6$

b **i** $\frac{dy}{dx} = 3x^2$

ii $\frac{dy}{dx} = -5x^4$

c **i** $\frac{dy}{dx} = 2x^3$

ii $\frac{dy}{dx} = -3x^7$

d **i** $\frac{dy}{dx} = x$

ii $\frac{dy}{dx} = -4x$

e **i** $\frac{dy}{dx} = 0$

ii $\frac{dy}{dx} = 1$

f **i** $\frac{dy}{dx} = -\frac{1}{6}$

ii $\frac{dy}{dx} = \frac{1}{3}x^8$

g **i** $\frac{dy}{dx} = \frac{14x^6}{5}$

ii $\frac{dy}{dx} = \frac{10x}{7}$

h **i** $\frac{dy}{dx} = 3x^5 - x + 4$

ii $\frac{dy}{dx} = 2x^4 + 4x^3 - 8x - 1$

i **i** $\frac{dy}{dx} = \frac{x^2}{2} + 2x - \frac{3}{4}$

ii $\frac{dy}{dx} = x^2 - \frac{3x^5}{8}$

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2 Find the following integrals:

a **i** $\int x^{\frac{2}{3}}\,\mathrm{d}x$

ii $\int x^{\frac{1}{3}}\,\mathrm{d}x$

b **i** $\int 5x^{\frac{2}{3}}\,\mathrm{d}x$

ii $\int 14x^{\frac{3}{4}}\,\mathrm{d}x$

c **i** $\int \frac{5}{4}x^{\frac{1}{4}}\,\mathrm{d}x$

ii $\int -\frac{3}{2}x^{\frac{1}{2}}\,\mathrm{d}x$

d **i** $\int -\frac{7x^{\frac{4}{3}}}{6}\,\mathrm{d}x$

ii $\int \frac{14x^{\frac{2}{5}}}{15}\,\mathrm{d}x$

e **i** $\int 5x - 7x^{\frac{5}{2}}\,\mathrm{d}x$

ii $\int 6 + \frac{x}{3} - 12x^{\frac{1}{2}}\,\mathrm{d}x$

f **i** $\int x^{-2}\,\mathrm{d}x$

ii $\int x^{-3}\,\mathrm{d}x$

g **i** $\int x^{-\frac{1}{3}}\,\mathrm{d}x$

ii $\int x^{-\frac{1}{4}}\,\mathrm{d}x$

h **i** $\int -4x^{-\frac{2}{3}}\,\mathrm{d}x$

ii $\int -5x^{-\frac{3}{2}}\,\mathrm{d}x$

i **i** $\int x^{-\frac{1}{2}} - \frac{6x^{-4}}{5}\,\mathrm{d}x$

ii $\int 3x^{-5} + 7x^{-\frac{1}{6}}\,\mathrm{d}x$

3 Find the following integrals:

a **i** $\int 3\,\mathrm{d}t$

ii $\int 7\,\mathrm{d}z$

b **i** $\int q^5\,\mathrm{d}q$

ii $\int r^{10}\,\mathrm{d}r$

c **i** $\int 12g^{\frac{3}{5}}\,\mathrm{d}g$

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ii $\int 6y^{\frac{7}{2}}\,\mathrm{d}y$

d i $\int 5a^{-\frac{3}{4}}\,\mathrm{d}a$

ii $\int 7p^{-\frac{4}{3}}\,\mathrm{d}p$

4 If $\mathrm{f}'(x) = \dfrac{x^3}{2} - 6x^{-\frac{5}{3}} + 2$, find $\mathrm{f}(x)$.

5 Find $\int x^{\frac{3}{2}} - x^{-\frac{3}{2}}\,\mathrm{d}x$.

6 Find $\int \dfrac{x^5 - 3x^{-3} + x}{2}\,\mathrm{d}x$.

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Section 2: Simplifying into terms of the form ax^n

Just as for differentiation, before you can integrate products or quotients of functions they need to be converted into terms of the form ax^n, often by using the laws of indices.

Fast forward

In Student Book 2 you will learn methods for integrating some products and quotients.

WORKED EXAMPLE 15.7

Find $\int (2x-1)(x+3)\ \mathrm{d}x$.

$$\int (2x-1)(x+3)\ \mathrm{d}x = \int 2x^2+5x-3dx$$ Expand the brackets.

$$= \frac{2}{3}x^3+\frac{5}{2}x^2-3x+c$$ Then integrate.

Tip

You can't integrate products and quotients of functions separately; that is:

$$\int \mathrm{f}(x)\mathrm{g}(x)\ \mathrm{d}x \neq \int \mathrm{f}(x)\ \mathrm{d}x \times \int \mathrm{g}(x)\ \mathrm{d}x$$

$$\int \frac{\mathrm{f}(x)}{\mathrm{g}(x)}\ \mathrm{d}x \neq \frac{\int \mathrm{f}(x)\ \mathrm{d}x}{\int \mathrm{g}(x)\ \mathrm{d}x}$$

WORK IT OUT 15.1

Three students are trying to find the following integral:

$$\int 2x\left(x^2-2\right)\ \mathrm{d}x$$

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$$\int 2x\left(x^2-2\right)\ \mathrm{d}x$$

$$= \int 2x^3-4x\ \mathrm{d}x$$

$$= \frac{2x^4}{4}-\frac{4x^2}{2}+c$$

$$= \frac{x^4}{2}-2x^2+c$$

Solution 2

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$$\int 2x\left(x^2-2\right)\ \mathrm{d}x$$
$$=\frac{2x^2}{2}\left(\frac{x^3}{3}-2x\right)+c$$
$$=x^2\left(\frac{x^3}{3}-2x\right)+c$$

Solution 3

$$\int 2x\left(x^2-2\right)\ \mathrm{d}x$$
$$=2x\int x^2-2\,\mathrm{d}x$$
$$=2x\left(\frac{x^3}{3}-2x\right)+c$$

WORKED EXAMPLE 15.8

Find $\int \frac{1}{x^2}\ \mathrm{d}x$.

$\int \frac{1}{x^2}\,dx=\int x^{-2}\,dx$ — First rewrite the function in the form x^n using the laws of indices.

$=-x^{-1}+c$ — Then integrate.

WORKED EXAMPLE 15.9

Find $\int 5x^2\sqrt[3]{x}\,\mathrm{d}x$.

$\int 5x^2\sqrt[3]{x}\,dx=\int 5x^2x^{\frac{1}{3}}\,dx$ — First rewrite the function in the form ax^n using the laws of indices.

$=\int 5x^{\frac{7}{3}}\,dx$

$=5\times\frac{3}{10}x^{\frac{10}{3}}+c$ — Then integrate: dividing by $\frac{10}{3}$ is the same as multiplying by $\frac{3}{10}$.

$=\frac{3}{2}x^{\frac{10}{3}}+c$

WORKED EXAMPLE 15.10

Find $\int \frac{(x-3)^2}{\sqrt{x}}\ \mathrm{d}x$.

$\int \frac{(x-3)^2}{\sqrt{x}}dx=\int \frac{x^2-6x+9}{\sqrt{x}}dx$ — Expand the brackets first.

$$= \int \frac{x^2 - 6x + 9}{x^{\frac{1}{2}}} dx$$

Then replace $\sqrt{x}$ with $x^{\frac{1}{2}}$.

$$= \int x^{2-\frac{1}{2}} - 6x^{1-\frac{1}{2}} + 9x^{-\frac{1}{2}}\, dx$$

Simplify using the laws of indices.

$$= \int x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 9x^{-\frac{1}{2}}\, dx$$

$$= \frac{2}{5}x^{\frac{5}{2}} - 6 \times \frac{2}{3}x^{\frac{3}{2}} + 9 \times 2x^{\frac{1}{2}} + c$$

$$= \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + 18x^{\frac{1}{2}} + c$$

Then integrate: dividing by $\frac{3}{2}$ is the same as multiplying by $\frac{2}{3}$; dividing by $\frac{1}{2}$ is the same as multiplying by 2.

EXERCISE 15B

1 Find y for the following:

a i $\int \sqrt{x}\, \mathrm{d}x$

ii $\int \sqrt[4]{x}\, \mathrm{d}x$

b i $\int 6\sqrt[5]{x}\, \mathrm{d}x$

ii $\int \frac{\sqrt[3]{x}}{6}\, \mathrm{d}x$

c i $\int \frac{1}{x^3}\, \mathrm{d}x$

ii $\int -\frac{1}{x^7}\, \mathrm{d}x$

d i $\int -\frac{8}{x^5}\, \mathrm{d}x$

ii $\int \frac{10}{3x^6}\, \mathrm{d}x$

e i $\int \frac{1}{\sqrt{x}}\, \mathrm{d}x$

ii $\int \frac{1}{\sqrt[3]{x}}\, \mathrm{d}x$

f i $\int -\frac{15}{\sqrt[6]{x}}\, \mathrm{d}x$

ii $\int \frac{9}{8\sqrt[4]{x}}\, \mathrm{d}x$

2 Find the following integrals:

a i $\int (2x+3)(x+1)\, \mathrm{d}x$

ii $\int 3x^2(x-2)\, \mathrm{d}x$

b i $\int \sqrt{x}(5x+4)\, \mathrm{d}x$

ii $\int \sqrt[3]{x}(2x^2+3)\, \mathrm{d}x$

c **i** $\int (\sqrt[4]{x} + 3x)^2 \, \mathrm{d}x$

ii $\int \left(\sqrt{x} - x^2\right)^2 \, \mathrm{d}x$

d **i** $\int \left(x - \frac{1}{x}\right)^2 \, \mathrm{d}x$

ii $\int \left(2x + \frac{1}{x^2}\right)\left(2x - \frac{1}{x^2}\right) \, \mathrm{d}x$

3 Find the following integrals:

a **i** $\int \frac{7x - 2}{x^2} \, dx$

ii $\int \frac{2 + 5x}{4x^3} \, \mathrm{d}x$

b **i** $\int \frac{\sqrt{x} - 3}{x^2} \, \mathrm{d}x$

ii $\int \frac{x^2 - 6x}{\sqrt{x}} \, \mathrm{d}x$

4 Find $\int 10x\sqrt{x} \, \mathrm{d}x$.

5 Find $\int x^2 \left(\sqrt[3]{x} - 12\right) \, \mathrm{d}x$.

6 Find $\int 9\sqrt[4]{x^5} \, \mathrm{d}x$.

7 Find $\int \frac{1}{3\sqrt[3]{x^2}} \, \mathrm{d}x$.

8 Find $\int \left(\frac{1}{\sqrt{x}} - 1\right) \, \mathrm{d}x$.

9 Find $\int \frac{1}{3x^3} + \frac{1}{4x^4} \, \mathrm{d}x$.

10 $\mathrm{f}(x) = \frac{6 + x}{\sqrt[3]{x}}$.

a Express $\mathrm{f}(x)$ in the form $ax^n + bx^m$.

b Hence find $\int \frac{6 + x}{\sqrt[3]{x}} \, \mathrm{d}x$.

11 $y = \frac{(x + 2)(x - 2)}{2\sqrt{x}}$

a Show that $y = ax^{\frac{3}{2}} + bx^{-\frac{1}{2}}$ where a and b are constants to be found.

b Hence find $\int y \, \mathrm{d}x$.

12 Show that $\int 12\sqrt{x} - \frac{4}{\sqrt{x}} \, \mathrm{d}x = 8\sqrt{x}(x - 1) + c$.

13 Find $\int \frac{(x + 4)^2}{8x\sqrt{x}} \, \mathrm{d}x$.

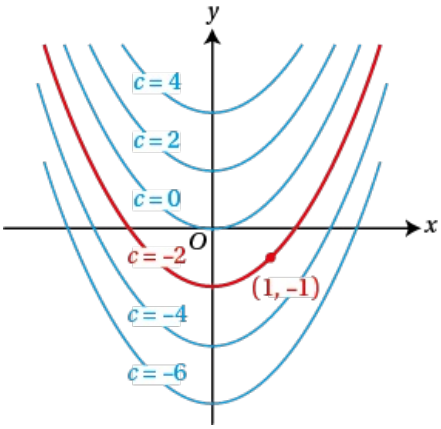
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Section 3: Finding the equation of a curve

Consider again $\frac{dy}{dx} = 2x$ which you encountered at the start of this chapter. You know that the original function has equation $y = x^2 + c$ for some constant value c.

If you are also told that the curve passes through the point $(1, -1)$, you can specify which of the family of curves your function must be, and find c.

Worked example 15.11 illustrates the general procedure for finding the equation of a curve given its gradient function.



WORKED EXAMPLE 15.11

The gradient of a curve is given by $\frac{dy}{dx} = 3x^2 - 8x + 5$ and the curve passes through the point $(1, -4)$. Find the equation of the curve.

$y = \int 3x^2 - 8x + 5\,dx$
$= x^3 - 4x^2 + 5x + c$

To find y from $\frac{dy}{dx}$ you need to integrate. Don't forget $+c$.

When $x = 1, y = -4$:
$-4 = (1)^3 - 4(1)^2 + 5(1) + c$
$-4 = 1 - 4 + 5 + c$
$c = -6$

Substitute in $x = 1$, $y = -4$ in order to find c.

$\therefore y = x^3 - 4x^2 + 5x - 6$

State the final equation.

Key point 15.4

To find the equation for y given the gradient $\frac{dy}{dx}$ and one point (p, q) on the curve:

- integrate $\frac{dy}{dx}$ to get an equation for y in terms of x, rememberin $g + c$
- find the value of c by substituting $x = p$ and $y = q$ into the equation
- rewrite the equation for y, using the value of c that has been found.

Sometimes you know the second derivative; in that case, you need to integrate twice to find the equation of the original curve. Be careful: you will need to add a different constant each time you integrate.

Fast forward

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Equations with second derivatives occur frequently in the work on velocity and acceleration – see Chapter 19.

WORKED EXAMPLE 15.12

A curve has second derivative $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3 - 6x$.

It passes through the point $(1, 2)$ and its gradient at this point is -3. Find the equation of the curve.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \int 3 - 6x \, \mathrm{d}x$
$= 3x - 3x^2 + c$

Integrating the second derivative once gives the first derivative.

when $x = 1, \frac{\mathrm{d}y}{\mathrm{d}x} = -3$
$-3 = 3 - 3 + c$
$c = -3$

You can find c by using the given value of the gradientat $x = 1$.

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = 3x - 3x^2 - 3$

State the full equation for the derivative.

$y = \int 3x - 3x^2 - 3 \, \mathrm{d}x$
$= \frac{3x^2}{2} - x^3 - 3x + d$

Now integrate again to find y. There will be another constant.

when $x = 1, y = 2$:
$2 = \frac{3}{2} - 1 - 13 + d$
$d = \frac{9}{2}$

Now use the given x and y values to find d.

$\therefore y = \frac{3x^2}{2} - x^3 - 3x + \frac{9}{2}$

Finally state the equation for y.

WORK IT OUT 15.2

Three students attempt to solve the following problem:

A curve has $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3x^2$ When $x = 2$, $y = 3$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ Find the equation of the curve.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$\frac{\mathrm{d}y}{\mathrm{d}x} = \int 3x^2 \, \mathrm{d}x$
$= x^3 + c$
$y = \int x^3 + c \, \mathrm{d}x$
$= \frac{x^4}{4} + cx + c$

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When $x = 2, y = 3$:

$3 = 4 + 2c + c$
$3 = 4 + 3c$
$c = -\frac{1}{3}$

$\therefore y = \frac{x^4}{4} - \frac{1}{3}x - \frac{1}{3}$

Solution 2

$\frac{\mathrm{d}y}{\mathrm{d}x} = \int 3x^2 \, \mathrm{d}x$
$= x^3 + c$

$y = \int x^3 + c \, \mathrm{d}x$
$= \frac{x^4}{4} + cx$

When $x = 2, y = 3$:

$3 = 4 + 2c$
$c = -\frac{1}{2}$

$\therefore y = \frac{x^4}{4} - \frac{1}{2}x$

Solution 3

$\frac{\mathrm{d}y}{\mathrm{d}x} = \int 3x^2 \, \mathrm{d}x$
$= x^3 + c$

When $x = 2$:

$\frac{\mathrm{d}y}{\mathrm{d}x} = -1$:

$-1 = 2^3 + c$
$c = -9$

$y = \int x^3 - 9 \, \mathrm{d}x$
$= \frac{x^4}{4} - 9x + d$

When $x = 2, y = 3$:

$3 = 4 - 18 + d$
$d = 17$

$\therefore y = \frac{x^4}{4} - 9x + 17$

EXERCISE 15C

1 Find the equation of the curve if:

a **i** $\frac{dy}{dx} = x$ and the curve passes through $(-2, 7)$

ii $\frac{dy}{dx} = 6x^2$ and the curve passes through $(0, 5)$

b **i** $\frac{dy}{dx} = \frac{1}{x^2}$ and the curve passes through $(1, -1)$

ii $\frac{dy}{dx} = \frac{1}{x^2}$ and the curve passes through $(1, 3)$

c **i** $\frac{dy}{dx} = 3x - 5$ and the curve passes through $(2, 6)$

ii $\frac{dy}{dx} = 3 - 2x^3$ and the curve passes through $(1, 5)$

d **i** $\frac{dy}{dx} = 3\sqrt{x}$ and the curve passes through $(9, -2)$

ii $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$ and the curve passes through $(4, 8)$

2 Find the equation of the curve if:

a **i** $\frac{d^2y}{dx^2} = 6x$ and the curve has gradient 2 at the point $(1, 5)$

ii $\frac{d^2y}{dx^2} = \frac{6}{x^3}$ and the curve has gradient -1 at the point $(1, 3)$

b **i** $f''(x) = 1 - 2x, f'(2) = 1$ and $f(2) = -1$

ii $f''(x) = 3x^2 - x, f'(1) = 3$ and $f(1) = 10$

3 A curve has gradient $\frac{dy}{dx} = x - \frac{1}{x^2}$ and passes through the point $(1, 3)$. Find the equation of the curve.

4 $f'(x) = \frac{2x - 1}{\sqrt{x}}$ and $f(4) = 2$. Find $f(x)$.

5 $f'(x) = \sqrt{x}(5x - 4)$ and $f(1) = \frac{19}{3}$. Find $f(4)$.

6 The gradient of a curve at any point is directly proportional to the x-coordinate of that point. The curve passes through point A with coordinates $(3, 2)$. The gradient of the curve at A is 12. Find the equation of the curve.

7 The gradient of a curve is $\frac{dy}{dx} = x2 - 4$.

a Find the x-coordinate of the maximum point, justifying that it is a maximum.

b Given that the curve passes through the point $(0, 2)$ show that the y-coordinate of the maximum point is $7\frac{1}{3}$.

8 A curve has second derivative $\frac{d^2y}{dx^2} = 2 - x^2$.

At the point $(1, 3)$ the gradient of the curve is 5. Find the equation of the curve in the form $y = f(x)$.

9 $f''(x) = 12x - \frac{5}{\sqrt[3]{x}}, f'(1) = -\frac{9}{2}$ and $f(0) = 2$.

Find $f(8)$.

10 The gradient of the normal to a curve at any point is equal to the square of the x-coordinate at that point. If the curve passes through the point $(2, 3)$ find the equation of the curve in the form $y = f(x)$.

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Section 4: Definite integration

Until now you have been carrying out a process known as **indefinite integration**: indefinite in the sense that you have an unknown constant each time, for example:

$$\int x^2 \, \mathrm{d}x = \frac{1}{3}x^3 + c$$

There is also a process called **definite integration,** which gives a numerical answer. Evaluate the indefinite integral at two points and take the difference of the two results:

$$\begin{aligned}\int_2^3 x^2 \, \mathrm{d}x &= \left[\frac{1}{3}x^3 + c\right]_2^3 \\ &= \left(\frac{1}{3} \times 3^3 + c\right) - \left(\frac{1}{3} \times 2^3 + c\right) \\ &= 6\frac{1}{3}\end{aligned}$$

Tip

The constant of integration cancels in the subtraction, so there is no need to include it in the calculation at all.

The numbers 2 and 3 are known as the **limits of integration**; 2 is the lower limit and 3 is the upper limit.

Notice the square bracket notation, which means that the integration has taken place but the limits have not yet been applied.

Tip

You may be able to evaluate definite integrals on your calculator. Even when you are asked to find the exact value of an integral, you can use your calculator to check the answer.

WORKED EXAMPLE 15.13

Evaluate $\int_2^4 x^3 - 2x \, \mathrm{d}x$.

$\int_2^4 x^3 - 2x \, dx = \left[\frac{1}{4}x^4 - x^2\right]_2^4$ — Integrate, using square brackets to indicate that the integration has taken place but the limits are still to be applied.

$= \left(\frac{1}{4} \times 4^4 - 4^2\right) - \left(\frac{1}{4} \times 2^4 - 2^2\right)$ — Evaluate at the upper and lower limits and subtract.

$= (64 - 16) - (4 - 4)$

$= 48$

WORKED EXAMPLE 15.14

Find the exact value of $\int_8^9 \frac{3}{\sqrt{x}} \, \mathrm{d}x$.

$\int_8^9 \frac{3}{\sqrt{x}} \, dx = \int_8^9 3x^{-\frac{1}{2}} \, dx$ — First rewrite in the form ax^n.

$= \left[3 \times 2x^{\frac{1}{2}}\right]_8^9$ — Then integrate: dividing by $\frac{1}{2}$ is the same as multiplying by 2.

$= \left[6x^{\frac{1}{2}}\right]_8^9$

$= \left(6 \times 9^{\frac{1}{2}}\right) - \left(6 \times 8^{\frac{1}{2}}\right)$ — Evaluate at the upper and lower limits and subtract.

$= 6\sqrt{9} - 6\sqrt{8}$ — Write the fractional power as a root in order to evaluate.

$= 18 - 12\sqrt{2}$ — Remember that $\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$.

EXERCISE 15D

1 Evaluate the following definite integrals, giving exact answers.

a i $\int_2^6 x^3 \, dx$

ii $\int_4^5 x^4 \, dx$

b i $\int_{-2}^2 3x^5 \, dx$

ii $\int_{-3}^3 2x^2 \, dx$

c i $\int_{-3}^{-1} 6x^2 - 3 \, dx$

ii $\int_{-4}^{-2} 3x^2 - 4 \, dx$

d i $\int_1^4 x^2 + x \, dx$

ii $\int_{-2}^1 3x^2 - 5x \, dx$

e i $\int_4^9 2\sqrt{x} \, dx$

ii $\int_8^{27} 6\sqrt[3]{x} \, dx$

f i $\int_1^{16} \frac{6}{\sqrt{x}} \, dx$

ii $\int_{-3}^{-1} \frac{3}{x^2} \, dx$

g i $\int_{-4}^{-2} 1 + \frac{2}{x^2} \, dx$

ii $\int_{-3}^{-1} 2x - \frac{1}{x^2} \, dx$

2 Evaluate this following integral, giving your answer in the form $a + b\sqrt{2}$: $\int_2^8 3\sqrt{x} - 2x \, dx$

3 Show that $\int_1^3 9\sqrt{x} - \frac{4}{\sqrt{x}} \, dx = a + b\sqrt{3}$, where a and b are integers to be found.

4 Find the following in terms of k: $\int_1^k 2 - \frac{1}{x^2} \, dx$

5 Find the following in terms of a: $\int_{-a}^{2} x^2 (4x - 3) \ \mathrm{d}x$

6 Find the value of a such that $\int_{1}^{a} \sqrt{t} \ \mathrm{d}t = 42$.

7 $\int_{-2}^{p} 1 - \frac{10}{3x^2} \ \mathrm{d}x = -\frac{2}{3}$

Find all possible values of p.

8 Given that $\int_{3}^{9} \mathrm{f}(x) \ \mathrm{d}x = 7$ evaluate $\int_{3}^{9} 2\mathrm{f}(x) + 1 \ \mathrm{d}x$.

9 Given that $\int_{-1}^{4} \mathrm{f}(x) \ \mathrm{d}x = 10$ evaluate $\int_{-1}^{4} \frac{\mathrm{f}(x)}{2} + x - 1 \ \mathrm{d}x$.

10 **a** **i** Evaluate $\int_{3}^{5} 2x \ \mathrm{d}x$

ii Evaluate $\int_{5}^{3} 2x \ \mathrm{d}x$.

b **i** Find an expression for $\int_{b}^{a} 2x \ \mathrm{d}x$.

ii Find an expression for $\int_{a}^{b} 2x \ \mathrm{d}x$.

c Suggest a general relationship between $\int_{b}^{a} \mathrm{f}(x) \ \mathrm{d}x$ and $\int_{a}^{b} \mathrm{f}(x) \ \mathrm{d}x$.

11 **a** Find the definite integral $\int_{1}^{a} x^2 \ \mathrm{d}x$ in terms of a.

b Let $\mathrm{f}(a) = \int_{1}^{a} x^2 \ \mathrm{d}x$. Find $f'(a)$.

c Find $f'(a)$ for each of the following:

i $\mathrm{f}(a) = \int_{2}^{a} x^2 \ \mathrm{d}x$

ii $\mathrm{f}(a) = \int_{-1}^{a} x^2 \ \mathrm{d}x$

What do you notice?

d Find $f'(a)$ for each of the following, and comment on your results:

i $\mathrm{f}(a) = \int_{0}^{a} 4x^3 - 2x \ \mathrm{d}x$

ii $\mathrm{f}(a) = \int_{1}^{a} \frac{3}{x^2} \ \mathrm{d}x$

iii $\mathrm{f}(a) = \int_{-2}^{a} 1 - 3x^2 \ \mathrm{d}x$

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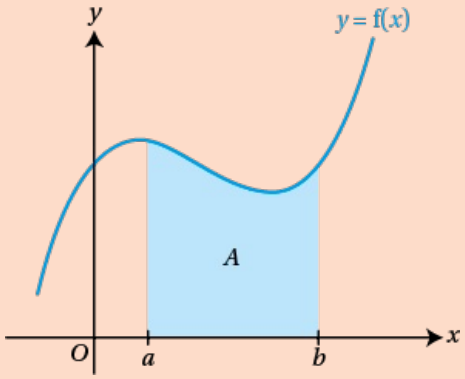
Section 5: Geometrical significance of definite integration

Now you have a method that gives a numerical value for an integral, the natural question to ask is: What does this number mean? The (somewhat surprising) answer is that the definite integral represents the area under a curve. More precisely, $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ is the area enclosed between the graph of $y = \mathrm{f}(x)$, the x-axis and the lines $x = a$ and $x = b$.

Did you know?

The ancient Greeks had methods for finding gradients of curves and areas under graphs. However, it took over 2000 years to develop the theory which formally proves that the two are related; that areas can be found by reversing differentiation. This was finally accomplished in the 17th century by Isaac Newton and Gottfried Leibniz.

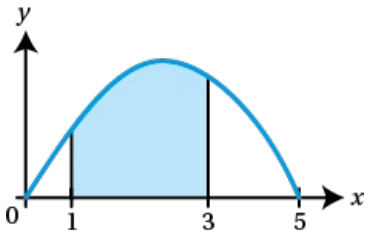
Key point 15.5



$$A = \int_a^b \mathrm{f}(x)\ \mathrm{d}x$$

WORKED EXAMPLE 15.15

Find the exact area enclosed between the graph of $y = 5x - x^2$, the x-axis and the lines $x = 1$ and $x = 3$.



Sketch the graph and shade the required area.

$$\text{Area} = \int_1^3 5x - x^2 dx$$

Write down the required definite integral. The limits are given by the x -coordinates at the end of the shaded region.

$$= \left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_1^3$$
$$= \left(\frac{5}{2} \times 9 - \frac{1}{3} \times 27\right) - \left(\frac{5}{2} - \frac{1}{3}\right)$$
$$= \frac{34}{3}$$

Integrate and write in square brackets, then evaluate the expression at the limits and subtract.

When the curve is entirely below the x-axis the integral will give you a negative value. The area should be taken to be the positive value.

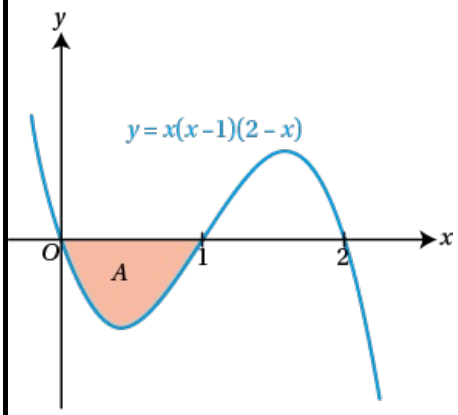
Fast forward

A In Student Book 2 you will learn that this can be written using the modulus function:

$$A = \int_a^b |f(x)|\,dx$$

WORKED EXAMPLE 15.16

Find the area A shaded in the diagram.



$$\int_0^1 x\,(x-1)\,(2-x)\;dx = \int_0^1 -x^3 + 3x^2 - 2x\,dx$$

Expand the brackets.

$$= \left[-\frac{1}{4}x^4 + x^3 - x^2\right]_0^1$$

Then integrate and evaluate at the limits.

$$= \left(-\frac{1}{4} + 1 - 1\right) - (0)$$

$$= -\frac{1}{4}$$

The integral gives a negative value because the area is under the x-axis. The area is a positive number.

$$\therefore A = \frac{1}{4}$$

The relationship between integrals and areas is a bit more complicated when some parts of the curve are above the axis and other parts are below it. Those parts above the axis contribute positively to the area, but parts below the axis contribute negatively. To calculate the total area you must separate out the sections above the axis and those below the axis.

Explore

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Another difficulty with finding areas arises when the graph has a vertical asymptote. For example, try working out $\int_{-2}^{2} \frac{1}{x^2}\,\mathrm{d}x$. This is called an 'improper integral'. It turns out that sometimes regions that appear to be infinite still have a finite area.

WORKED EXAMPLE 15.17

a Find $\int_1^4 x^2 - 4x + 3\ \mathrm{d}x$.

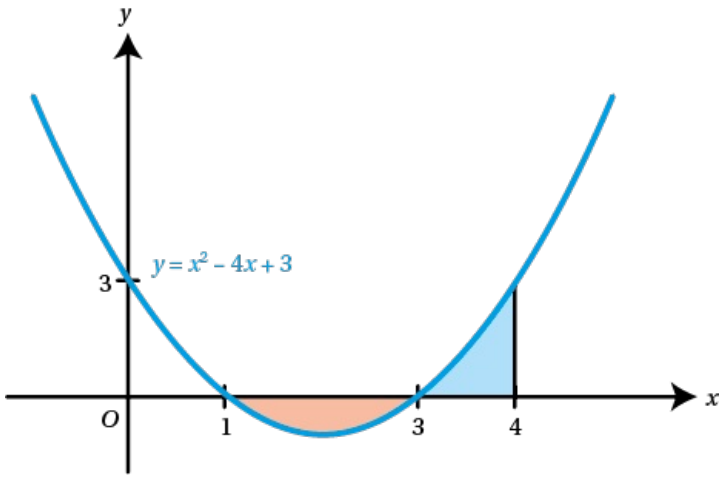
b Sketch the graph of $y = x^2 - 4x + 3$. Hence find the area enclosed between the x-axis, the curve $y = x^2 - 4x + 3$ and the lines $x = 1$ and $x = 4$.

a

$$\int_1^4 x^2 - 4x + 3\ dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_1^4$$
$$= \left(\frac{1}{3}(4)^3 - 2(4)^2 + 3(4)\right) - \left(\frac{1}{3}(1)^3 - 2(1)^2 + 3(1)\right)$$
$$= \left(\frac{4}{3}\right) - \left(\frac{4}{3}\right)$$
$$= 0$$

Integrate and evaluate at the upper and lower limits.

b



You can see that the required area is made up of two parts, so evaluate each of them separately.

$$\int_1^3 x^2 - 4x + 3\ dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_1^3$$
$$= (0) - \left(\frac{4}{3}\right)$$
$$= -\frac{4}{3}$$

$\therefore$ Area below the axis is $\frac{4}{3}$

The integral for the part of the curve below the axis is negative, but the area must be positive.

$$\int_3^4 x^2 - 4x + 3dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_3^4$$
$$= \left(\frac{4}{3}\right) - (0)$$
$$= \frac{4}{3}$$

$\therefore$ Area above the axis is $\frac{4}{3}$

The area of the part above the axis is found as normal.

Total area $= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

Now add the two areas together.

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The fact that the integral in Worked example 15.17 was zero can be interpreted as meaning that the area above the axis is exactly cancelled by the area below the axis.

Tip

Always sketch the graph first (if it is not given) when asked to find an area.

WORK IT OUT 15.3

Find the total area enclosed by the graph of $y = x^3 - 5x^2 + 6x$ and the x-axis.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

The x-intercepts are 0, 2, 3.

$$\int_0^3 x^3 - 5x^2 + 6x \, \mathrm{d}x$$

$$= \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2\right]_0^3 = \frac{9}{4}$$

Solution 2

The x-intercepts are 0, 2, 3.

$$\int_0^2 x^3 - 5x^2 + 6x \ \mathrm{d}x$$

$$+ \int_2^3 x^3 - 5x^2 + 6x \ \mathrm{d}x$$

$$= \left(\frac{8}{3} - 0\right) + \left(\frac{9}{4} - \frac{8}{3}\right) = 2.25$$

Solution 3

The x-intercepts are 0, 2, 3.

$$\int_0^2 x^3 - 5x^2 + 6x \, \mathrm{d}x = \frac{8}{3} - 0 = \frac{8}{3}$$

$$\int_2^3 x^3 - 5x^2 + 6x \, \mathrm{d}x = \frac{9}{4} - \frac{8}{3}$$

$$= -\frac{5}{12}$$

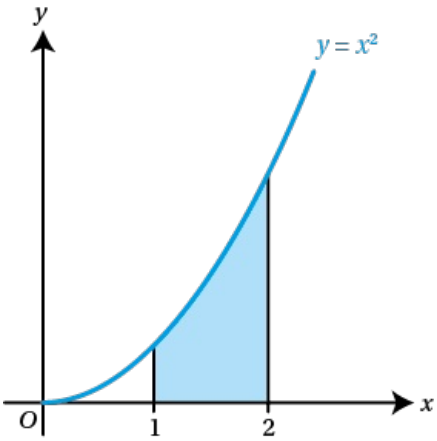
$$\text{Area} = \frac{8}{3} + \frac{5}{12} = \frac{37}{12}$$

EXERCISE 15E

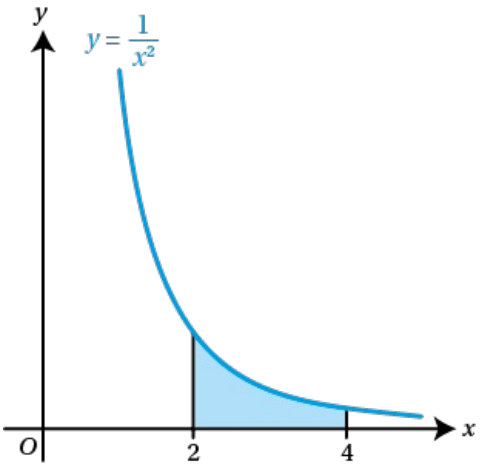
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1 Find the shaded areas.

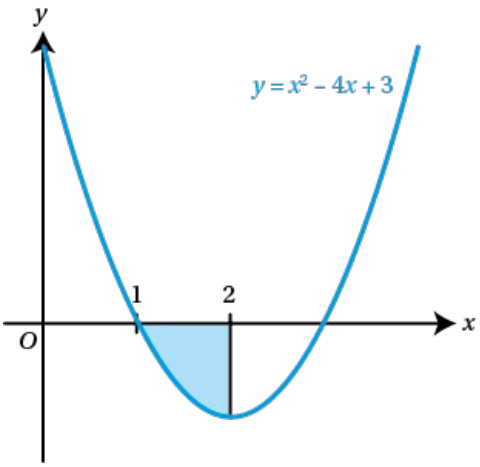
a i



ii

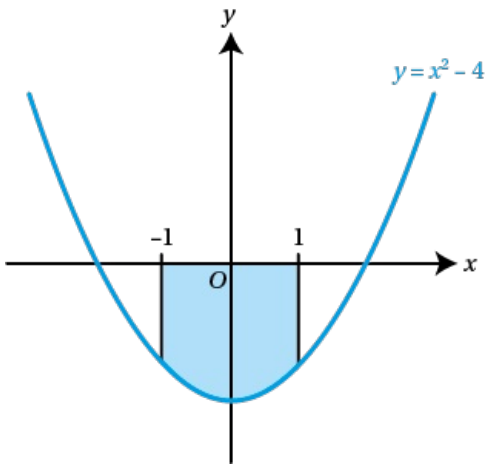


b i

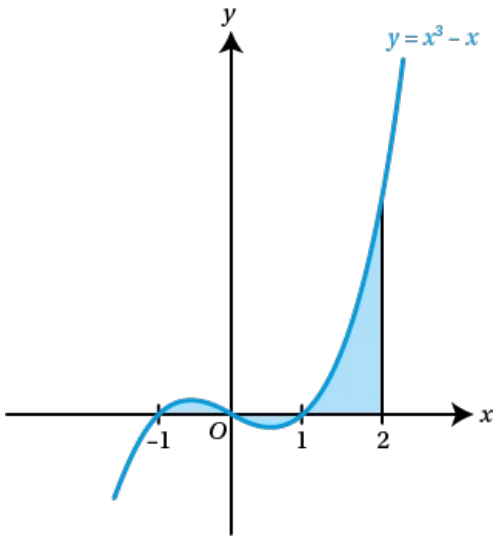


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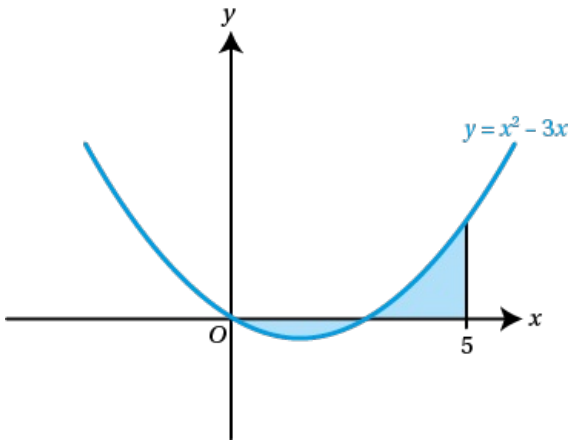
ii



c i



ii



2 Find the following areas. You may want to sketch the graph first.

a i Between the curve $y = x^2 + 2$, the x-axis and the lines $x = 2$ and $x = 5$.

ii Between the curve $y = 2x^2 + 1$, the x-axis and the lines $x = 1$ and $x = 2$.

b i Enclosed between the graph of $y = 4 - x^2$ and the x-axis.

ii Enclosed between the graph of $y = x^2 - 1$ and the x-axis.

c i Enclosed between the curve $y = x^2 - 9x$ and the x-axis.

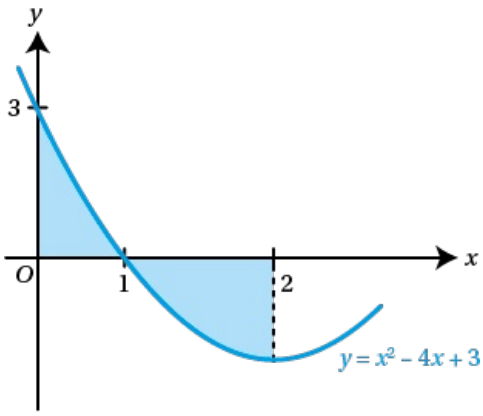
ii Enclosed between the curve $y = x^3 - 3x^2 + 2x$ and the x-axis.

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Worksheet

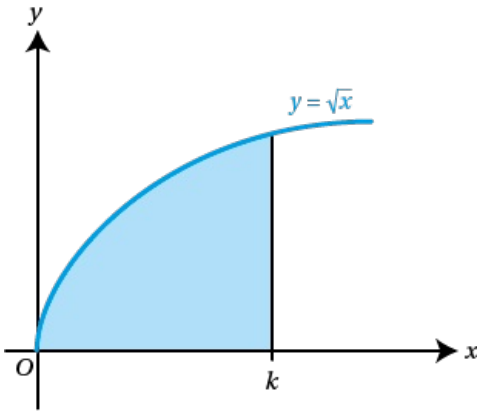
For a reminder and more practice of questions like this on finding areas, see Support sheet 15.

3 A part of the graph of $y = x^2 - 4x + 3$ is shown here. The curve crosses the x-axis at $x = 1$. The shaded region is enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 2$.



Find the area of the shaded region.

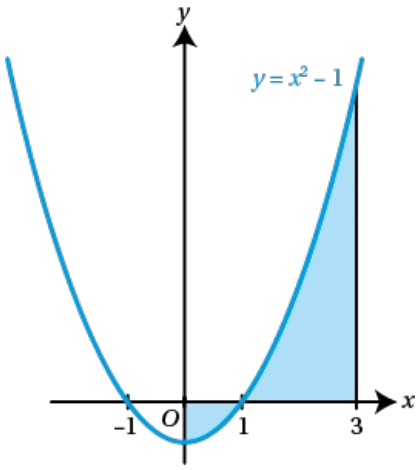
4 The diagram shows the graph of $y = \sqrt{x}$. The shaded area is 18.



Find the value of k.

5 **a** Find $\int_0^3 x^2 - 1 \, \mathrm{d}x$.

b The graph of $y = x^2 - 1$ is shown in the diagram. The shaded region is bounded by the curve, the x-axis and the lines $x = 0$ and $x = 3$.



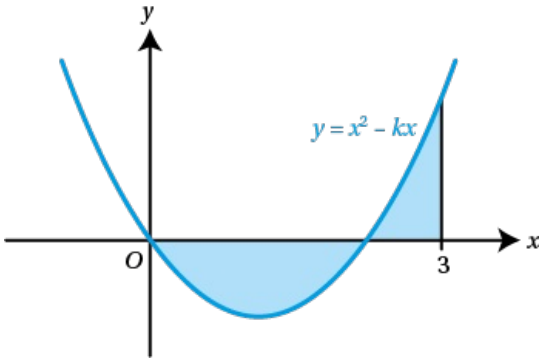
Find the area of the shaded region.

6 **a** Factorise $x^2 - 7x + 10$.

b Find the area enclosed by the curve $y = 7x - x^2 - 10$ and the x-axis.

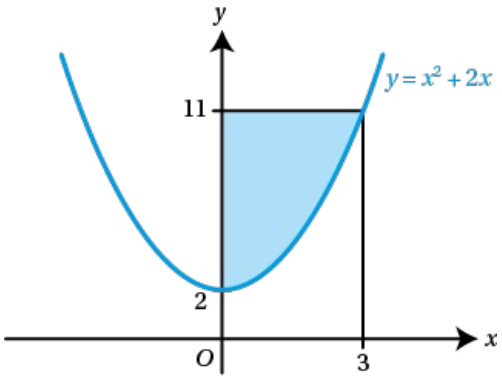
7 **a** Write down the coordinates of the points where the graph of $y = x^2 - kx$ crosses the x-axis.

b The area shaded in the diagram is bounded by the curve $y = x2 - kx$, the x-axis and the lines $x = 0$ and $x = 3$. The area below the x-axis equals the area above the x-axis. Find the value of k.



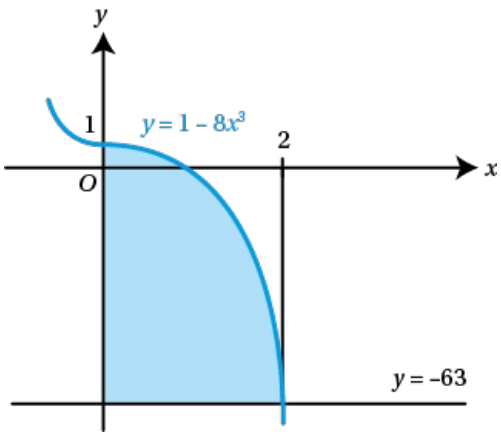
8 **a** Evaluate $\int_0^3 x^2 + 2 \, \mathrm{d}x$.

b The diagram shows a part of the curve with equation $y = x^2 + 2$. The shaded region is bounded by the curve, the y-axis and the line $y = 11$.

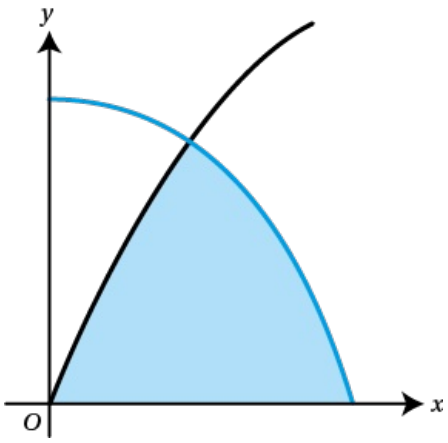


Find the area of the shaded region.

9 Find the area enclosed between the curve $y = 1 - 8x^3$, the y-axis and the line $y = -63$.

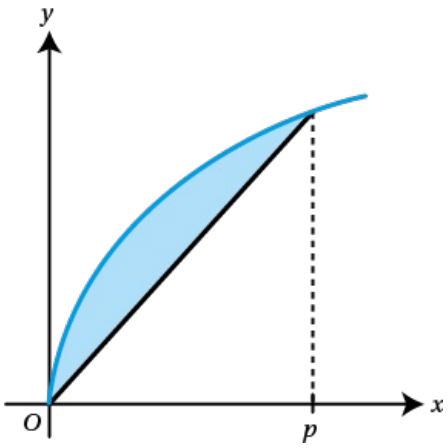


10 The diagram shows the graphs of $y = 36 - x^2$ and $y = 18x - x^2$. The shaded region is bounded by the two curves and the x-axis.



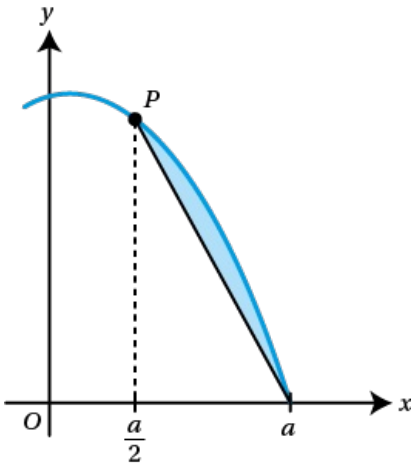
Find the area of the shaded region.

11 The curve in the diagram has equation $y = \sqrt{x}$.



Find the shaded area in terms of p.

12 The diagram shows a part of the parabola $y = a^2 - x^2$. Point P has x-coordinate $\frac{a}{2}$.



Find the shaded area in terms of a.

Checklist of learning and understanding

- Integration is the reverse process of differentiation.
- If you know $\frac{\mathrm{d}y}{\mathrm{d}x}$, the indefinite integral gives an expression for y with an unknown constant of integration. You can find this constant if you know the coordinates of a point on the curve.
- For all rational $n \neq -1$:

$$\int x^n \,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + c$$

- The definite integral $\int_a^b \mathrm{f}(x) \,\mathrm{d}x$ is found by evaluating the integrated expression at the upper limit b and then subtracting the integrated expression evaluated at the lower limit a.
- The **area** between the curve, the x-axis and lines $x = a$ and $x = b$ is given by $A = \int_a^b \mathrm{f}(x)\,\mathrm{d}x$ provided that the part of the curve between $x = a$ and $x = b$ lies entirely above the x-axis.
- If the curve goes **below** the x-axis, then the integral of the part below the axis will be **negative**. To find the total area you need to find areas of the parts below and above the axis separately.

Mixed practice 15

1 A curve has gradient $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x - \sqrt{x}$ and passes through the point $(4, -1)$. Find the equation of the curve.

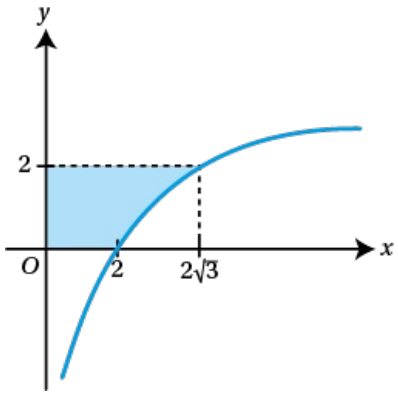
2 Find the indefinite integral $\int \frac{1 + x\sqrt{x}}{x^2}\ \mathrm{d}x$.

3 Given that $\mathrm{f}'(x) = (1 - x)(\sqrt{x} + 2)$, and that $\mathrm{f}(1) = 3$, find an expression for $\mathrm{f}(x)$.

4 **a** Find the exact value of $\int_2^{2\sqrt{3}} 3 - \frac{12}{x^2}\ \mathrm{d}x$.

Give your answer in the form $a + b\sqrt{3}$, where a and b are integers.

b The curve in the diagram has equation $y = 3 - \frac{12}{x^2}$. The curve crosses the x-axis at $x = 2$. The shaded region is bounded by the curve, the y-axis and the lines $y = 0$ and $y = 2$.



Find the area of the shaded region.

5 **a** Find $\int \left(x^2 - 2x + 5\right)\ \mathrm{d}x$.

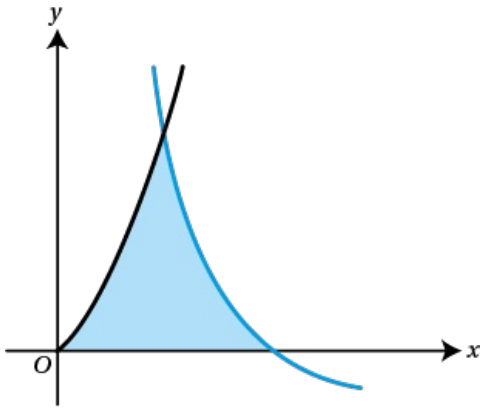
b Hence find the equation of the curve for which $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - 2x + 5$ and which passes through the point $(3, 11)$.

OCR, GCE Mathematics, Paper 4722, June 2012
[Question part reference style adapted]

6 Given that $\mathrm{f}(x) = \int 4x + 5\ \mathrm{d}x$, find $\mathrm{f}'(x)$.

7 **a** Find $\int \left(x^2 + 4\right)(x - 6)\ \mathrm{d}x$.

b The diagram shows the curve $y = 6x^{\frac{2}{3}}$ and part of the curve $y = \frac{8}{x^2} - 2$, which intersect at the point $(1, 6)$.
Use integration to find the area of the shaded region enclosed by the two curves and the x-axis.



OCR, GCE Mathematics, Paper 4722, January 2012

8 $f'(x) = \dfrac{4x^2 - 3\sqrt{x}}{x}$ **and** $f(1) = 2$**.**

Find $f(4)$.

9 $\int_1^a 2x - 3 \, dx = 6, a > 0$

What is the value of a?

10 **a** Find the positive value of a for which $\int_0^a x^3 - x \, dx = 0$.

b For this value of a, find the total area enclosed between the x-axis and the curve $y = x^3 - x$ for $0 \leqslant x \leqslant a$.

11 Find the area enclosed between the graph of $y = k^2 - x^2$ and the x-axis, giving your answer in terms of k.

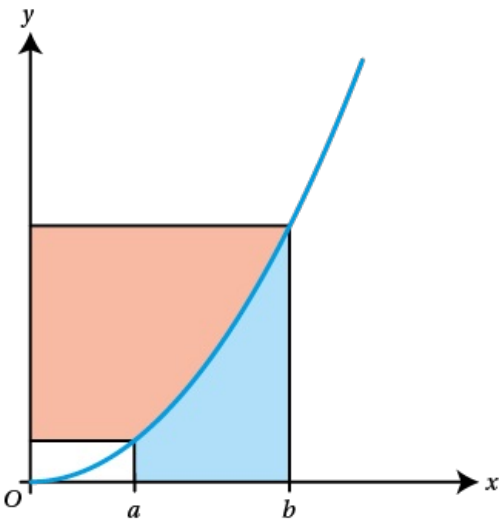
12 Let $f(x) = 2x^3 - 3x^2 - 3x + 2$.

a **i** Show that $(x - 2)$ is a factor of $f(x)$ and hence factorise f(x) completely.

ii Sketch the graph of $y = f(x)$, labelling clearly the points where the curve crosses the coordinate axes.

b Find the exact area enclosed by the x-axis and the graph of $y = f(x)$.

13 The diagram shows the graph of $y = x^n \text{for} n > 1$.



a **i** Write down an expression for the area of the white rectangle.

ii B is the area of the blue shaded region. Find an expression for B in terms of a, b and n.

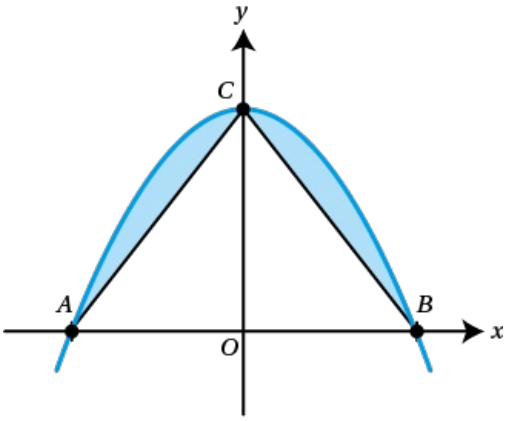
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b The pink area is three times as large as the blue area. Find the value of n.

Tip

You don't need an expression for the pink area to do this question.

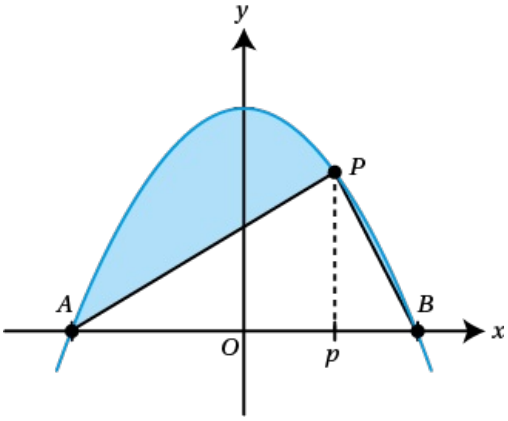
14 The diagram shows a parabola with equation $y = a^2 - x^2$.
The parabola crosses the x-axis at points A and B, and the y-axis at point C.



a i Write down the coordinates of A, B and C.

ii Find, in terms of a, the area of the shaded region.

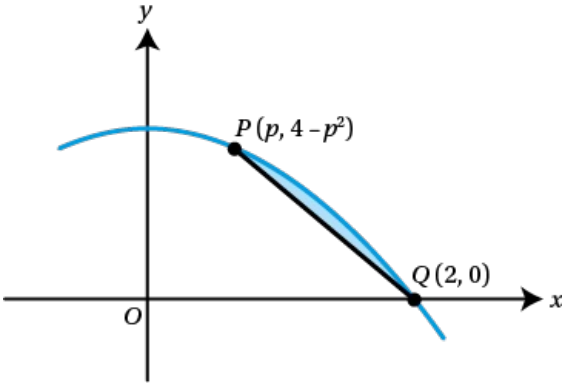
b Point P lies on the parabola. The x-coordinate of P is p.



The value of p varies between the x-coordinates of A and B.

Find the minimum value of the shaded area.

15 A part of the curve with equation $y = 4 - x^2$ is shown in the diagram. Point P has coordinates $(p, 4 - p^2)$ and point Q has coordinates $(2, 0)$.



The shaded region is bounded by the curve and the chord PQ. Show that the area of the

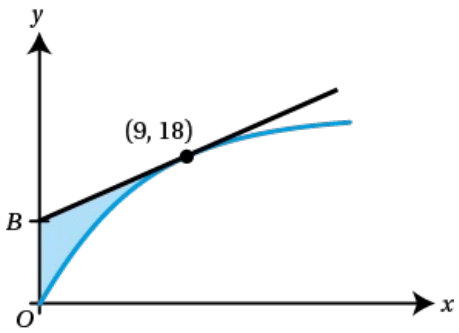
shaded region is $\frac{1}{6}(2-p)^3$.

16 The function f(x) has a stationary point at $(3, 19)$ and $\mathrm{f}''(x)$ $\mathrm{f}''(x) = 6x + 6$.

a Determine the nature of the stationary point at $(3, 19)$.

b Find an expression for $\mathrm{f}(x)$.

17 The diagram shows the graph of $y = 6\sqrt{x}$ and the tangent to the graph at the point $(9, 18)$. The tangent crosses the y-axis at the point B.



Find the area of the shaded region.

Worksheet

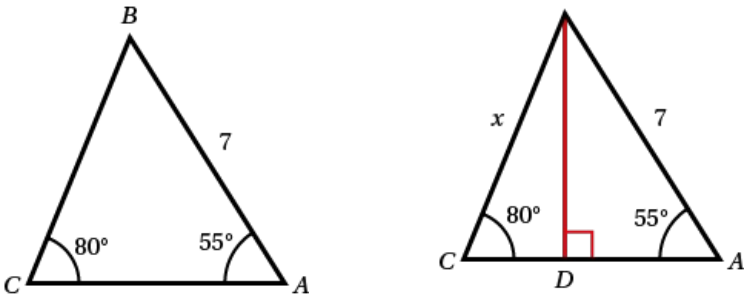
For more challenging question on integration and areas see Extension sheet 15.

FOCUS ON … PROOF 2

Breaking the problem down

The aim of this section is to show you how you can prove new results by breaking them down into previously known ones. You will use as examples the sine and cosine rules.

We start with a specific example. Consider a triangle with $\mathrm{AB}{=}7$, $\angle \mathrm{BAC} = 55^\circ$ and $\angle \mathrm{ACB} = 80^\circ$. What is the length BC?



There are no right angles in the diagram, but we can create some by drawing the line BD perpendicular to AC.

We now have two right-angled triangles, which share the side BD.

- Write an expression for BD using triangle ABD.
- Write an expression for BD using triangle BCD.
- Comparing the two expressions for BD, we get:

  $$x \sin 80^\circ = 7 \sin 55^\circ \quad (*)$$

  and rearranging gives:

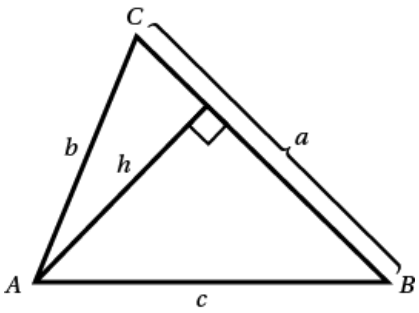
  $$x = \frac{7 \sin 55^\circ}{\sin 80^\circ} = 5.82$$

Tip

When writing a proof, we need to know which results we can assume. Here we will assume that you can use trigonometry in right-angled triangles.

Questions

1. Use the given example to write a general proof of the sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
2. How is the diagram different if the angle C is obtuse? Does the proof still work?
3. Use this diagram, and your knowledge of right-angled triangles and Pythagoras' theorem, to prove the cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$.



FOCUS ON … PROBLEM SOLVING 2

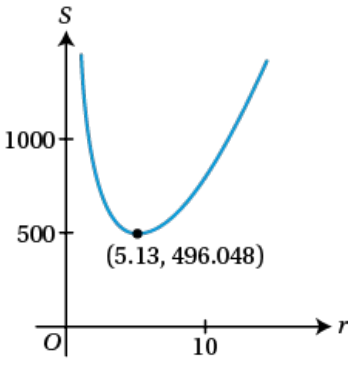
Choosing variables

Solving a problem often starts with writing some equations to represent the situation. While in exam questions you are often told things like 'express the volume in terms of r', in real applications you need to decide for yourself which variables are relevant to the question. Sometimes there is more than one possible choice, and some choices need simpler equations than others.

This section focuses on selecting variables and writing equations. Some of the resulting equations cannot be solved algebraically, so you will solve the equations using technology.

WORKED EXAMPLE

A closed cylindrical can has a fixed volume of $540\pi\text{cm}^3$. Find the minimum possible surface area of the can.

Let r be the radius and h the height of the cylinder.	Define variables: the surface area depends on the radius and the height of the can.
Surface area: $S = 2\pi r^2 + \pi rh$	Write an expression for the surface area.
Volume: $V = \pi r^2 h = 540\pi$	The surface area depends on two variables. You can eliminate one of them by using the expression for the volume.
$\Rightarrow h = \dfrac{540}{r^2}$	You now have a choice: you can express h in terms of r, or r in terms of h.
$\Rightarrow S = 2\pi r^2 + \pi r\left(\dfrac{540}{r^2}\right)$	The latter would involve square roots, so choose the former.
$S = 2\pi r^2 + \dfrac{540\pi}{r}$	
	You can use graphing software to sketch the graph of S against r and find the minimum value.
The graph shows that the minimum value of the surface area is $496\ \text{cm}^2$.	

Questions

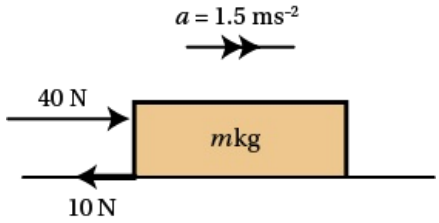
1 Two people attempt to push-start a car on a horizontal road. One person pushes with a force of 100 N; the other with a force of 80 N. The car starts to accelerate constantly at 0.15 m s^{-2}. Assuming

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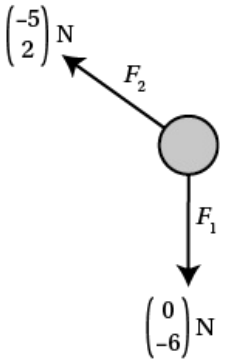
these are the only horizontal forces acting, find the mass of the car.

2 A sledge of mass m kg is pushed horizontally through the snow by a force of 40 N. There is resistance to its motion of magnitude 10 N as shown in the diagram.

If the sledge is accelerating at 1.5 m s^{-2}, find its mass.



3 Two forces F_1 and F_2 act on a particle as shown.



A third force, F_3, is added so that the resultant force on the particle is 2 N to the right.

Find:

a the magnitude of F_3

b the direction F_3 makes with the direction of motion.

FOCUS ON … MODELLING 2

The sunrise equation

The ‘sunrise equation’ can be used to calculate the approximate time the sun rises and sets at a specified location on Earth on a specified day.

$$H = \left| \frac{1}{15} \arccos \left[-\tan(L) \tan \left(23.44 \sin \left(\frac{360(D+284)}{365} \right)^\circ \right) \right] \right|$$

In this equation:

- H is the number of hours before local noon when sunrise occurs. ‘Local noon’ is when the sun is directly overhead at that particular location. (You may need to add an hour for summer time.)
- L is the latitude. It is measured in degrees, between -90° and 90°; points in the northern hemisphere have positive latitude.
- D is the day of the year, with $D = 1$ being 1 January.
- The modulus sign, | |, means that you take the positive value even if the value of arccos is negative.
- Sunset occurs H hours after local noon.

As well as calculating the times of sunrise and sunset, the equation can be used to generate a ‘Day and night world map’, which shows the parts of the world that are in daylight at a particular point in time.



Questions

1 London is at the latitude of around 53° north. Work out the time that the formula predicts sunrise will occur on:

a 16 February

b 5 July.

2 Predict the time of sunset, relative to local noon, in Rio de Janeiro (about 22° south) on 12 August. What would you need to consider to find the actual time of sunset?

3 Sydney, Australia, is at about 33° south. Find the approximate length of the day (the time between

sunrise and sunset) in Sydney on:

a 8 March

b 26 August.

4 State the largest possible value of $23.44 \sin x$. Hence find the length of the longest day in Cairo, located at 30° north.

5 Use the sunrise equation to show that, on the days of the vernal and autumnal equinoxes (20 March and 22 September) there are exactly 12 hours of daylight, regardless of the location on Earth.

6 Show that, when $L > 66.5$, there are values of D for which it is not possible to find H. What does this mean?

7 Plot the graph of H against D when:

a $L = 55$ (for example, Edinburgh)

b $L = 70$ (for example, Hammerfest, Norway).

8 Plot the graph of H against L for:

a 6 September

b 6 January.

9 What value of D should you use for 29 February in a leap year? Does the exact choice make much of a difference?

10 The constant 365 in the formula represents the number of the days in a year. A year actually has 365.25 days. Would using this value make significant difference to any of the answers you found?

11 Try to interpret the other constants in the equation.

12 What other modelling assumptions do you think have been made in forming this equation?

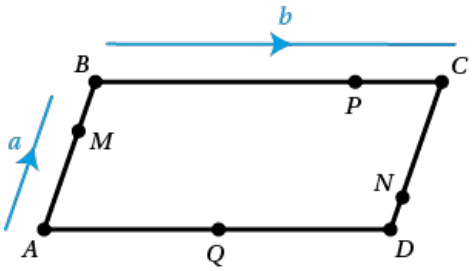
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CROSS-TOPIC REVIEW EXERCISE 2

1 For the function $\mathrm{f}(x) = ax^3 + bx^3 + 4x - 3$, given that $\mathrm{f}'(2) = 0$ and $\mathrm{f}''(2) = 10$, find $a, b \in \mathbb{R}$.

2 Find the exact period of the function $\mathrm{f}(x) = \sin 4x + \sin 6x$.

3 In this diagram, $\overrightarrow{AB} = \overrightarrow{DC} = \mathbf{a}$ and $\overrightarrow{BC} = \overrightarrow{AD} = \mathbf{b}$. Q is the midpoint of AD and points M, N, P and Q are such that $AM : MB = 2 : 1, DN : NC = 2.7, BP : PC = 3.1$.

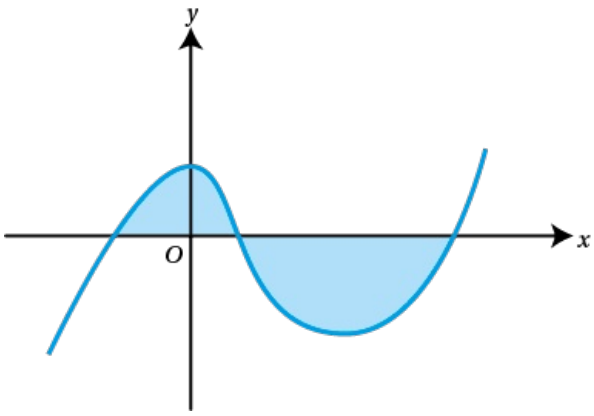


a Express $\overrightarrow{MP}$ and $\overrightarrow{QN}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

b Hence show that MP and QN are parallel.

4 Do not use a calculator in this question. Find the exact solutions of the equation $\sin 2\theta = \sqrt{3}\cos 2\theta$ for $0 \leqslant \theta \leqslant 180^\circ$.

5 A polynomial is defined by $\mathrm{f}(x) = x^3 - 5x^2 - x + 5$.



a Show that $(x + 1)$ is a factor of $\mathrm{f}(x)$, and factorise $\mathrm{f}(x)$ completely.

b The graph of $y = \mathrm{f}(x)$ is shown in the diagram. Find the exact value of the shaded area.

6 Consider the function: $\mathrm{f}(x) = \dfrac{(x-2)(x-6)}{\sqrt{x}}$.

a Show that this can be written in the form $\mathrm{f}(x) = x^a - 8x^b + 12x^c$ giving the values of the real numbers a, b and c.

b Find the equation of the normal to $\mathrm{f}(x)$ at the point $x = 4$.

c The normal intersects the x-axis at the point P and the y-axis at the point Q.

i State the coordinates of P and Q.

ii Give the exact area of the triangle POQ.

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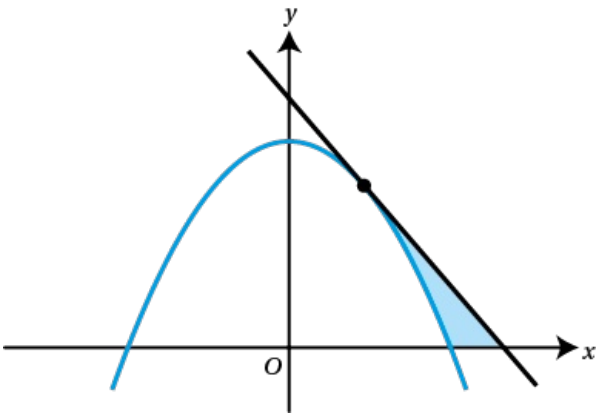
7 **a** Show that $\frac{1}{\cos x} - \cos x = \sin x \tan x$.

b Given that $\sin x - \cos x = \frac{1}{\sin x} - \frac{1}{\cos x}$

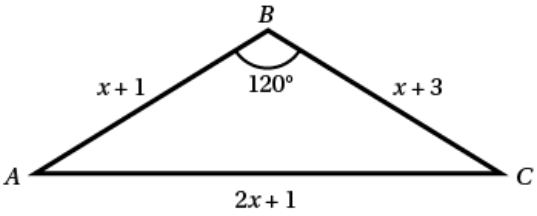
i Show that $\tan x = 1$.

ii Hence find the value of $x \in [180^\circ, 360^\circ]$.

8 The diagram shows the graph of $y = 9 - x^2$ and the tangent to the graph at $x = 1$. Find the shaded area.



9 The triangle in the diagram has sides $AB = x + 1$, $BC = x + 3$, and $CA = 2x + 1$ and angle $B = 120^\circ$.



Find the value of x.

10 **a** Expand and simplify $(x + h)^4 - x^4$.

b Hence prove from first principles that the derivative of x^4 is $4x^3$.

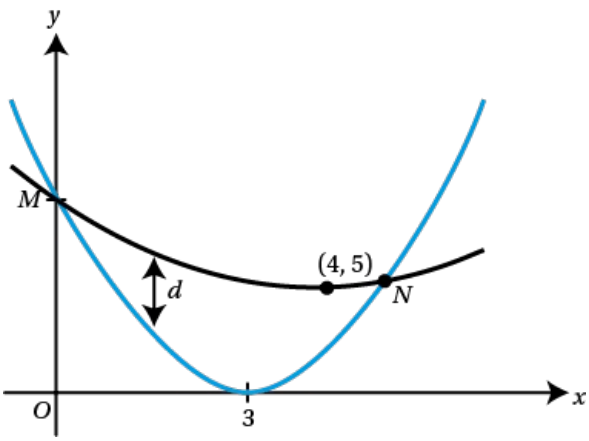
11 A curve has gradient $f'(x) = 2x - 6$ and passes through the point $(2, 0)$.

a Find the equation of the curve.

b Find the equation of the normal to the curve at the point $(2, 0)$.

c Find the coordinates of the points where the normal intersects the curve again.

12 The diagram shows curves with equations $y = (x - 3)^2$ and $y = a(x - p)^2 + q$.



a Find the values of a, p and q.

b Find the coordinates of the intersection points, M and N, of the two curves.

c The vertical distance between the two curves is denoted by d, as shown in the diagram. Write an expression for d in terms of x. Hence find the maximum vertical distance between two curves, on the part of the curves between points M and N.

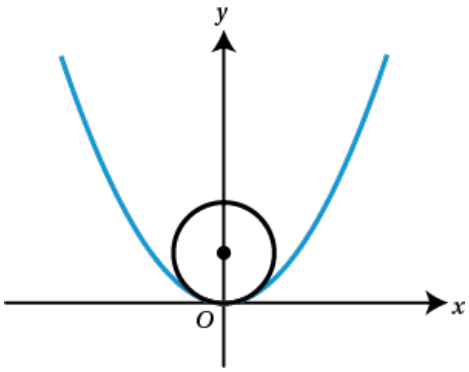
13 The point $A\,(3, 4)$ lies on the circle with equation $x^2 + y^2 = 25$. The point B lies on the parabola $y = 10 - x^2$. The tangent to the circle at A is parallel to the tangent to the parabola at B. Find the coordinates of B.

14 **a** The polynomial $\mathrm{f}(x)$ is defined by $\mathrm{f}(x) = x^3 - x^2 - 3x + 3$.
Show that $x = 1$ is a root of the equation $\mathrm{f}(x) = 0$, and hence find the other two roots.

b Hence solve the equation $\tan^3 x - \tan^2 x - 3\tan x + 3 = 0$ for $0 \leqslant x \leqslant 180^\circ$.

15 Given that $A = \arcsin\left(\frac{1}{3}\right)$, find the exact value of $\cos A$.

16 The diagram shows a parabola with equation $y = ax^2$ and a circle, with the centre on the y-axis, that passes through the origin. The radius of the circle is r.



a Show that the y-coordinates of any intersections of the circle and the parabola satisfy the equation $y^2 - 2yr + \frac{y}{a} = 0$.

b Hence find, in terms of a, the largest value of r for which the circle and the parabola have only one common point.

17 **a** Simplify $\left(a^2 + b^2\right)^2 - 2a^2b^2$.

Hence show that $\cos^4\theta + \sin^4\theta = 1 - 2\ \sin^2\theta\cos^2\theta$.

b Show that $(\cos\theta - \sin\theta)^4 = \left(1 - 2\sin^2\theta\cos^2\theta\right)^2$.

18 **a** Sketch the graph of $y = \tan\left(\frac{1}{2}x\right)$ for $-360^\circ \leqslant x \leqslant 360^\circ$.

On the same axes, sketch the graph of $y = 3\cos\left(\frac{1}{2}x\right)$ for $-360^\circ \leqslant x \leqslant 360^\circ$, indicating the point of intersection with the y-axis.

b Show that the equation $\tan\left(\frac{1}{2}x\right) = 3\cos\left(\frac{1}{2}x\right)$ can be expressed in the form

Hence solve the equation $\tan\left(\frac{1}{2}x\right) = 3\cos\left(\frac{1}{2}x\right)$ for $-360^\circ \leqslant x \leqslant 360^\circ$.

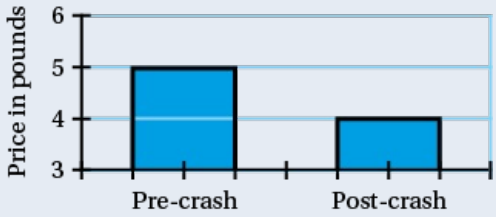
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16 Working with data

In this chapter you will learn how to:

- interpret statistical diagrams including stem-and-leaf diagrams, histograms, scatter diagrams, cumulative frequency curves and box-and-whisker plots
- calculate standard deviation for data
- work with grouped data
- understand correlation and use a regression line

Before you start...

GCSE	You should know how to interpret basic statistical diagrams such as pie charts and bar charts.	1 Find the percentage decrease in the stock price after a financial crash in the following bar chart. 
GCSE	You should know how to calculate the mean, median and mode of a set of data.	2 Find the mean, median and mode of: $1, 1, 4, 5, 9, 10$
GCSE	You should know how to calculate the range and interquartile range of a set of data.	3 Find the range and interquartile range of: $12, 15, 18, 18, 19, 16, 14, 20, 12$

Why collect data?

Statistics is an incredibly important part of mathematics in the real world. It provides the tools to collect, organise and analyse large amounts of data. For example, the government uses statistical analysis to plan public services such as building schools and hospitals.

At first, it might seem as if statistics is just about representing data through either diagrams or calculations. However, as you progress through statistics you will see that it is about making inferences about a larger group on the basis of a **sample**. Medical trials and pre-election polls are both examples of using samples to make a prediction about a whole population.

Although many statistical methods were developed more than 200 years ago, technological advances in the last few decades have enabled the analysis of much larger quantities of data, making statistics one of the most common applications of mathematics.

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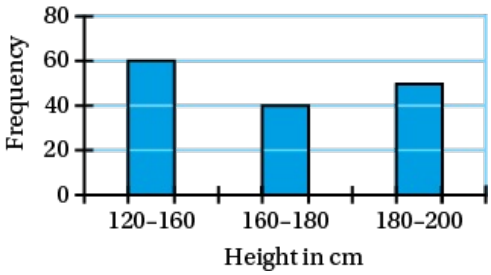
Section 1: A reminder of statistical diagrams

Histograms

Large amounts of data are often sorted into groups (called classes). With **continuous data**, these classes need to cover all possible data values in the range; this means that there are no gaps between classes. For example, if you have a sample of heights between 120 cm and 200 cm, you might split them into classes as follows:

Group	$120 \leqslant h < 160$	$160 \leqslant h < 180$	$180 \leqslant h < 200$
Frequency	60	40	50

You could draw this bar chart for this data.



However, this bar chart is misleading. The first class has the greatest frequency, but that might be due to the fact that it covers a wider range of height than the other classes. You can overcome this problem by using a **histogram**, which looks a lot like a bar chart but the horizontal scale is continuous and the frequency is equal to area rather than height. This means that the vertical scale is **frequency density**. The equation for frequency density is given in Key point 16.1.

Tip

Notice the use of inequality signs to define the classes. In practice, continuous data is usually rounded. It is therefore important to know where to put someone whose height is recorded as 180 cm.

Gateway to A Level

For a reminder of and practice with basic statistical diagrams, see Gateway to A Level section T.

Worksheet

You can use bar charts, pie charts, dot plots, histograms and cumulative frequency diagrams to explore real data in the Large Data Set section on the Cambridge Elevate digital platform.

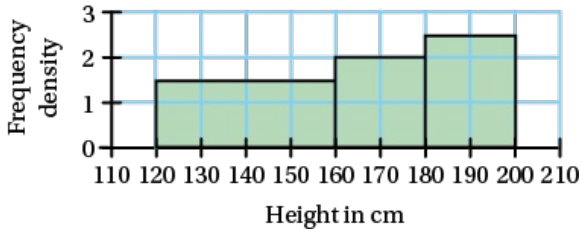
Key point 16.1

$$\text{Frequency density} = \frac{\text{frequency}}{\text{class width}}$$

Tip

The class width is the difference between the largest and smallest possible value in the class.

If you plot the previous data on a histogram, it shows much more clearly the shape of the distribution.

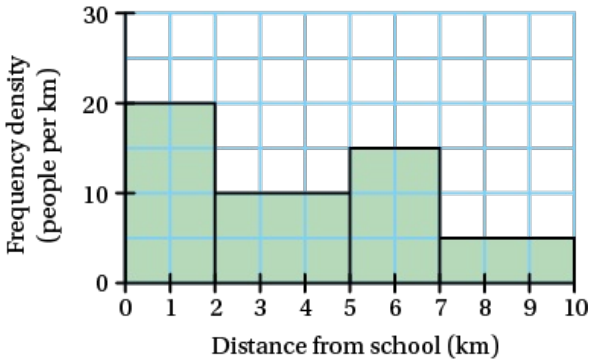


Tip

If all the class widths are equal, the histogram will have the same shape as the bar chart. However, it is still preferable to use frequency density on the vertical axis.

WORKED EXAMPLE 16.1

Use this histogram to estimate the probability of living between 5 km and 6 km away from school.



Class	0 **to** 2	2 **to** 5	5 **to** 7	7 **to** 10
width	2	3	2	3
Frequency density	20	10	15	5
Frequency	40	30	30	15

Convert the frequency densities to frequencies using the equation: frequency = frequency density × class width

So the total frequency is $40 + 30 + 30 + 15 = 115$

The area from 5 **to** 6 **is** $1 \times 15 = 15$ **people, so the probability is** $\frac{15}{115} \approx 0.13$

Did you know?



Some of the first users of statistical diagrams (in the 18th and 19th centuries) include Florence Nightingale and the political economist William Playfair. The term ‘histogram’ was coined by the statistician Karl Pearson. Florence Nightingale (1820–1910) used statistical evidence to convince the government to improve healthcare provision for military personnel during the Crimean War.

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Cumulative frequency graphs

A **cumulative frequency diagram** is a curve showing the data values on the horizontal axis and cumulative frequency on the vertical axis. You have already met cumulative frequency diagrams in your previous studies. Although these are diagrams, their main purpose is not to provide a visual representation of the data – histograms are much better at that. They are mainly used to estimate the **median** and **quartiles** of grouped data.

Cumulative frequency is the total number of data items less than or equal to a particular value.

WORKED EXAMPLE 16.2

This table shows the masses of a sample of eggs. Find the cumulative frequency for the upper bound of each class.

Mass of eggs, x, in g	Frequency
$100 < x \leqslant 120$	26
$120 < x \leqslant 140$	52
$140 < x \leqslant 160$	84
$160 < x \leqslant 180$	60
$180 < x \leqslant 200$	12

Mass of eggs, x, in g	Cumulative frequency
$x \leqslant 120$	26
$x \leqslant 140$	78
$x \leqslant 160$	162
$x \leqslant 180$	222
$x \leqslant 200$	234

The cumulative frequency is found by adding on the number in each class to the cumulative frequency of the previous class and is recorded against the upper bound for the class.
For example, 26 eggs had mass up to 120 g, $26 + 52 = 78$ had mass up to 140 g, etc.

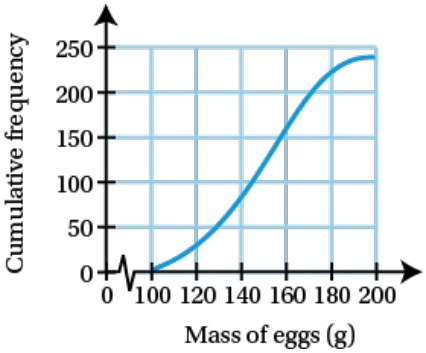
Once you have the cumulative frequency you can draw a cumulative frequency diagram. This has the data values along the horizontal axis and the cumulative frequency up the vertical axis. In Worked example 16.2, you can see that 26 eggs weighed 120 g or less, 78 eggs weighed 140gor less, and so on. Therefore you should plot the cumulative frequency against the upper bound of each class.

Also, notice that you have an additional point you can plot. You know from the data table that there are no eggs that weigh less than 100 g. You can then produce this cumulative frequency diagram.

Once you have the cumulative frequency graph you can use it to estimate the median and the quartiles. The median is the value corresponding to the middle data item. On the graph, this corresponds to drawing

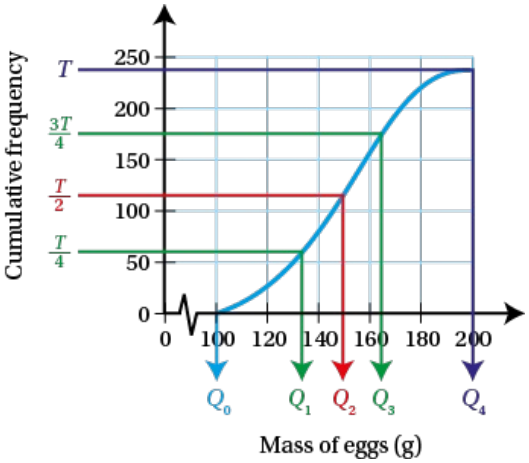
a horizontal line at half the total frequency until it meets the curve, then a vertical line down to find the median value.

To find the quartiles you follow a similar process but with horizontal lines at one quarter and three quarters of the total frequency.



WORKED EXAMPLE 16.3

Estimate the median and interquartile range of the eggs data in Worked example 16.2.



The total frequency (T) is 234, so draw lines across from the vertical axis at $0.5 \times 234 = 117$ for the median, and at $0.75 \times 234 = 175.5$ and $0.25 \times 234 = 58.5$ for the upper and lower quartiles. Where these horizontal lines meet the cumulative frequency curve, draw down to the horizontal axis to find the values of the median and quartiles.

Median $(Q_2) \approx 150\ g$
Upper quartile $(Q_3) \approx 165\ g$
Lower quartile $(Q_1) \approx 135\ g$
IQR $= Q_3 - Q_1 \approx 30\ g$

Read off the values on the horizontal axis shown by the constructed lines. The accuracy of the quartiles will depend on the scale of the graph and the smoothness of the line drawn. Here, the values are given to the nearest 5.

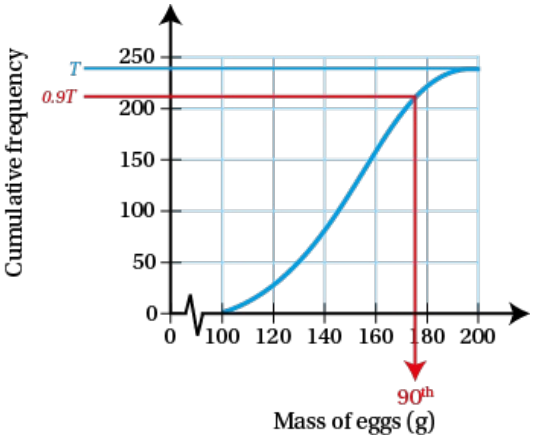
The median and quartiles are specific examples of **percentiles**, which tell you what data value is a given percentage of the way through the data when it is put in order. The median is therefore the 50th percentile and the lower quartile is the 25th percentile.

WORKED EXAMPLE 16.4

a Find the 90th percentile of the mass of eggs in Worked example 16.2.

b 10% of eggs are classed as extra large. At what mass is an egg classed as extra large?

The total frequency is 234, so draw a line across from the vertical axis at $0.9 \times 234 \approx 211$.



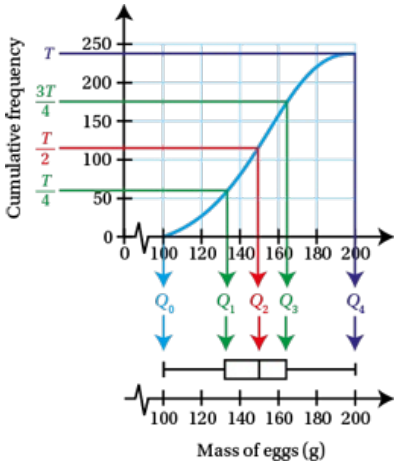
a From the diagram the 90th percentile is approximately 175 g.

Read off the values on the horizontal axis shown by the constructed line.

b Eggs with mass greater than 175 g are classified as extra large.

Box-and-whisker plots

A useful way of visually representing the information found from a cumulative frequency diagram is a **box-and-whisker plot**. The 'box' extends from the lower quartile to the upper quartile, with the vertical line marking the median. The 'whiskers' connect the ends of the box to the maximum and minimum data values.



Fast forward

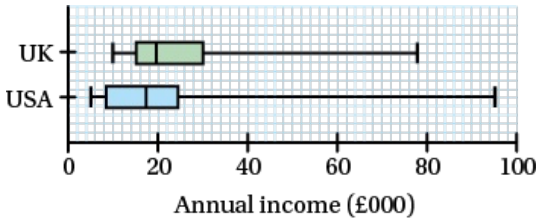
In Section 5 you will see that if the data contains outliers you can mark them as crosses on the box-and-whisker plot.

Box-and-whisker plots are useful when comparing two sets of data.

WORKED EXAMPLE 16.5

These box-and-whisker plots show the incomes of a large sample of people in the UK and the USA. Make two comments comparing the distributions of incomes.

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The median income in the UK is slightly higher than in the USA, so people get paid a little more on average.

You should make one comment on average...

Both the range and the interquartile range of the USA income is higher, so they have a larger spread of incomes.

and one comment on spread.

Stem-and-leaf diagrams

The diagrams you have met so far are all used for grouped data. They are very good for showing the distribution of the data, but the detail of the individual results is lost. A **stem-and-leaf diagram** is a way of recording all the data values by sorting it in rows according to the first digit(s), thus organising them into groups to show the overall distribution. This is only practical for moderately large sets of data.

Stem-and-leaf diagrams can be used to compare two distributions, or to find the median and quartiles.

WORKED EXAMPLE 16.6

This back-to-back stem-and-leaf diagram shows the heights (in cm) of a group of 20 boys and a group of 25 girls.

boys		girls
8	12	6 6 7 9 9
9 8 4	13	0 1 1 3 4 7 7 8 8
9 9 7 6 4 4 1 0 0	14	1 2 2 2 5 8 9
3 3 2 1 1 1 0	15	0 0 1 4

key: $12|6$ means 126 (cm)

a How tall is the tallest girl?

b Find the median height of the boys.

c Compare the distributions of heights of the two groups.

a The tallest girl in the list has height recorded as $15|4$, which means 154 cm.

The 'key' beside the diagram tells you what the numbers mean.

b The 10th value is 146 and the 11th 147, so the median height of the boys is 146.5 cm.

There are 20 boys, so the median is the average of the 10th and 11th numbers.
Count along to find them, remembering that the boys' heights on each line go from right to left!

c Boys are taller on average.

You can comment on three things: the average...

The spread of the heights is similar for both groups.

the spread ...

The girls' distribution is more symmetrical.

and the shape of the distribution.

EXERCISE 16A

1 For each of the following data sets:

a draw a histogram

b draw a cumulative frequency diagram

c estimate from your cumulative frequency diagram the median and interquartile range

d draw a box-and-whisker plot.

A. x is the time, in minutes, taken to travel to work.

x	Frequency
$0 < x \leqslant 15$	17
$15 < x \leqslant 30$	23
$30 < x \leqslant 45$	42
$45 < x \leqslant 60$	21
$60 < x \leqslant 90$	5

B. x is the age, in years, of residents in a village.

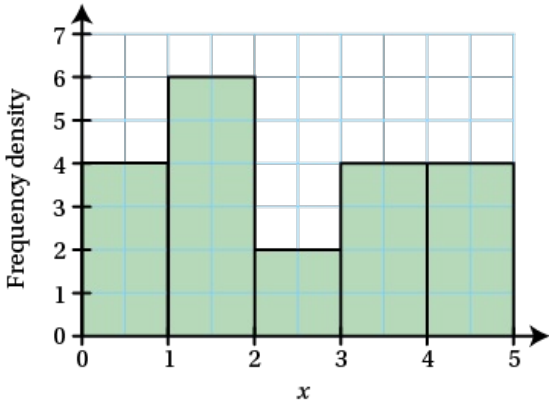
x	Frequency
$0 < x \leqslant 15$	19
$15 < x \leqslant 30$	15
$30 < x \leqslant 45$	7
$45 < x \leqslant 60$	5
$60 < x \leqslant 90$	4

Tip

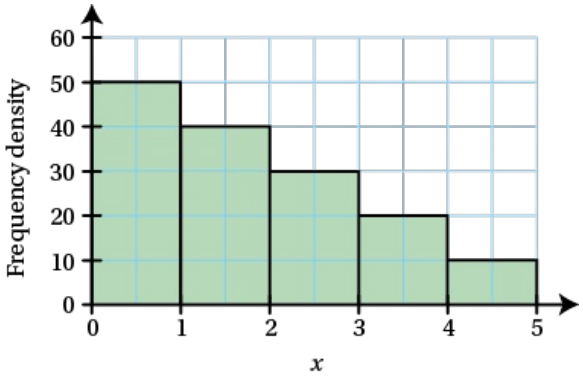
Technology should normally be used to draw statistical diagrams so that you can focus on interpreting them. However, you may find that question 1 is helpful in developing your understanding.

2 For each of these histograms find the probability that the value of x is between 1 and 2.

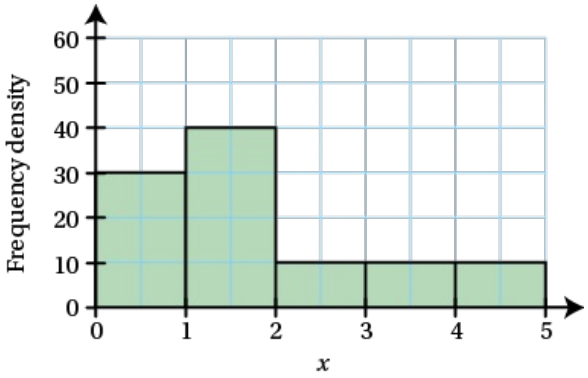
a **i**



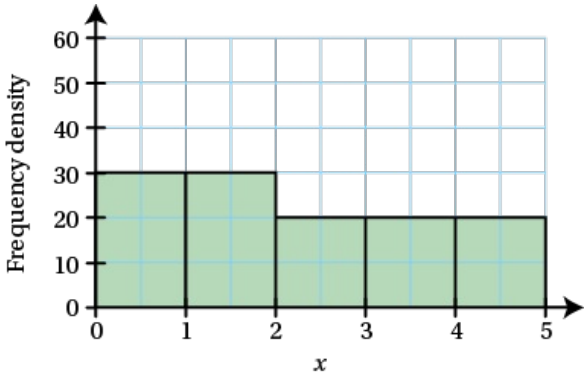
ii



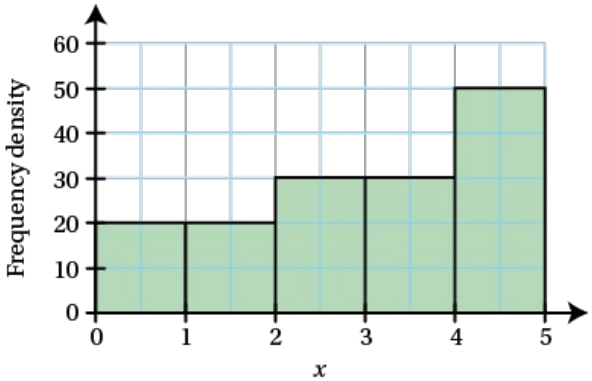
b **i**



ii

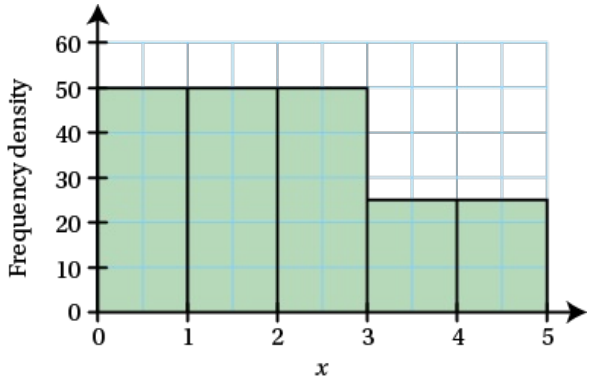


c **i**

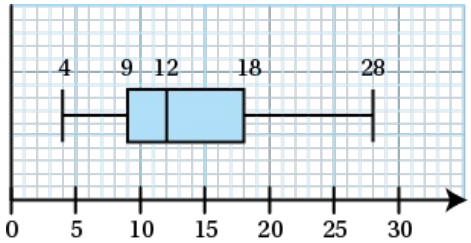


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ii



3 From the box-and-whisker plot shown, state the median and interquartile range.



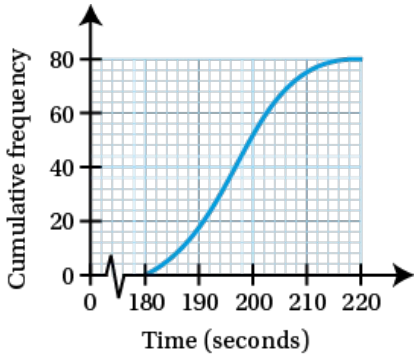
4 80 students were asked to solve a simple word puzzle and their times, in seconds, were recorded. The results are shown on the cumulative frequency graph.

a Estimate the median.

The middle 50% of students took between c and d seconds to solve the puzzle.

b Estimate the values of c and d.

c Hence estimate the interquartile range.

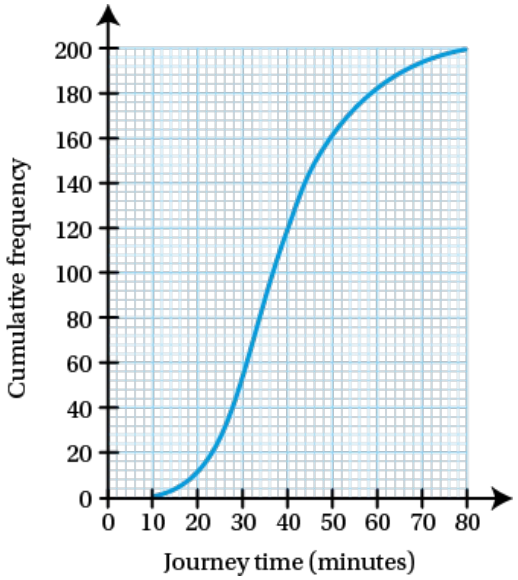


5 This cumulative frequency curve indicates the amount of time 200 students spend travelling to school.

a Estimate the percentage of students who spend between 30 and 50 minutes travelling to school.

b If 80% of the students spend more than x travelling to school, estimate the value of x.

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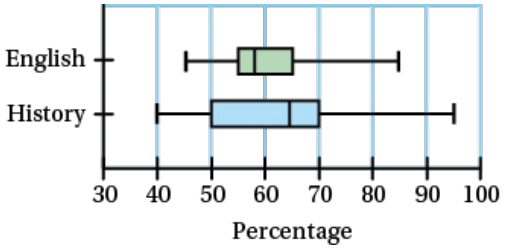
6 These box-and-whisker plots show the results of students in a History test and an English test.

a Compare the results in the two tests.

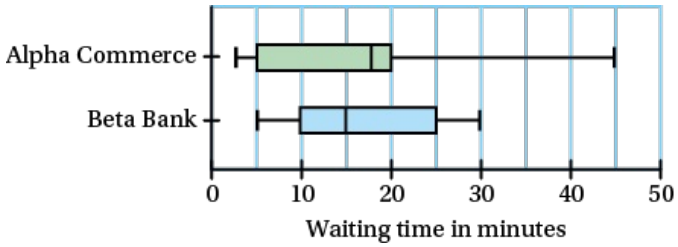
b What is the probability that a randomly chosen student scores more than 50% in the History test?

c State one further piece of information you would need to know to decide if the History test was easier than the English test.

d State one important feature of the data that is not conveyed by the box-and-whisker plot.



7 These box-and-whisker plots show waiting times for two telephone banking services.



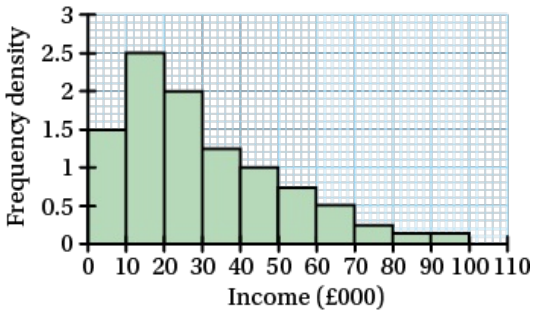
a What is the interquartile range of the waiting time for Beta Bank?

b If I need my calls to be answered within 5 minutes, which bank should I choose? What is the probability of getting the call answered within 5 minutes?

c If I need my calls to be answered within 15 minutes, which bank should I choose? What is the probability of getting the call answered within 15 minutes?

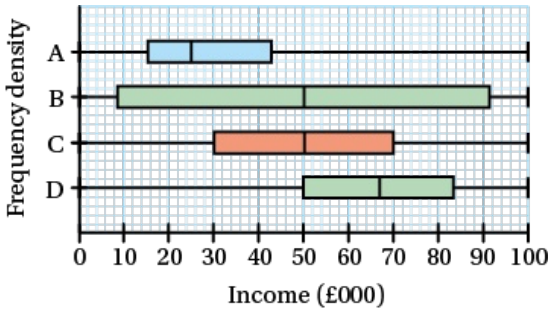
8 The histogram shows the wages of employees in a company.

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a Use the histogram to estimate the probability of a randomly chosen employee earning between £ $20\,000$ and £$25\,000$.

b The diagram shows four box-and-whisker plots labelled A, B, C and D. Explain which one corresponds to the data in the histogram.



9 The stem-and-leaf diagram shows the times, in seconds, that two groups of 20 students took to complete a task.

Group 1		Group 2
8 8 6	1	6 6 7 9 9
9 8 5 4 4 3 2	2	0 1 1 3 4 7 8 8
9 9 7 6 4 4 1 0 0 0	3	1 2 2 4 8 9

key: $1|4$ **means** 14s

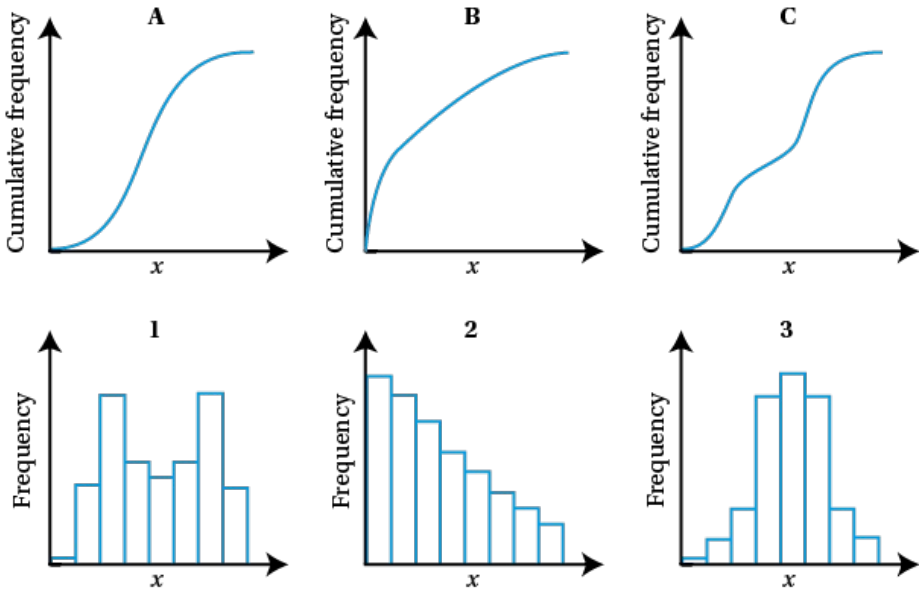
a Find the median and interquartile range of the times for Group 2.

b Calculate the mean time for Group 2.

c Without doing any further calculations, write two comments comparing the times for the two groups.

10 Match each histogram with the cumulative frequency diagram coming from the same data.

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11 A histogram is drawn for the following data.

x	$0 < x \leqslant 2$	$2 < x \leqslant 5$
Frequency	a	ab

The height of the bar over the $0 < x \leqslant 2$ group is h. Find, in terms of b and h only, the height of the bar over the $2 < x \leqslant 5$ group.

Section 2: Standard deviation

The **range** is the difference between the largest and smallest values in a data set and the **interquartile range** is the difference between the lower and upper quartiles of a data set. These are both measures of spread. However, **neither** takes into account **all** of the data. There is another measure called the **standard deviation**, usually given the symbol σ (the Greek letter lower case sigma). This does take into account all of the data. It is a measure of the mean difference of each data point from the mean.

Gateway to A Level

For a reminder of measures of average and spread, see Gateway to A Level section U.

Consider the data $2,\ 5,\ 8$. The mean, which is usually given the symbol $\bar{x}$, is 5.

You can look at the difference of each data point from the mean.

x	$x - \bar{x}$
2	-3
5	0
8	3

Worksheet

You will practise using spreadsheets to calculate measures of average and spread in the Large Data Set section on the Cambridge Elevate digital platform.

The mean of the differences is zero because the negative values cancel out the positive values. This will always be the case so it is no good as a measure of spread. However, if you square the difference you eliminate the negative values.

x	$(x - \bar{x})^2$
2	9
5	0
8	9

The average is given by adding up all the values of $(x - \bar{x})^2$ and dividing by n, the number of data items. In mathematics the symbol Σ (the Greek letter capital sigma), is used to mean 'add up all the terms'. In this case, the sum of the numbers in the right column is 18 so their average is $18 \div 3 = 6$.

Explore

Try finding out about other possible measures of spread. If, instead of squaring, you simply ignore any negative signs you will get another measure of the spread called the absolute deviation. However, this is not used very widely because it does not have some of the beautiful properties of the standard deviation, which you will meet if you study Further Mathematics.

You then need to undo the squaring to get a measure that has the same units as x. This means that the standard deviation for this set of data is $\sqrt{6}$.

Key point 16.2

Standard deviation: $\sigma = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$

Standard deviation can be thought of as the 'root mean square deviation from the mean'. Look at the formula in Key point 16.2 to see how this is an accurate definition. There is an alternative formula for the standard deviation which is sometimes easier to calculate.

Focus on ...

Focus on ... Proof 3 shows you the proof that the formulae in Key points 16.2 and 16.3 are equivalent.

Key point 16.3

$$\sigma = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2}$$

This can be written as $\sigma^2 = \overline{x^2} - \bar{x}^2$

Tip

Notice that the mean of x^2 is different from the mean squared.

The **variance**, σ^2, is the square of the standard deviation. It has some very useful mathematical properties.

You now know how to calculate different measures of spread, including: range, interquartile range and standard deviation. Worked example 16.7 illustrates all three of these. There are several alternative ways to find quartiles from a list of data values; some give slightly different answers. We illustrate one possible method here, but you can use whichever way you are already familiar with.

Fast forward

You will learn more about variance in the Statistics option in Further Mathematics.

Gateway to A Level

For more practice of finding quartiles see Gateway to A Level section U.

WORKED EXAMPLE 16.7

In this question you must show detailed reasoning.
Find the range, interquartile range and standard deviation of the following numbers:
$1,\ 12,\ 9,\ 9,\ 15,\ 7,\ 5$

range $= 15 - 1 = 14$

Interquartile range:
$1,\ 5,\ 7,\ 9,\ 9,\ 12,\ 15$
$LQ = 5, UQ = 12$, so $IQR = 7$

Order the data then split into two halves. Because there is an odd number, you discard the middle number. Then find the midpoint of each half.

Standard deviation:

	x	x^2
	1	1

Square each data value.

	5	25
	7	49
	9	81
	9	81
	12	144
	15	225
Sums:	58	606

$$\sigma = \sqrt{\frac{606}{7} - \left(\frac{58}{7}\right)^2}$$
$$= \sqrt{17.918}$$
$$\approx 4.23 \text{ (3 s.f.)}$$

Find the sum of x and of x^2.

In many naturally occurring measurements, about two thirds of the data will be less than one standard deviation away from the mean. This can serve as a useful quick check (even though you should not draw any firm conclusions from a small data set): in worked example 16.7, two data items (1 and 15) out of seven are more than 4.23 away from the mean (8.29). In a large data set, nearly all the data will be within two standard deviations of the mean, and anything more than three standard deviations away is very unusual.

Fast forward

Most calculators also have an option to calculate s_{n-1} which is used to estimate the population standard deviation from a sample. You will only use this if you study the Statistics option in Further Mathematics.

Although you need to understand how to use the formula, in practice you can use your calculator to find the mean and standard deviation. Be careful: different calculators use different symbols for standard deviation (for example, σ or s_n). Make sure you know which option to choose! Whatever symbol your calculator uses, in your solution you should use σ for standard deviation.

You will often be given data in the form of summary statistics, where you are told the sum of all the data values and the sum of their squares. You can then use the formula from Key point 16.3 to find the standard deviation.

WORKED EXAMPLE 16.8

Deepti records the length of time, t minutes, it takes her to cycle to school on 18 separate days. Her results are summarised as follows: $\sum t = 295.2, \sum t^2 = 5222.6$. Find the mean and standard deviation of the times.

$$\bar{t} = \frac{295.2}{18} = 16.4 \text{ minutes}$$

The mean is the sum of all the values divided by the number of values.

$$\sigma^2 = \frac{\sum t^2}{18} - (\bar{t})^2$$
$$= \frac{5222.6}{18} - 16.4^2$$
$$= 21.18$$

Use the formula from Key point 16.3. Find the variance first and then square root the answer. Note that you have already calculated the mean.

$$\sigma = \sqrt{21.18} = 4.6 \text{ minutes}$$

The units of standard deviation are the same as the units of the original data.

Comparing sets of data

You already know how to use the median and inter-quartile range to compare two sets of data. The advantage of using mean and standard deviation is that they take into account all of the data, whereas the IQR only tells you about the spread of the middle half. On the other hand, if a data set contains outliers, mean and standard deviation may not be a good representation of the whole data set.

Fast forward

In Section 5 you will learn how to use standard deviation to identify outliers.

WORKED EXAMPLE 16.9

A school football team consists of eleven players with a mean age 16.3 years and standard deviation 0.6 years. The school orchestra has 20 members with a mean age 15.8 years and standard deviation 1.4 years.

a Compare the ages of the football team players and the orchestra members.
The orchestra conductor is a teacher who is 62 years old.

b Find the mean age of all the orchestra members, including the conductor.

c A student says that the orchestra members are on average older than the football team members.
Comment on this statement.

a The football team are older on average, but have a smaller spread of ages.

You should make one comment comparing an average and one comment comparing spread of ages.

b $$\text{new mean} = \frac{20 \times 15.8 + 62}{21}$$
$$= 18 \text{ years}$$

You should make one comment comparing an average and one comment comparing spread of ages.

c This is not a useful comparison, because the mean for the orchestra is affected by one extreme value.

The mean is not a good measure of average age of the orchestra because one person is much older than the rest.

WORK IT OUT 16.1

Find the standard deviation of the following data: $2, 4, 5, 9$.
Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution A

$$\bar{x} = \frac{2+4+5+9}{4} = 5$$
$$\Sigma(x = \bar{x})^2 = (-3)^2 + (-1)^2 + 0^2 + 4^2 = 26$$
$$\sigma = \sqrt{\frac{26}{4}} \approx 2.55$$

Solution B

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$\Sigma x = 20$ so $\Sigma x^2 = 400$

$\overline{x^2} = \dfrac{400}{4} = 100$

$\bar{x} = \dfrac{20}{4} = 5$

So standard deviation is $100 – 5^2 = 75$

Solution C

From calculator, standard deviation is 2.94.

EXERCISE 16B

1 For each set of data, calculate the standard deviation and interquartile range.

a **i** $19.0, 23.4, 36.2, 18.7, 15.7$

ii $0.4, -1.3, 7.9, 8.4, -9.4$

b **i** $28, 31, 54, 28, 17, 30$

ii $60, 18, 42, 113, 95, 23$

c **i** $1, 2, 1, 3, 5$

ii $3, -2, 4, -2, 5, 2$

2 Sets of data are summarised by the information given. For each set of information find the standard deviation.

a **i** $\Sigma(x - \bar{x})^2 = 42.9, n = 10$

ii $\Sigma(x - \bar{x})^2 = 8.9, n = 10$

b **i** $\Sigma x = 49, \Sigma x^2 = 339,\ n = 8$

ii $\Sigma x = 329, \Sigma x^2 = 22135, n = 8$

c **i** $\bar{x} = 66.6, \overline{x^2} = 4512.6$

ii $\bar{x} = 24.8, \overline{x^2} = 1072.4$

Worksheet

For more practice finding standard deviation from summary statistics see Support sheet 16.

3 The ordered set of data $5, 5, 7, 8, 9, x, 13$ has interquartile range equal to 7.

a Find the value of x.

b Find the standard deviation of the data set.

4 10 data items have a sum of 468 and the sum of the squares of the data is $27\,172$.

a Find the mean of the data.

b Find the variance of the data.

5 The speed, x, in mph, of 10 serves by Tim, a professional tennis player, is summarised as follows:

$\Sigma x = 1245$, $\Sigma x^2 = 156403$

a Find the mean speed of the serves.

b Find the standard deviation in the speed of the serves.

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c Andy is another professional tennis player. The variance in the speed of Andy's serves is $89.6\ \text{mph}^2$. Which player appears to be more consistent in their serving speed?

6 The scores in a Physics test were : $81, 36, 73, 78, 74, 75$.

a Find the standard deviation of these scores.

b The standard deviation of the results of the same set of students in a Chemistry test was 5.91. Give two reasons why it would not be appropriate to use a comparison of these two standard deviations to determine whether students were more consistent in Physics or Chemistry.

7 Consider the five numbers, $2, 5, 9, x$ and y. The mean of the numbers is 5 and the variance is 6. Find the value of xy.

Tip

In question 6, you can use a calculator to find the standard deviation.

8 The mean IQ of a class of 9 students is 121 and the variance is 226. Another student joins the class and the variance changes to 239.4. What are the possible values of the IQ of the new student?

9 **a** Explain why for any piece of data $x - \bar{x}$ is less than the range.

b By considering the formula $= \sqrt{\dfrac{\Sigma(x-\bar{x})^2}{n}}$ prove that the standard deviation is always strictly less than the range.

Worksheet

You can use Extension sheet 16 to explore different ways of comparing sets of data.

Section 3: Calculations from frequency tables

It is very common to summarise large amounts of data in a frequency table. This is a list of all the values that the data takes, along with how often they occur. You could convert this into a list of all the data values and calculate the statistics as you did previously. However, there is a formula that you can use to find the mean more quickly.

Tip

Your calculator may be able to work with frequency tables. You can always use it to check your answer, but you may also be required to use the formula.

Key point 16.4

$\bar{x} = \dfrac{\Sigma fx}{n}$ where f is the frequency of each x value and n is the total frequency.

You can work out $\overline{x^2}$ in a similar way, giving the following formula for the variance.

Key point 16.5

$$2 = \frac{\Sigma\, fx^2}{n} - \bar{x}^2$$

WORKED EXAMPLE 16.10

Find the median, mean and standard deviation of the number of passengers observed in cars passing a school.

Passengers	Frequency
0	32
1	16
2	2
3 or more	0

median = average of 25th and 26th numbers so median $= 0$

There are $32 + 16 + 2 = 50$ data items so the median is the average of 25th and 26th data items. Both of these equal 0.

$\Sigma fx = (32 \times 0) + (16 \times 1) + (2 \times 2)$
$= 20$ passengers

The total number of passengers is: none in the first group, 16 in the second group and 4 in the third group.

$\bar{x} = 20 \div 50 = 0.4$

The total frequency is $32 + 16 + 2 = 50$. Always sense check your answer – here, 0.4 passengers per car seems reasonable. It does not have to be rounded as means do not have to be achievable numbers.

$\Sigma fx^2 = (32 \times 0) + (16 \times 1) + (2 \times 4) = 24$

To find the standard deviation you need to first calculate the mean of the squares.

$\sigma^2 = \frac{24}{50} - 0.4^2 = 0.32$

$\sigma = 0.566 \text{ to } 3 \text{ s.f.}$

Using Key point 16.5.

In the calculations in Worked example 16.9, you knew the exact data values, but when you are dealing with grouped data, you no longer have this level of precision. In order to assess the mean and standard deviation, your best and simplest assumption is that all the original values in a group were located at the centre of the group, called the **mid-interval value**. To find the centre of the group you take the mean of the largest and the smallest possible values in the group.

WORKED EXAMPLE 16.11

Find the mean and standard deviation of the mass of eggs produced by a chicken farm. Explain why these answers are only estimates.

Mass of eggs, in g	Frequency
$[100, 120)$	26
$[120, 140)$	52
$[140, 160)$	84
$[160, 180)$	60
$[180, 200)$	12

x	f	fx	fx^2
110	26	2860	314 600
130	52	6760	878 800
150	84	12 600	1 890 000
170	60	10 200	1 734 000
190	12	2280	433 200
Sum:	234	34 700	5 250 600

Make a table assuming each data value lies in the centre of its group.

$\bar{x} = \frac{\Sigma fx}{n}$

$= \frac{34\,700}{237} = 148.3\text{g} \left(3 \text{ s.f.}\right)$

Your calculator should be able to give you the values of Σfx and Σfx^2. You should write those down as evidence of your method.

$2 = \frac{\Sigma fx^2}{n} - \bar{x}^2$

$= \frac{5250600}{234} - 148.3^2 = 448.4$

Therefore $\sigma = 21.2\,\text{g}$ $(3\,\text{s.f.})$

Always do a sense check – in this case 21.2 g seems like a sensible measure of spread as it is significantly smaller than the range. You can also check that about two thirds of the data (about 150 values) lie within one standard deviation of the mean (between about 130 and 170).

These answers are only estimates because we have assumed that all the data in each group is at the centre, rather than using the actual data values.

Rewind

This bracket notation to describe intervals was introduced in Chapter 1, Section 2.

Sometimes the endpoints of the intervals shown in the table are not the actual smallest and largest possible values in that group. For example, when measuring length in centimetres it is common to round the values to the nearest integer, so $10 - 15$ actually means $[9.5,\ 15.5)$. To find the mid-interval values you must first identify the actual interval boundaries.

WORKED EXAMPLE 16.12

Estimate the mean of the following data, which relates to the age of a sample of young people in years. Give your answer accurate to 3 significant figures.

Age	Frequency
10 to 12	27
13 to 15	44
16 to 19	29

Group	x	f	fx
[10,13)	11.5	27	310.5
[13,16)	14.5	44	638
[16,20)	18	29	522
Sum:		100	1470.5

Carefully decide on the upper and lower interval boundaries. There should be no 'gaps' between the groups, because age is continuous. Age is a little bit tricky because you are 17 years old until your 18th birthday.

$$\bar{x} = \frac{\sum fx}{n} = \frac{1470.5}{100} = 14.7 \text{ (3 s.f.)}$$

Enter midpoints and frequencies into your calculator.

EXERCISE 16C

1 Calculate the mean, standard deviation and the median for each of these data sets.

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a **i**

x	Frequency
0	16
1	22
2	8
3	4
4	0

ii

x	Frequency
-1	10
0	8
1	5
2	1
3	1

b **i**

x	Frequency
10	7
12	19
14	2
16	0
18	2

ii

x	Frequency
0.1	16

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0.2	15
0.3	12
0.4	9
0.5	8

2 Calculate estimates of the mean and standard deviation of each of the following sets of data. Use statistical functions on your calculator.

a **i** x is the time taken to complete a puzzle in seconds.

x	Frequency
$[0,\ 15)$	19
$[15,\ 30)$	15
$[30,\ 45)$	7
$[45,\ 60)$	5
$[60,\ 90)$	4

ii x is the mass of plants in grams.

x	Frequency
$[50,\ 100)$	17
$[100,\ 200)$	23
$[200,\ 300)$	42
$[300,\ 500)$	21
$[500,\ 1000)$	5

b **i** x is the length of fossils found in a geological dig, to the nearest centimetre.

x	Frequency
0 to 4	71
5 to 10	43

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11 to 15	22
16 to 30	6

ii x is the power consumption of light bulbs, to the nearest watt.

x	Frequency
90 to 95	17
96 to 100	23
101 to 105	42
106 to 110	21
111 to 120	5

c **i** x is the age of children in a hospital ward.

x	Frequency
0 to 2	12
3 to 5	15
6 to 10	7
11 to 16	6
17 to 18	3

ii x is the value of tips paid in a restaurant, rounded down to the nearest pound.

x	Frequency
0 to 5	17
6 to 10	29
11 to 20	44
21 to 30	16
31 to 50	8

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3 A group is described as '$17 - 20$'. State the upper and lower boundaries of this group if it is measuring:

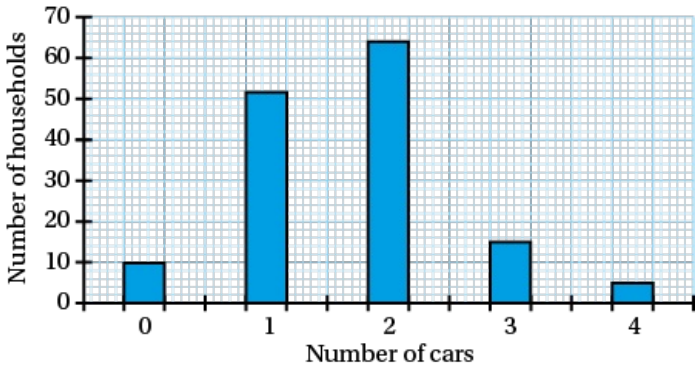
a age in completed years

b number of pencils

c length of a worm to the nearest centimetre

d hourly earnings, rounded up to a whole number of pounds.

4 The bar chart shows the outcome of a survey into the number of cars owned in each household in a small town called Statham.



a Copy and complete this table using the graph:

Number of cars	0	1	2	3	4
Number of households	10	52			

b Hence find the mean and standard deviation of the number of cars in a household.

c The survey is also conducted in a nearby town called Mediton where the mean is found to be 2.17 cars per household with standard deviation 1.12. Make two comparisons, in context, between the two towns.

5 The mass of food eaten by 50 dogs during a week was measured to the nearest kg:

Mass (kg)	1 or 2	3 or 4	5 or 6	7 or 8
Frequency	12	20	16	2

a Estimate the mean and standard deviation of the masses.

b State two ways in which the accuracy of these estimates could be improved if the observations were to be repeated.

6 In a sample of 50 boxes of 12 eggs, the number of broken eggs per box is shown.

Number of broken eggs per box	0	1	2	3	4	5	6
Number of boxes	17	8	7	7	6	5	0

a Find the median number of broken eggs per box.

b Calculate the mean number of broken eggs per box.

c Calculate the variance of the data.

d The packaging process for the eggs is changed. In a new sample of 50 boxes the mean number of broken eggs was 1.92 and the variance was 2.84. Give one reason to support someone who argues:

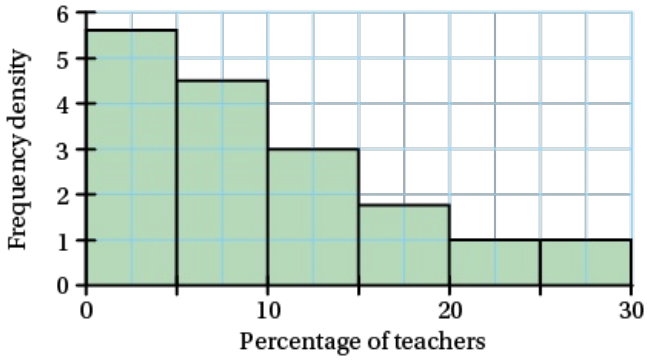
i the new packaging process is better

ii the new packaging process is worse

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iii there is no difference between the two packaging processes.

7 A student is investigating the ways teachers from different schools travel to work. For a large number of schools, he recorded the percentage of teachers who cycle to work. He summarised the information in a histogram.



a Use the graph to calculate estimates of the mean and standard deviation of the percentage of teachers who cycle to work.

b What assumptions have you made in your calculations in part **a**?

c Explain with reference to the graph why the median will be below the mean for this set of data.

8 The standard deviation of this data is $0.8k$ where k is a positive constant.

x	Frequency
1	8
p	2

a Explain why there will be two possible values for p for each value of k.

b Find expressions for p in terms of k.

c Given that the mean is 2.2, find the value of k.

9 The mean of the data in the table is 32 and the variance is 136. Find the possible values of p and q.

x	Frequency
20	12
40	q
p	8

10 Amy and Bob are both playing a game on their computer. Amy's average score on both level one and level two is higher than Bob's. Show that it is possible for Bob to still have a higher overall average across levels one and two.

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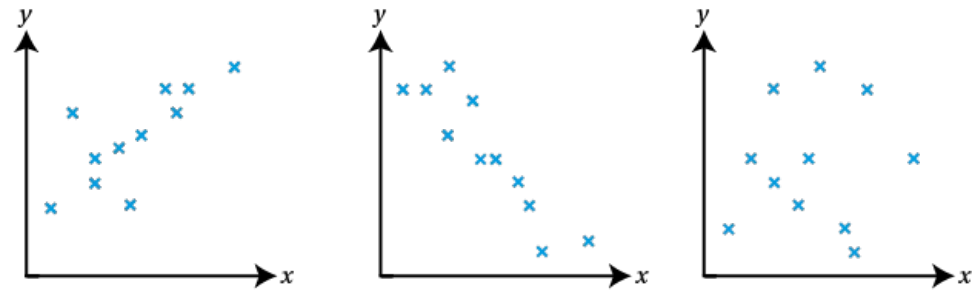
Section 4: Scatter diagrams and correlation

So far, you have only been interested in one variable at a time, such as someone's height or their IQ. But now you would like to see whether there is a relationship between two variables. By gathering two data values from an individual source, say a person's age and mass, you can investigate any potential relationship. Data that comes in pairs in this fashion is said to be **bivariate**.

When you have two sets of data, their relationship might be independent – when you know one variable it gives you no information about the other one. For example, the IQ and the house number of a randomly chosen person. Alternatively, the variables may be in a fixed relationship – when you know one variable you know exactly what the other one will be. For example the length of a side of a randomly created cube and the volume of the cube. However, usually it is somewhere in between – if you know one value you can make a better guess at the value of the other variable, but not be certain. For example, your mark in paper 1 and mark in paper 2 of a Maths exam. The **correlation** of the two variables describes where the relationship lies on this spectrum.

In this course, you shall focus on linear correlation – the extent to which two variables are related by a relationship of the form $y = mx + c$. If the gradient of the linear relationship is positive you describe the correlation as positive, and if the gradient is negative you describe the correlation as negative.

These relationships are often best illustrated using a scatter diagram:



Strong positive correlation: as x increases y generally increases

Strong negative correlation: as x increases y generally decreases

No correlation: no clear relationship between x and y

Rather than simply describing the relationship in words you can find a numerical value to represent the linear correlation. The **correlation coefficient**, r, is a measure of the strength of the relationship between two variables. It can take values between -1 and 1.

You need to know how to interpret the value of r:

Value of r	Interpretation
$r \approx 1$	Strong positive linear correlation
$r \approx 0$	No linear correlation
$r \approx -1$	Strong negative linear correlation

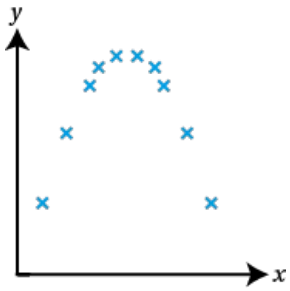
If $r = \pm 1$ it is perfect correlation – the data lies exactly along a straight line.

Explore

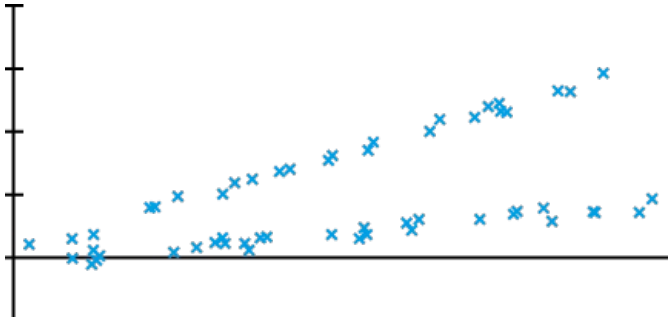
There are many different measures of correlation. This one is often referred to as Pearson's product moment correlation coefficient (or PPMCC). Find out about other measures such as

Spearman's Rank or Kendall's Tau.

Just because $r = 0$ does not mean that there is no relationship between the two variables – it just means that there is no **linear** relationship. This graph shows data which has a correlation coefficient of zero, but there is clearly a relationship:



Scatter diagrams can also reveal if there are two separate groups within the data:



Worksheet

You will find more practice of interpreting scatter graphs in the Large Data Set section on the Cambridge Elevate digital platform.

While the product moment correlation coefficient can give a measure of correlation between two variables, it is important to realise that just because r might be close to ± 1, a change in one variable does **not** necessarily **cause** a change in the other. Such a correlation might be simply coincidence or due to a third, hidden variable.

For example, there might be a strong correlation between ice cream sales and instances of drowning at beaches in a given location. Clearly, eating more ice cream does not cause drowning; instead, the hidden variable of temperature could cause both to rise.

Tip

You may want to remember the phrase 'correlation does not imply causation'.

WORKED EXAMPLE 16.13

Decide which of the following graphs has a correlation coefficient of -0.94. Justify your answer.

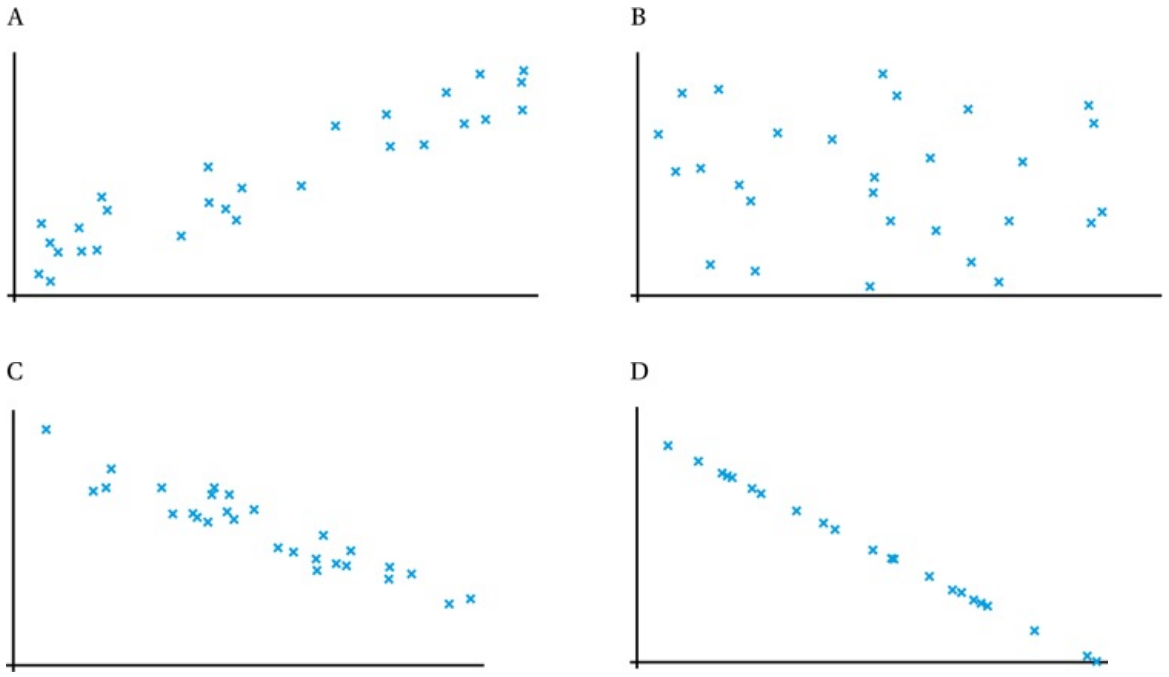


Diagram A has a positive correlation.

Diagram B has no correlation.

Diagram C has strong negative correlation.

Diagram D has perfect negative correlation.

$r = -0.94$ **indicates strong (but not perfect) negative correlation which is in diagram C.**

Once you have established from the correlation coefficient that there is a linear relationship it is often useful to draw in a line of best fit, sometimes called a **regression line**.

When interpreting regression lines you should always consider whether the data actually follows a linear trend – either by looking at the correlation coefficient or the scatter diagram. You should also be aware that if you are using the regression line to predict values outside the range of the observed data – called **extrapolation** – your answer may be less valid.

Explore

Find out about a method for calculating lines of best fit called 'least squares regression'.

Fast forward

A In Student Book 2, Chapter 19, you will see a method for deciding if a calculated correlation coefficient is evidence of a genuine linear relationship between the two variables. Until then, you will not have to deal with correlation coefficients where the interpretation is ambiguous.

EXERCISE 16D

1 For each of the following sets of bivariate data describe the correlation you would expect to see.

a The distance someone lives from school and the time it takes them to travel to school.

b A person's height and their income.

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c The age of a car and the number of miles driven.

d The distance travelled by a car going at constant speed and the time it has travelled.

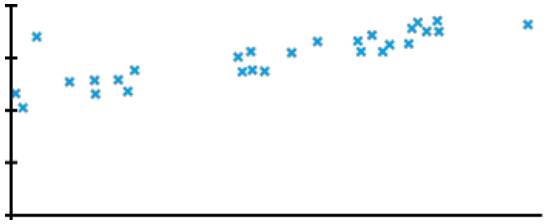
e The age of a school student and the time taken to run $100\,\mathrm{m}$.

f The value of a house and the number of bedrooms.

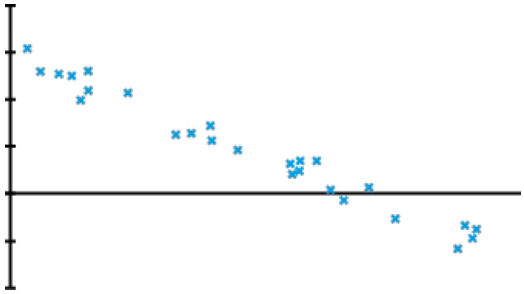
g The average age of adults in a village and the percentage of adults who cycle to work.

2 Describe the correlation shown in each of these scatter diagrams.

a



b



c



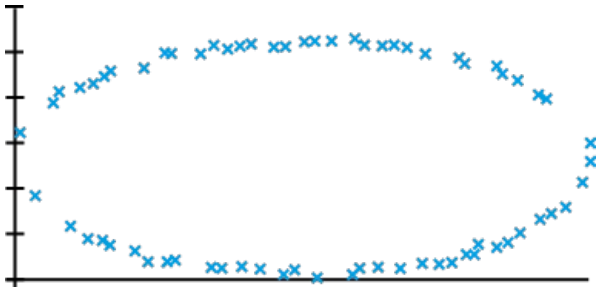
d



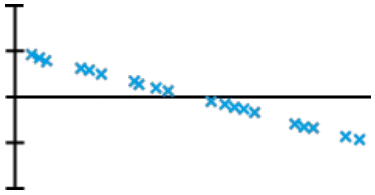
e



f



g



h



3 The correlation coefficient between the speed of a computer processor and its life expectancy is 0.984, based on a sample of 50 processors.

a Interpret this correlation coefficient.

b Does this result imply that processor speed affects the life expectancy? Explain your answer.

4 A road safety group has tested the braking distance of cars of 20 different ages. The correlation coefficient between a car's age and braking distance is 0.92.

a Interpret the correlation coefficient.

b Nicole says that this provides evidence that older cars tend to have longer stopping distances. State with a reason whether you agree with her.

5 The masses of babies (Y kg) at age X months is measured for a sample of 100 babies between 0 and 18 months.

a The correlation coefficient is found to be 0.88. Describe what this suggests about the relationship between mass and age for babies.

b The equation of the line of best fit is found to be $Y = 0.5X + 3.2$. Interpret in context the meaning in this equation of:

i 0.5

ii 3.2

c Explain why this line of best fit would not be an appropriate model to predict the mass of a 14-year-old boy.

6 The number of years since starting primary school (x) and 100 m times (t seconds) of 20 students was measured. The output from a spreadsheet gave the following information:

$t = 24.8 - 0.609x$ $\quad$ $r = -0.24$ $\quad$ $r^2 = 0.0576$

a Interpret in context the values in the output of:

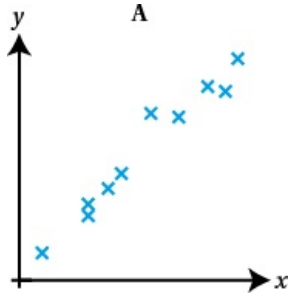
i 24.8

ii 0.609

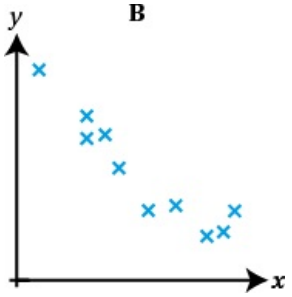
b Give two reasons why it would not be appropriate to use this model to predict the 100 m time of a 60-year-old.

7 Match the scatter diagrams with the following values of r:

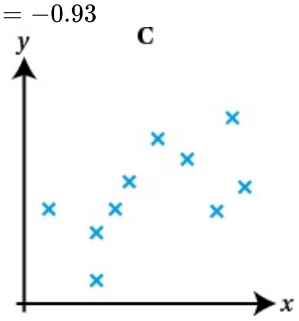
a $r = 0.98$



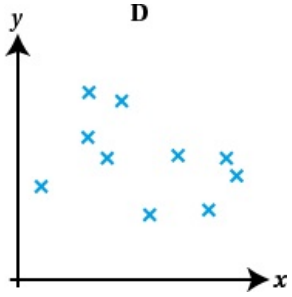
b $r = -0.34$



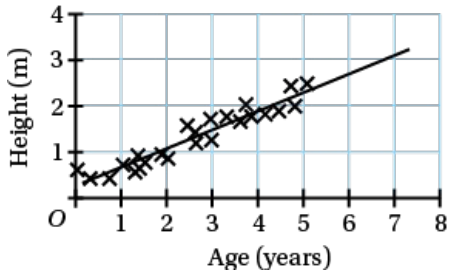
c $r = -0.93$



d $r = 0.58$



8 The heights and ages of 30 trees in a forest were measured and plotted on this scatter diagram.



The correlation coefficient is 0.96.

a Use the line of best fit to estimate the height of a tree that is

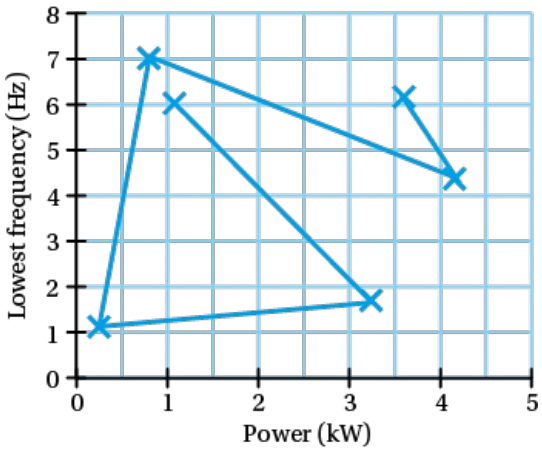
i 3 years old

ii 7 years old.

b Comment on the validity of your answers to part **a**.

9 This graph shows a connected scatter plot of the lowest frequency produced by a sample of 6 speakers against their power.

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a Use the line provided to estimate, where possible, the lowest frequency if the power is:

i $3.5\,\text{kW}$

ii $2\,\text{kW}$

b Explain why it is not appropriate to connect the data using straight lines in this way.

c Describe a situation in which it would be appropriate to connect the points using straight lines in this way.

10 Decide which of the following statements are true for the bivariate data and which are false?

a If $r = 0$ there is no relationship between the two variables.

b If $Y = kX$ then $r = 1$.

c If $r < 0$ then the gradient of the line of best fit is negative.

d As r increases then so does the gradient of the line of best fit.

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Section 5: Outliers and cleaning data

When dealing with real-world data there are sometimes errors, missing data or extreme values that can distort your results. In this section you will look at some standard ways to identify problematic data and how to deal with it.

Often the most useful thing to do is to look at your data graphically. If the underlying pattern is strong, **outliers** can become obvious. For example, on the following scatter diagram, the red point does not seem to follow the trend of the other points.

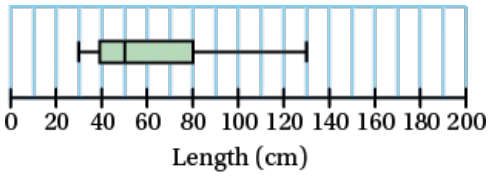


There are also some standard calculations that you can use to check for outliers.

The first is that an outlier is any number more than 1.5 interquartile ranges away from the nearest quartile.

WORKED EXAMPLE 16.14

A group of people were asked to name as many characters from *Harry Potter* as possible in one minute. The results are illustrated in this box-and-whisker plot. Determine if there are any outliers in the data set using the definition that an outlier is any data value lying more than 1.5 interquartile ranges away from the nearest quartile.



Lower quartile: 7 Upper quartile: 13 Interquartile range: 6	Read off the quartiles from the plot to find the IQR.
Upper quartile plus 1.5 IQR $= 13 + 1.5 \times 6 = 22$ Lower quartile minus 1.5 IQR $= 7 - 1.5 \times 6 = -2$	Find the largest and smallest values that are not outliers.
Smallest value is 1 which is not an outlier. Largest value is 24 which is an outlier.	Look at largest and smallest values.
Hence there is at least one outlier in the data set.	The box plot does not show the exact distribution of the data values along the right-hand ‘whisker’, so it’s impossible to tell whether there are any other outliers.

Another possible method is to classify anything more than two standard deviations from the mean as an outlier.

WORKED EXAMPLE 16.15

The wages of workers in a factory are (in thousands of pounds):

x	10	16	20	145
f	18	14	4	1

Use the definition of an outlier as anything more than two standard deviations from the mean to determine if the wage of £$145\,000$ is an outlier.

Use your calculator:

Use your calculator to find the mean and standard deviation.

$\bar{x} = 17$

$\sigma = 21.633\ldots$

Two standard deviations from the mean is $17 + 2 \times 21.633\ldots = 60.266\ldots$

Use the full accuracy from your calculator rather than the quoted rounded value.

$145 > 60.266\ldots$ so $£145000$ is an outlier.

Once you have found that something is an outlier you must then decide whether or not to include it in your calculation. This often requires you to look at the data in context.

If the outlier is clearly an error (for example, the wrong units being used or an impossible value) then it should be excluded from the data.

If there are several outliers it might be a distinctly different group which should be analysed separately.

Worksheet

You will find practice of how to deal with missing data in a real data set in the Large Data Set section on the Cambridge Elevate digital platform.

Otherwise, it might simply be that there is an unusual value in your data. This does not mean that it is an error. Unless you have a good reason to exclude it you should keep it in your analysis but report the presence of outliers.

When using large data sets, you will often find that some of the data are missing. You should consider how the missing values may affect the results of your calculations.

Tip

You may be asked to add to diagrams in order to interpret data.

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Did you know?

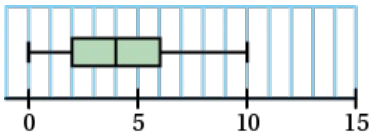


There is a famous story (probably not entirely accurate) that NASA satellites first 'discovered' a hole in the ozone layer in the 1970s but an automatic error checker decided it was an anomalous reading so it was ignored until nearly ten years later. Sometimes outliers are potentially the most interesting part of the data.

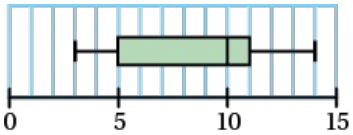
EXERCISE 16E

1 Determine, using the definition of an outlier as more than 1.5 IQR from the nearest quartile, if there are outliers in the following sets of data.

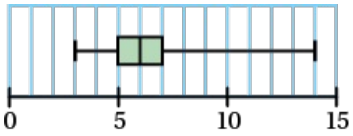
a **i**



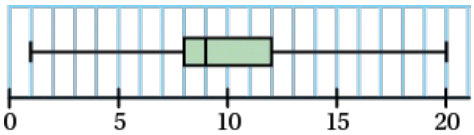
ii



b **i**



ii



2 Determine if the following data contain outliers, defined as data more than 2 standard deviations from the mean.

a **i** $1,\ 5,\ 5,\ 7,\ 14$

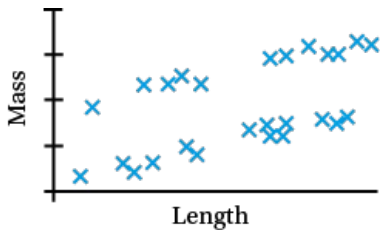
ii $80,\ 85,\ 90,\ 90,\ 125$

b **i** $-3,\ 8,\ 8,\ 10,\ 14,\ 15,\ 16$

ii $0,\ 26,\ 26,\ 28,\ 29,\ 30,\ 64$

3 A biologist collects mass and length information on a sample of ants from a forest. He finds the correlation coefficient is 0.22, so concludes that there is not a strong linear relationship between mass and length. A colleague suggests that he should draw a scatter diagram to illustrate the data, which is shown here.

What statistical advice would you give the biologist?

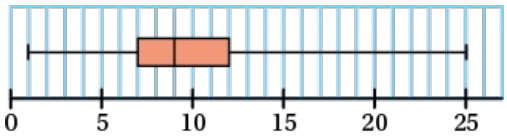


4 30 people attempted to complete a level on a computer game. This box-and-whisker plot shows the number of attempts required.

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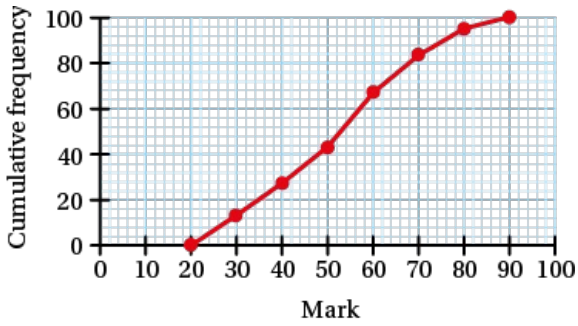
a Find the interquartile range for this set of data.

b If outliers are defined as more than 1.5 IQR from the nearest quartile, then show that there must be an outlier in the data.



5 The following cumulative frequency graph shows the marks of 100 students in a test.

Use the definition that an outlier is more than 1.5 IQR from the nearest quartile to show that there are no outliers in this data.



6 A survey is conducted in a large school to find out the mean height of all students in years 7 to 13. In each situation described below the data set contains some missing data. Explain whether the calculated mean is likely to be an underestimate, an overestimate or close to the actual mean.

a The basketball team were away.

b 5% of the whole school were ill.

c All of Year 7 were out on a Geography trip.

7 All 560 Year 12 students at a college did a Maths test, on which the maximum possible mark was 70. Unfortunately, one teacher forgot to record the marks for his class of 18 students. The mean mark for the remaining 542 students is 48. Find the minimum and maximum possible mean mark for all 560 students.

8 This scatter diagram shows the number of matches on a tennis court in a week (N>) against the average temperature in that week (T).

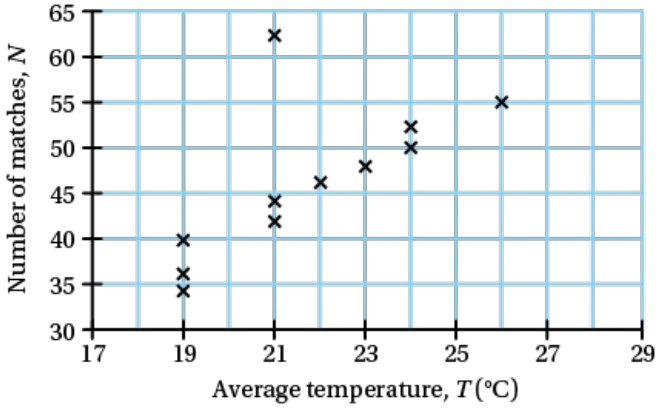
The mean number of matches per week is 46.4 with standard deviation 7.80.

a Use of the tennis court is known to increase noticeably during the first week of the Wimbledon tennis tournament. What was the average temperature during that week?

b The data point corresponding to the first week of Wimbledon is removed. Without further calculation, determine:

i Will the mean number of matches per week increase or decrease?

ii Will the standard deviation in the number of matches increase or decrease?



9 A runner uses a smartwatch to track the time taken (t in minutes) and distance covered (d in km)

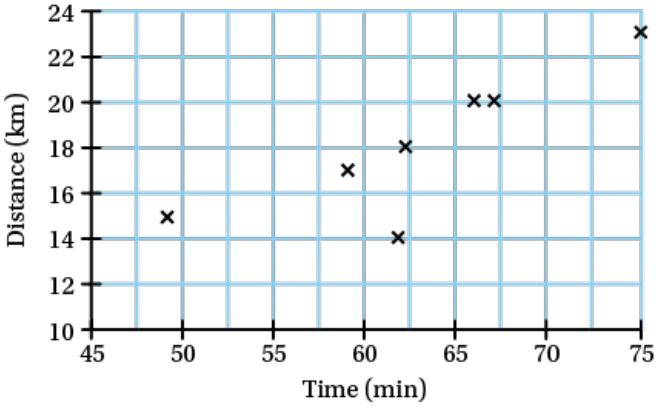
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on her run each day for a seven day week.

a The times taken are summarised by $\Sigma t = 440$, $\Sigma t^2 = 28040$. Find the mean and standard deviation of the times taken.

b The data is tabulated and illustrated in the scatter diagram as follows:

	Time (min)	Distance (km)
Monday	62	18
Tuesday	49	15
Wednesday	75	23
Thursday	62	14
Friday	66	20
Saturday	59	17
Sunday	67	20



Six of the days were sunny but on one day it was raining and windy. Which day was this?

c Remove the day with bad weather and recalculate the mean and standard deviation of the times.

d The equation of the regression line for the sunny days is $d = 0.314t - 0.957$. Use this equation to estimate the speed of the runner.

e Would it be appropriate to use this regression line to estimate the time the runner would take to complete a 40 km race? Justify your answer.

10 **a** A doctor measures the level of infection markers in 6 samples of blood. The mean of the values is 18 with a variance of $\frac{631}{3}$. The largest value is 50. Show that this is an outlier, using the definition that outliers are more than 2 standard deviations away from the mean.

b Find the mean and standard deviation if the value of 50 is removed.

11 **a** Find the maximum and minimum value of $8x^2 - 576x + 11826$ for $0 \leqslant x \leqslant 50$.

b A group of nine students took a test, marked out of 50. The mark for one of the students did not get recorded. For the remaining eight students, the mean mark was 36 and the standard deviation was 4.5. Find the smallest and largest possible standard deviation for all nine students.

12 In this question, define an outlier as being more than two standard deviations from the mean.

a The standard deviation of a set of data is 9 with range 50. Prove that there must be an outlier in the data.

b Consider the following data set:

x	Frequency

0	p
30	q

Show that 30 is an outlier if $p > 4q$.

c Hence show that if a data set has a standard deviation of 9 with range 30 it is possible for there to be an outlier. (Hint: try $q = 3$.)

d Show using a counter example that if a data set has standard deviation 9 and range 30 it is not certain that there is an outlier. (Hint: Consider a data set with three values: 0, 15 and 30.)

Checklist of learning and understanding

- **Histograms** are a useful visual summary of data, giving an immediate impression of centre and spread.
- In a histogram, the area represents frequency. The vertical axis represents **frequency density**.
- **Cumulative frequency diagrams** are useful for finding the median and the interquartile range and also facilitate the construction of a **box-and-whisker plot**, another good visual summary of data.
- A **stem-and-leaf diagram** shows the distribution of the data while displaying all the values.
- The centre of the data can be measured using the **mean**, **median** or **mode**. The spread around the centre can be measured using the **range**, **interquartile range** or **standard deviation**.
- The square of the standard deviation is called the **variance**.
- If the data has been grouped you can estimate the mean and standard deviation by assuming that every data value is at the centre of each class.
- The **correlation coefficient** is a value between –1 and 1 which measures the strength of linear relationship between two variables.
- The **regression line** can be drawn on a scatter diagram for a set of **bivariate data**. It should only be used if there is significant linear correlation and there is not too much extrapolation.
- Graphs or calculations can be used to identify **outliers** in data, which then may need to be removed.

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Mixed practice 16

1 A sample of discrete data is drawn from a population and given as $115,\ 108,\ 135,\ 122,\ 127,\ 140,\ 139,\ 111,\ 124$.

Find:

a the interquartile range

b the mean

c the variance.

2 $\Sigma(x-\bar{x})^2 = 12$ and $n = 3$. Find the value of the standard deviation.

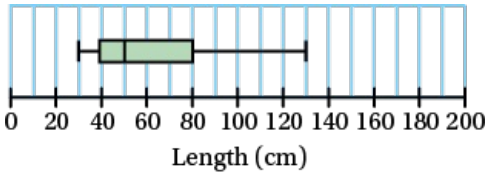
3 A student takes the bus to school every morning. She records the length of the time, in minutes, she waits for the bus on 12 randomly chosen days. The data set is summarised by $\Sigma x = 49$ and $\Sigma x^2 = 305.7$.

a Find the mean.

b Find the variance.

4 This box-and-whisker plot shows the lengths of corn snakes (x) in cm.

Use the definition that an outlier is more than 1.5 IQR from the closest quartile to find the range of values that would be outliers. Hence show that there are no outliers for this data.



5 **a** Use a counter example with two data items to show that $\overline{x^2} = \bar{x}^2$ is not always true.

b If $\overline{x^2} = \bar{x}^2$ find the standard deviation of the data. Hence provide an example of a set of data containing two items which has $\overline{x^2} = \bar{x}^2$.

6 40 people were asked to guess the length of a certain road. Each person gave their guess, l km, correct to the nearest kilometre. The results are summarised as follows:

l	$10-12$	$13-15$	$16-20$	$21-30$
Frequency	1	13	20	6

a **i** Use appropriate formulae to calculate estimates of the mean and standard deviation of l.

ii Explain why your answers are only estimates.

b A histogram is to be drawn to illustrate the data. Calculate the frequency density of the block for the $16-20$ class.

c Explain which class contains the median value of l.

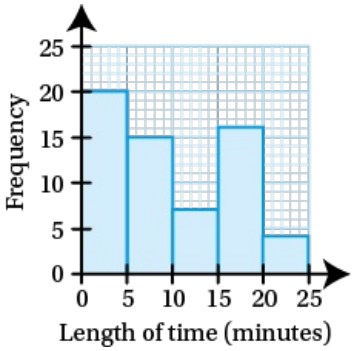
d Later, the person whose guess was between 10 km and 12 km changed his guess to between 13 km and 15 km. Without calculation, state whether the following will increase, decrease or remain the same:

i the mean of l

ii the standard deviation of l

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[Question part reference style adapted]

7 **a** From this chart calculate an estimate of the mean and the standard deviation of the data.



b The chart shows the distribution of the length of time, in minutes, Tom has to wait for a bus in the morning. Find the probability that, on a randomly selected day, he has to wait between 15 and 25 minutes.

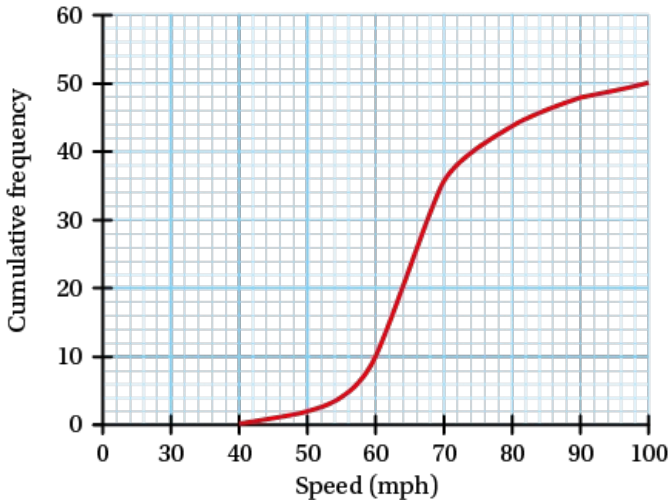
8 Jenny must sit 4 papers for an exam. All papers have an equal weight when their marks are combined. The mean of the first 3 papers Jenny has sat is 72% with a standard deviation of 8%.

a If she wants to get a mean of 75% overall what is the lowest percentage she can get in her fourth paper?

b What is the highest possible mean she can get?

c If she does get the highest possible mean, what is her new standard deviation?

9 This cumulative frequency diagram gives the speed, v, of 50 cars in mph as they travel past a motorway checkpoint.



a From the diagram find the median speed.

b Any car travelling above 75 mph will be stopped by the police. How many of these cars will be stopped?

c The middle 50% of speeds lie between a and b where $a < b$. Find the values of a and b.

d Copy and complete the following frequency table.

v	Frequency
$40 < v \leqslant 50$	2
$50 < v \leqslant 60$	8
$60 < v \leqslant 70$	
$70 < v \leqslant 80$	
$80 < v \leqslant 90$	

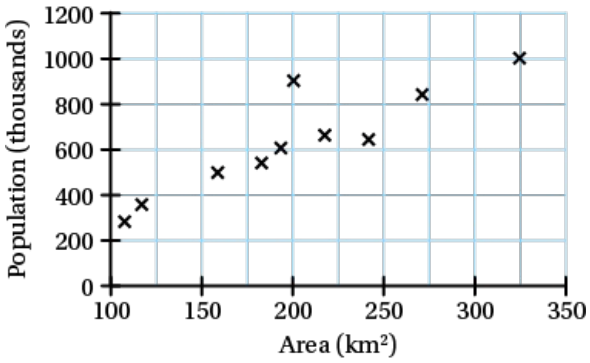
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$90 < v \leqslant 100$	

e Hence estimate the mean and the standard deviation in the speeds.

f Use the definition of an outlier as anything more than 2 standard deviations from the mean to show that some of the observed speeds are outliers. Decide, with justification, whether these speeds need to be removed for a valid analysis.

10 A geographer is studying data on the area (A) and population (P) of various cities in a country. He displays his data on a scatter diagram.



The mean population of the cities studied is 604 thousand.

a What is the advantage of displaying the data on a scatter diagram rather than two histograms?

b Describe the correlation of the data shown by the scatter graph.

c One of the cities completely fills an island so that as its population has grown it has not been able to expand. What is the area of the island?

d In the rest of the analysis, the city on the island is removed from the data. What effect does this have on the mean population of the cities studied? Explain your answer.

e The regression line of the remaining cities has equation $P = 3.05A - 3.26$.

Interpret the value 3.05 in the context of the cities.

f The capital of the country has an area of $1600\ \text{km}^2$. Why would it not be valid to use the regression line to predict the population of the capital?

11 The test marks of 14 students are displayed in a stem-and-leaf diagram, as shown.

0	
1	2 6
2	1 3 5
3	w x 4 8 y z
4	6 7 7

Key: $1|6$ means 16 marks

a Find the lower quartile.

b Given that the median is 32, find the values of w and x.

c Find the possible values for the upper quartile.

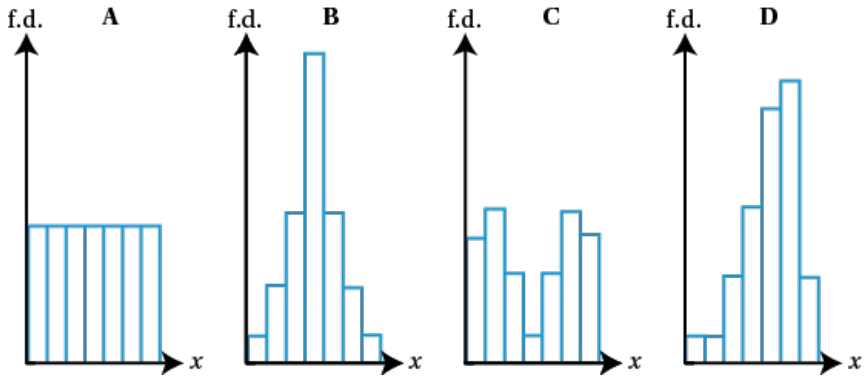
d State one advantage of a stem-and-leaf diagram over a box-and-whisker plot.

e State one advantage of a box-and-whisker plot over a stem-and-leaf diagram.

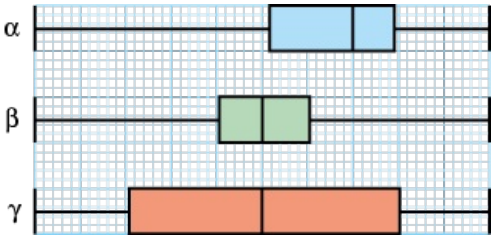
OCR, GCE Mathematics, Paper 4732, June 2012
[Question part reference style adapted]

12 The four populations A, B, C and D are the same size and have the same range.

Histograms for the four populations are shown.



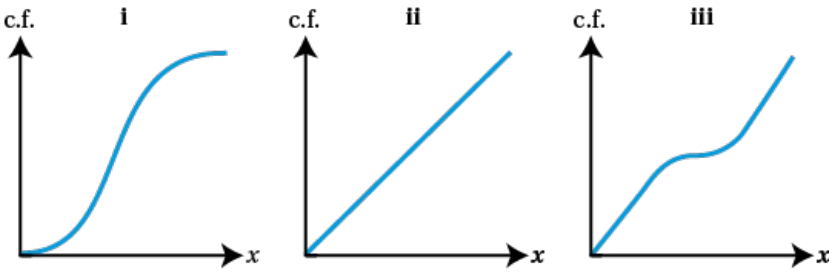
a Each of these three box-and-whisker plots corresponds to one of the four populations. Write the letter of the correct population for each of α, β and γ.



Tip

γ is a Greek letter pronounced gamma. This is the lower case form of the letter.

b Each of these three cumulative frequency diagrams corresponds to one of the four populations. Write the letter of the correct population for each of **i**, **ii** and **iii.**



13 The mean of a set of 10 data items is 115 and the variance is 154. Another piece of data is discovered and the new mean is 114. What is the new variance?

14 If the sum of 20 pieces of data is 1542, find the smallest possible value of Σx^2.

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17 Probability

In this chapter you will learn how to:

- work out combined probabilities when you are interested in more than one outcome
- work out the probability of a sequence of events occurring
- construct and use a table showing probabilities of all possible outcomes in a given situation (probability distributions)
- calculate probabilities in a situation when an experiment is repeated several times (binomial distribution).

Before you start...

GCSE	You should be able to find probabilities by listing all possible outcomes (sample space) of a single event or a combination of two events.	1 One spinner has the numbers 1 to 5 written on it, and another has the letters A to D. What is the probability of getting an A and a 3 when the two spinners are spun?
GCSE	You should know how to use tree diagrams to determine probabilities of successive events, and to calculate probabilities of combined events.	2 A bag contains 7 red and 3 yellow sweets. A sweet is taken out of the bag and eaten. This is done three times. Find the probability that 3 red sweets are picked.
Chapter 9	You should know how to calculate factorials and binomial coefficients.	3 Use your calculator to find the following: a $7!$ b $\frac{7!}{3!}$ c ${}^{7}C_{3}$ d $\binom{5}{5}$ e ${}^{10}C_{0}$

Why study probability?

Probability is the study of events that depend on chance. Knowing how likely a certain outcome is, even if you cannot predict it with certainty, is important in estimating the risk of events such as earthquakes and disease outbreaks.

You already know how to calculate probabilities of simple events, such as rolling a 6 on a dice or picking an ace from a pack of cards. In this chapter, you will review the concepts of independent and mutually exclusive events, and use them to calculate more complicated probabilities; for example, several events happening at the same time or one after the other.

Fast forward

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Probability theory also underpins the statistical tests you will meet in Chapter 18.

It is often useful to have a list of all possible outcomes in a given situation together with the probability of each outcome. This is called a **probability distribution**. A particularly important distribution, called the **binomial distribution**, can be used to model the number of successful outcomes in a series of repeated experiments. It has applications from medical trials to predicting election results from exit polls.

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Section 1: Combining probabilities

You are often interested in probabilities of more than one outcome. For example, your university offer may require you to get an A or a B in Mathematics. Suppose you are told that last year 33% of all candidates achieved an A and 18% achieved a B in Mathematics. You can then work out that the probability of getting an A or a B is $0.33 + 0.18 = 0.51$, or 51%. You can write this as $\mathrm{P}(A \text{ or } B) = 0.51$.

But what if, instead, your offer asks for an A in Mathematics **or** Economics? Last year 20% of all candidates got an A in Economics; so is the probability of getting an A in at least one of the subjects $0.33 + 0.20 = 0.53$? The answer is no, because those who got an A in Mathematics and those who got an A in Economics are not two separate groups of people – there are those who got an A in both. In fact, unless you know how many got two As, it is impossible to find the probability of getting this combination of grades.

The events 'getting an A in Mathematics' and 'getting a B in Mathematics' are **mutually exclusive**. This means that they cannot both happen at the same time; the probability of both happening together is zero. The events from the second example, 'getting an A in Mathematics' and 'getting an A in Economics', are not mutually exclusive because it is possible for both of them to happen at the same time.

Gateway to A Level

For a reminder of and practice with basic probability calculations, see Gateway to A Level section V.

Key point 17.1

Events A and B are mutually exclusive if it is impossible for both of them to happen at the same time:

$$\mathrm{P}(A \text{ and } B) = 0$$

If events are mutually exclusive, their probabilities can be added:

$$\mathrm{P}(A \text{ or } B) = \mathrm{P}(A) + \mathrm{P}(B)$$

WORKED EXAMPLE 17.1

A fair six-sided dice is rolled once. In each case, state whether the two events are mutually exclusive, and write down $\mathrm{P}(A)$, $\mathrm{P}(B)$ and $\mathrm{P}(A \text{ or } B)$.

a A: rolling a 5; B: rolling a 6.

b A: rolling an even number; B: rolling a prime number.

a A and B are mutually exclusive.

$\mathrm{P}(A) = \frac{1}{6}$, $\mathrm{P}(B) = \frac{1}{6}$

It is not possible to roll a 5 and a 6 at the same time.

$\mathrm{P}(A \text{ or } B) = \frac{2}{6}$

The probabilities can be added.

b A and B are not mutually exclusive.

It is possible to get a number that is both even and prime (2).

$\mathrm{P}(A) = \frac{3}{6}$, $\mathrm{P}(B) = \frac{3}{6}$

There are three even numbers $(2, 4, 6)$ and three prime number $(2, 3, 5)$.

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$\mathrm{P}(A \text{ or } B) = \frac{5}{6}$

The probabilities cannot be added; you have to count all possible outcomes $(2, 3, 4, 5, 6)$.

When two events are not mutually exclusive, there is a possibility that they can both happen at the same time. Can the probability of both events happening together be worked out from their individual probabilities?

Consider again the example of Mathematics and Economics grades. Let M be the event 'getting an A in Mathematics' and E the event 'getting an A in Economics'. As previously, suppose that $\mathrm{P}(M) = 0.33$ and $\mathrm{P}(E) = 0.20$. What is $\mathrm{P}(M \text{ and } E)$?

Fast forward

You may remember the formula $\mathrm{P}(A \text{ or } B) = \mathrm{P}(A) + (B) - \mathrm{P}(A \text{ and } B)$ from the GCSE course. You will meet it again if you study Venn diagrams in Student Book 2.

Here are two possible situations that fit in with these numbers. The numbers in cells show percentages. For example, in the first table, the percentage of candidates with an A in Mathematics is $15 + 18 = 33$: 15% also got an A in Economics and the other 18% did not.

		Economics grade		Total
		A	not A	
Maths grade	A	15	18	33
	not A	5	62	67
Total		20	80	100

		Economics grade		Total
		A	not A	
Maths grade	A	6.6	26.4	33
	not A	13.4	53.6	67
Total		20	80	100

Both examples have $\mathrm{P}(M) = 0.33$ and $\mathrm{P}(E) = 0.20$. But in the first situation $\mathrm{P}(M \text{ and } E) = 0.15$ and in the second $\mathrm{P}(M \text{ and } E) = 0.066$. This suggests that P($M$ and E) depends on more than just the individual probabilities of M and E. In fact, it depends on how the two events influence each other – how the probability of one event changes when you know the outcome of the other. This is called **conditional probability**.

Fast forward

You may have met conditional probability at GCSE, and you will meet it again in Student Book 2 if you study the full A level course.

There is one important special case when the probability of two events happening together can be easily calculated: when the two events do not affect each other. You say that the events are **independent**.

Explore

Explore applications of independence of random events. This is an important concept in both legal and medical trials

Key point 17.2

Events A and B are independent if knowing the outcome of A does not affect the probability of B.

For independent events:

$$\mathrm{P}(A \text{ and } B) = \mathrm{P}(A) \times \mathrm{P}(B)$$

If A and B are independent then A' and B' are also independent, as are A and B', and B and A'.

WORKED EXAMPLE 17.2

Two dice are rolled simultaneously. What is the probability that both dice show a prime number?

Let the two events be:

$A =$ 'The first dice shows a prime number'

$B =$ 'The second dice shows a prime number'

Then $\mathrm{P}(A) = \frac{3}{6}$, $\mathrm{P}(B) = \frac{3}{6}$

There are three prime numbers on a dice: $2, 3, 5$.

A and B are independent, so

$$\begin{aligned}\mathrm{P}(A \text{ and } B) &= \mathrm{P}(A) \times \mathrm{P}(B) \\ &= \frac{3}{6} \times \frac{3}{6} \\ &= \frac{1}{4}\end{aligned}$$

Knowing the result for the first dice does not affect probabilities for the second dice.

Tip

You can also answer the question in Worked example 17.2 by using a sample space table showing all possible outcomes.

	1	2	3	4	5	6
1		\	\		\	
2	/	X	X	/	X	/
3	/	X	X	/	X	/
4		\	\		\	
5	/	X	X	/	X	/
6		\	\		\	

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Each cell in the 6×6 table corresponds to a possible outcome of a roll of two dice. The outcomes where the first dice shows a prime number are marked /, and the outcomes where the second dice shows a prime number are marked \. This means that the outcomes where both numbers are prime are marked X.

You can see that 9 out of 36 possible outcomes have both dice showing prime numbers, so the required probability is $\frac{9}{36} = \frac{1}{4}$.

WORKED EXAMPLE 17.3

A biased coin has the probability $\frac{2}{3}$ of showing heads. The coin is tossed three times. Find the probability of getting either three heads or three tails.

$$\begin{aligned} \mathrm{P}(3\,\text{heads}) &= \mathrm{P}(\text{heads}) \times \mathrm{P}(\text{heads}) \times \mathrm{P}(\text{heads}) \\ &= \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \\ &= \frac{8}{27} \end{aligned}$$

The three tosses are independent, so multiply the probabilities for each one.

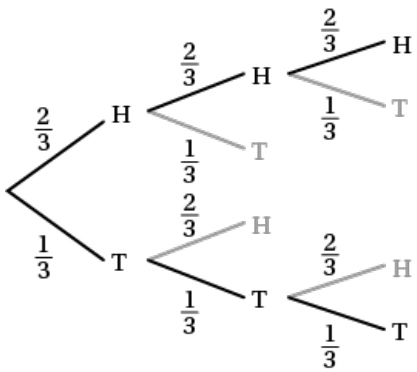
$$\begin{aligned} \mathrm{P}(3\,\text{tails}) &= \mathrm{P}(\text{tails}) \times \mathrm{P}(\text{tails}) \times \mathrm{P}(\text{tails}) \\ &= \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \\ &= \frac{1}{27} \end{aligned}$$

$$\mathrm{P}(3\,\text{heads or}\,3\,\text{tails}) = \frac{8}{27} + \frac{1}{27} = \frac{1}{3}$$

The events ‘getting three heads’ and ‘getting three tails’ are mutually exclusive, so find the probability of each and add them together.

Tip

In examples like Worked example 17.3 you may find it helpful to think of a tree diagram (which you may have met in your previous study, and will cover in more detail in Student Book 2).



One important example of mutually exclusive events is an event and its **complement**: in statistics, ‘complement’ means the event not happening. Since either an event or its complement must occur, the total probability is 1.

Gateway to A Level

For a reminder of tree diagrams for working with probabilities, see Gateway to A Level section W.

Key point 17.3

The complement of an event A is the event "not A" or A': $\mathrm{P}(A) + \mathrm{P}(A') = 1$.

The equation in Key point 17.3 is very useful because sometimes the complement is much simpler to evaluate than the event itself.

WORKED EXAMPLE 17.4

A fair dice is rolled five times. Find the probability of getting at least one six.

$\mathrm{P}(\text{at least one six}) = 1 - \mathrm{P}(\text{no sixes})$

There are several possible ways to get at least one six (you could get 1, 2, 3, 4 or 5).

But the complement of the event 'at least one six' is 'no sixes', which can happen in only one way: if each dice shows 'not a six'.

$= 1 - \left(\frac{5}{6}\right)^5$

$= 0.598\ (3\text{ s.f.})$

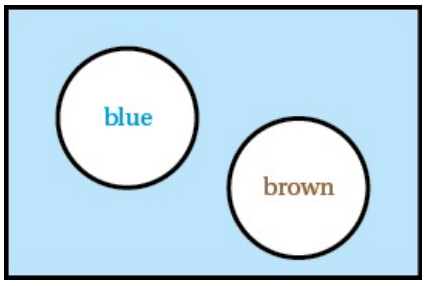
The five rolls are independent, so you need to multiply five probabilities of 'not a six'.

Tip

The complement of an event can be represented on a Venn diagram, which you may have met in your previous studies (and will cover in more detail in Student Book 2). For example, if it is known that 40% of people have blue eyes and 25% of people have brown eyes, then the event 'a person has neither blue nor brown eyes' is the complement of the event 'a person has blue eyes or brown eyes', and is represented by the shaded region on this Venn diagram:

The probability of this event is:

P(a person has neither blue eyes nor brown eyes) $= 1 - 0.4 - 0.25 = 0.35$



EXERCISE 17A

1 Which of these events are mutually exclusive? For those events that are mutually exclusive, find $\mathrm{P}(A \text{ or } B)$.

a On a fair six-sided dice:

i A: rolling a multiple of 3; B: rolling a multiple of 4.

ii A: rolling an even number; B: rolling a multiple of 5.

b One card is selected from a standard pack of 52 cards.

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i A: selecting a king; B: selecting a red card.

ii A: selecting an ace; B: selecting a spade.

c Two fair dice are rolled and the scores are added.

i A: the total is a multiple of 6; B: the total is less than 5.

ii A: the total is greater than 7; B: the total is less than 9.

d A bag contains four green and six yellow balls. A ball is taken out of the bag, its colour noted, and then returned to the bag. Another ball is then selected.

i A: both balls are green; B: both balls are yellow.

ii A: the first ball is green; B: the second ball is green.

e A bag contains four green and six yellow balls. Two balls are taken out without replacement.

i B: the first ball is green; B: the second ball is green.

ii A: both balls are green; B: both balls are yellow.

2 Which pairs of events from question 1 are independent? For those that are, calculate $\mathrm{P}(A \text{ and } B)$ and $\mathrm{P}(A) \times \mathrm{P}(B)$.

3 Two events, A and B, have probabilities $\mathrm{P}(A) = p$ and $\mathrm{P}(B) = q$.

a Write down an expression for P(A and B) in the following situations:

i A and B are independent

ii A and B are mutually exclusive.

b If two events are independent, can they also be mutually exclusive?

4 A coin is biased so that the probability of getting tails is $\frac{3}{4}$. The coin is tossed twice. Find the probability that:

a the coin shows heads both times

b the coin shows heads at least once.

5 Two fair six-sided dice are rolled. Find the probability that the product of the scores is 6.

6 Daniel has three blocks with letters C, A and T written on them. He arranges the blocks in a row randomly.

a Write down all possible arrangements of the three letters.

b Find the probability that the blocks make the word ‘CAT’ or ‘ACT’.

7 A fair six-sided dice is rolled once. Define events as:

A: the dice shows an even number; B: the dice shows a prime number.

a Find P(A and B).

b Determine whether events A and B are independent.

8 300 students in Years 9, 10 and 11 at a school were asked to say which of Biology, Chemistry and Physics is their favourite science. The results are shown in this table.

Year group	Biology	Chemistry	Physics	Total
Year 9	41	29	27	97
Year 10	35	36	34	105

Year 11	37	30	31	98
Total	113	95	92	300

a Find the probability that a randomly chosen student:

i prefers Chemistry

ii is in Year 11 and doesn't prefer Biology.

b Determine whether the event 'the student is in Year 9' and the event 'the student's favourite science is Physics' are independent.

9 A fair four-sided spinner, with numbers 1 to 4 written on it, is spun three times. Find the probability of getting either three 1s or three 4s.

10 A fair coin is tossed three times. Determine the probability that it shows:

a three tails

b at least one head.

11 The probability that a student is late for a lesson is 0.15, independently of any other students.

a Find the probability that at least one of the 12 students in a class is late.

b Is the assumption of independence reasonable in this case? Explain your answer.

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Section 2: Probability distributions

So far you have only been asked to calculate the probability of a specific event, or a combination of events happening. But sometimes you are interested in probabilities of all possible outcomes in a given situation. For example, if you roll two dice and add up the scores you can get 11 possible outcomes: any integer between 2 and 12. The probabilities of those outcomes are not all equal; a total of 7 is more likely than a total of 12.

The list of all possible outcomes together with their probabilities is called a probability distribution. This information is best displayed in a table. In the AS course you only work with **discrete** probability distributions – ones where all possible outcomes can be listed (though the list may be infinite!).

Fast forward

A If you study the A level course you will also meet continuous distributions, which can take any value in a given range, in Student Book 2.

WORKED EXAMPLE 17.5

Two fair dice are rolled and their scores are added. Find the probability distribution of the total.

Possible outcomes for the total:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The numbers in the table show the total score of the two dice.

The probability distribution of the total:

Total	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Each combination of scores is equally likely, so each has a probability of $\frac{1}{36}$.

So, for example, $\mathrm{P}(\text{total} = 5) = \frac{4}{36}$, because the total of 5 can be obtained in 4 different ways.

The table in Worked example 17.5 is an example of a **sample space table**, which is a way of listing all possible outcomes of an event.

The probabilities in a probability distribution cannot be just any numbers.

Focus on ...

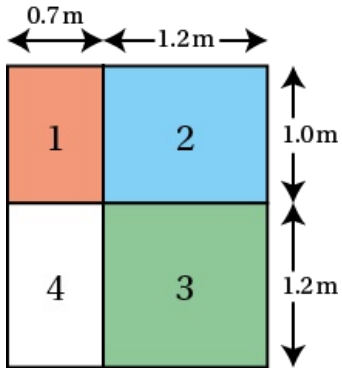
Probability distributions of combined events can be difficult to find theoretically. Focus on ... Modelling 3 shows you how to use computer simulation to find probability distributions.

Key point 17.4

The total of all the probabilities of a probability distribution must always equal 1.

WORKED EXAMPLE 17.6

In a game at a fair, a ball is thrown at a rectangular target. The dimensions of the target (in metres) are as shown in the diagram. The probability of hitting each region is proportional to its area. The prize for hitting a region is the number of chocolates equal to the number shown in that region. Find the probability distribution of the number of chocolates won.



Let x = the number of chocolates won

x	1	2	3	4
$\mathrm{P}(X = x)$	$0.7k$	$1.2k$	$1.44k$	$0.84k$

The probability is proportional to the area so write $p = k \times \text{area}$.

To find k, use the fact that the probabilities add up to 1.

$0.7k + 1.2k + 1.44k + 0.84k = 1$
$4.18k = 1$

$k = 0.239 (3 \text{ s. f.})$

So the distribution is:

x	1	2	3	4
$\mathrm{P}(X = x)$	0.167	0.287	0.344	0.201

Sometimes a probability distribution can be given by a formula.

Sometimes the notation used here can be confusing. X is the name of the **random variable** (a variable whose value depends on chance); x is the **value** it takes. For example, $\mathrm{P}(X = 2) = 0.287$.

WORKED EXAMPLE 17.7

A probability distribution is given by $\mathrm{P}(X = x) = kx$ for $x = 1, 2, 3$. Find $\mathrm{P}(X = 2)$.

$\mathrm{P}(X = 2) = 2k$

Replace x by the particular value of X.

To find k:
$k + 2k + 3k = 1$

Use the fact that all probabilities add up to 1.

$\Rightarrow k = \frac{1}{6}$

You may find it helpful to show the probability distribution in a table:

x	1	2	3
$\mathrm{P}(X = x)$	k	$2k$	$3k$

So $\mathrm{P}(X = 2) = 2 \times \frac{1}{6} = \frac{1}{3}$

EXERCISE 17B

1 For each of the following, draw a table to represent the probability distribution.

a A fair coin is thrown three times. T is the number of tails obtained.

b Two fair dice are thrown. D is the difference between the larger and the smaller score, or zero if they are the same.

c A fair dice is thrown once. X is calculated as half the result if the dice shows an even number, or one higher than the result if the dice shows an odd number.

d A bag contains six red and three green counters. Two counters are drawn at random from the bag without replacement. G is the number of green counters remaining in the bag.

e Karl picks a card at random from a standard pack of 52 cards. If he draws a diamond, he stops; otherwise, he replaces the card and continues to draw cards at random, with replacement, until he has either drawn a diamond or has drawn a total of 4 cards. C is the total number of cards drawn.

f Two fair four-sided spinners, each labelled 1, 2, 3 and 4, are spun. X is the product of the two values shown.

2 Find the missing value k for each probability distribution.

a i

x	3	7	9	11
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	k

ii

x	1	2	3	4
$P(X = x)$	$2k$	$3k$	$2k$	k

b i

x	1	2	3	4
$P(X = x)$	$2k$	$3k$	$2k$	k

ii

x	1	2	3	4
$P(X = x)$	$\frac{k}{2}$	k	$\frac{k}{3}$	k

c i $\mathrm{P}(Y = y) = ky$ for $y = 1, 2, 3, 4$

ii $\mathrm{P}(X = x) = \dfrac{k}{x}$ for $x = 1,\ 2,\ 3,\ 4$

3 A six-sided dice is biased with the probabilities of each outcome as shown in the table:

Score	1	2	3	4	5	6
Probability	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{4}$	$2k$	$3k$

a Find the value of k.

b The dice is rolled once. Find the probability that the score is more than 4.

4 The number of students absent from a Mathematics lesson on any particular day follows the probability distribution:

Number of absentees	0	1	2	3	4	5
Probability	0.15	0.21	k	0.26	0.12	0.07

a Find the value of k.

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b Find the probability that at most 2 students are absent.

5 Ben and Anna both take three shots at a goal. The table shows the probability distribution of the number of goals each of them scores.

Number of goals	0	1	2	3
Probability	0.2	0.3	0.4	0.1

a Find the probability that Anna scores at least one goal.

b Find the probability that both Ben and Anna score three goals.

6 A fair four-sided spinner, with numbers 1 to 4 written on it, is spun twice and the scores are added.

a Find the probability distribution of the total.

b Find the probability that the total is at least 6.

7 A teacher randomly selects how many questions to set for homework after each lesson. The probability distribution of the number of questions is:

Number	2	3	4	5	6
Probability	a	0.1	0.3	0.3	b

The probability that the teacher sets fewer than four questions is 0.2.

Find the values of a and b.

8 A probability distribution of X is given by $\mathrm{P}(X = x) = k(x + 1)$ for $x = 2, 3, 4, 5, 6$.

a Show that $k = 0.04$.

b Find $\mathrm{P}(X \geqslant 4)$.

9 A four-sided dice is biased. The probability of each possible score is shown.

Score	1	2	3	4
Probability	$\frac{1}{3}$	$\frac{1}{4}$	k	$\frac{1}{5}$

a Find the value of k.

b Find the probability that the total score is four after two rolls.

10 Ronnie and Jimmy are playing snooker. They both try to pot two balls.

For Ronnie, the probability distribution of the number of successful pots is:

Number of pots	0	1	2
Probability	0.1	0.2	0.7

For Jimmy, the probability distribution is:

Number of pots	0	1	2
Probability	0.3	0.5	0.2

Assuming that their performance is independent, find the probability that, between them:

a they pot exactly one ball

b they pot at least one ball.

11 $\mathrm{P}(X \leqslant x) = kx^2$ for $x = 1, 2, 3$.

a Find the value of k.

b What is the probability distribution of X?

12 $\mathrm{P}(X = x) = \frac{1}{9}$ for integers in the range $[-k, k]$.

a Find the value of k.

b If $Y = X^2$ find the probability distribution of Y.

13 On particular day, 20 babies are born at a certain hospital. Denote by G the number of girls.

a State two assumption you need to make so that G can be modelled by the distribution $\mathrm{B}(20, 0.5)$.

b Under these assumptions, find the probability that more than 15 of the babies are girls.

14 At a certain college, it is known that the average probability of a student being late in the morning is 0.07. Let X denote the number of students in a class of 24 who are late on a particular morning.

a In order to model X by a binomial distribution, it is necessary to assume that the probability of being late is the same for every student. Is this a reasonable assumption?

b State a further condition required for X to follow a binomial distribution. Would it be reasonable to assume that this condition is met in this context?

Section 3: The binomial distribution

Some probability distributions occur in lots of different contexts. Consider the following questions:

- A fair six-sided dice is rolled four times. What is the probability of getting exactly two fives?
- A fair coin is tossed ten times. What is the probability that it shows heads at least six times?
- A multiple choice test has 20 questions, each with five possible answers, only one of which is right. A student guesses the answer to each question, with an equal probability of guessing any answer correctly. What is the probability that he gets fewer than five correct answers?

All of these questions involve a similar scenario:

- An action is repeated several times (a dice roll, a coin toss, an attempt to answer a question). These repeats are called 'trials'.
- Possible outcomes can be split into two groups, usually labelled 'success' and 'failure' (five or not a five; heads or tails; right or wrong answer).
- You are interested in the probability of one of the outcomes happening a given number of times.

This type of situation is very common, so it is worth asking whether there is a general rule or formula for calculating the probability. In order to establish the formula, you must first ensure that certain conditions are satisfied.

Key point 17.5

The **binomial distribution** models the number of successful outcomes from repeated trials, provided the following conditions are satisfied:

- the number of trials is fixed
- each outcome can be classified as either 'success' or 'failure'
- the trials are independent of each other
- the probability of success is the same in each trial.

If n is the number of trials, p the probability of a success and X denotes the number of successes, you can write $X \sim \mathrm{B}(n, p)$.

The probabilities for the binomial distribution can be found using your calculator. You need to specify the number of trials (n), the probability of success (p) in a single trial and the required number of successes.

WORKED EXAMPLE 17.8

Decide whether each of the following situations can be modelled using the binomial distribution. If not, say which of the conditions is not satisfied. If yes, find the required probability.

a A fair dice is rolled until it shows a six. Find the probability of getting two fours.

b Tom and Jerry play eight games of chess. The probability that Tom wins a game is 0.6, independently of any other game. Find the probability that Tom wins exactly four games.

c A student is trying to answer 20 quiz questions. The probability of getting the first question right is 0.9, but the probability halves for each subsequent question. What is the probability that he answers 10 questions correctly?

d In a particular village, 63% of five-year-olds attend the local primary school. What is the probability that in a group of 15 friends, at least 10 attend that school?

e The probability that it rains on any particular day is 0.3. Assuming the days are independent, find

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the probability that it rains on more than four days in a week.

a Not binomial; the number of trials is not constant.	The first thing to check is that the situation has a fixed number of trials.
b Binomial, $n = 8$, $p = 0.6$ $P(X = 4) = 0.232$	All the conditions are satisfied. Use your calculator to find the probability.
c Not binomial; the probability of success is not constant.	There is a fixed number of trials, but the probability changes every time.
d Not binomial; the trials are not independent.	If one child attends the school it is more likely that their friends do as well.
e Binomial, $n = 7, p = 0.3$ $P(X > 4) = P(X = 5, 6 \text{ or } 7)$	All the conditions are satisfied. The required probability is for 5, 6 or 7 days.
$= P(X = 5) + P(X = 6) + P(X = 7) = 0.0288$	The three outcomes are mutually exclusive, so their probabilities can be added.

In practice, it may not always be clear that all of the necessary conditions are satisfied, and you may need to make some assumptions in order to be able to use the binomial distribution. For example, if you want to use the binomial distribution to calculate the probability of guessing 8 out of 20 correct answers in a quiz, you need to assume that the different guesses are independent of each other, and that the probability of guessing correctly does not change from one question to the next.

The conditions 'fixed number of trials' and 'outcomes classified as either success or failure' are usually given as a part of the description of the situation in the question. For example, you may be told that a quiz has 20 questions and that you are interested in the number of correct answers. The other two conditions, 'independent trials' and 'constant probability of success', are additional modelling assumptions that you need to make if they are not explicitly given.

You may also need to make further assumptions about the exact values of n and p. For example, when calculating the probability of getting five heads out of 12 coin tosses, you may need to assume that the coin is fair so that $p = \frac{1}{2}$.

Tip

You only need to assume things that are not explicitly given in the question. For example, if you are told that a coin is fair, then $\mathrm{p} = \frac{1}{2}$ would not be an additional assumption.

WORK IT OUT 17.1

On historical evidence, Carol estimates that the probability that it rains on any particular day is 0.08. Carol wants to calculate the probability that it rains on exactly three days in a week. What further assumptions does she need to make so that the number of rainy days in a week can be modelled by the distribution $\mathrm{B}(7, 0.08)$?

Which of the following answers is correct? Explain the error with the incorrect answers.

Solution 1

One day being rainy is independent of any other day being rainy.

Solution 2

The number of days in a week is constant.

Solution 3

The probability that it rains on any given day is constant.

WORKED EXAMPLE 17.9

Alessia is going to roll 10 dice and wants to find the probability of getting more than two 5s. She says that, in order to justify a binomial distribution as a model for the number of 5s, she must assume that the rolls are independent of each other, and that all the dice are fair.

a Explain which part of Alessia's statement is incorrect.

b What assumption does she need to make about the probability of getting a 5 on each dice?

Daniel says that the number of 5s can be modelled by the distribution $\mathrm{B}\left(10, \frac{1}{6}\right)$.

c State two assumptions that Daniel needs to make.

a It is not necessary to assume that the dice are fair.

It is true that the rolls need to be independent. This is almost certainly justified in the case of rolling dice repeatedly.

The binomial distribution only requires that the probability of success is constant, but does not specify a value for it.

b That the probability of rolling a 5 is the same for each dice.

For example, if all the dice were biased so that the probability of rolling a five was $\frac{1}{2}$ for each dice, the conditions for binomial distribution would still be satisfied.

c The rolls are independent of each other. All the dice are fair.

Daniel wants to use a specific probability, so he does need to assume that all the dice are fair. (The assumption of independence is almost certainly justified in this case.)

Note that he does not need to assume that the number of rolls is constant, as the question already states that there are ten rolls.

Cumulative probabilities

In the final part of Worked example 17.8 you were asked to find the probability of more than one outcome. This is straightforward when there are only three probabilities to add up, but what if there were 20 trials and you wanted to find the probability of more than 11 successes? Would you have to add all the probabilities from $\mathrm{P}(X = 12)$ to $\mathrm{P}(X = 20)$?

Tip

Make sure you can distinguish between the single probability and the cumulative probability functions on your calculator.

Fortunately, calculators usually have a function to calculate the probability of getting up to (and including) a specified number of successes – this is called a **cumulative probability**. For example, if $n = 20$ and $p = 0.3$ you can find that $\mathrm{P}(X \leqslant 7) = 0.772$. You can also find the probability of getting more than a certain number of successes, for example:

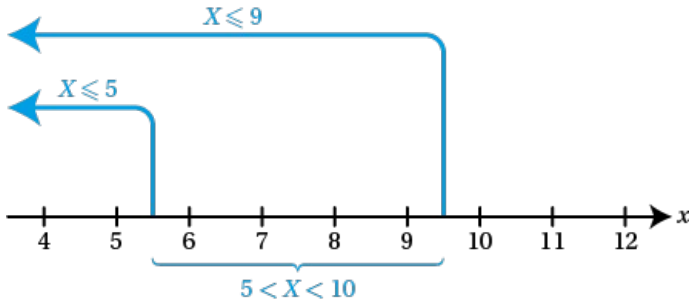
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$$\begin{aligned} \mathrm{P}(X > 11) &= 1 - \mathrm{P}(X \leqslant 11) \\ &= 1 - 0.995 \\ &= 0.005 \end{aligned}$$

If you want to find the probability of getting more than 5 but fewer than 10 successes, this is:

$$\mathrm{P}\left(5 < X < 10\right) = \mathrm{P}\left(X \leqslant 9\right) - \mathrm{P}\left(X \leqslant 5\right)$$

You can see this by looking at the number line.



WORKED EXAMPLE 17.10

Anita shoots at a target 15 times. The probability that she hits the target on any shot is 0.6, independently of the other shots. Find the probability that she hits the target more than 5 but at most 10 times.

Let X be the number of times Anita hits the target. Then

$X \sim B(15, 0.6)$

There is a fixed number of trials and you are interested in the number of successes, so this is a binomial distribution.

$$\begin{aligned} \mathrm{P}(5 < X \leqslant 10) &= \mathrm{P}(X \leqslant 10) - \mathrm{P}(X \leqslant 5) \\ &= 0.7827 - 0.0338 \\ &= 0.749 \end{aligned}$$

Express the required probability in terms of cumulative probabilities

Tip

When you are using a calculator to find probabilities, you should still use correct mathematical notation (not calculator notation) in your answer. You must show what distribution you used and which probabilities you have found from your calculator.

WORK IT OUT 17.2

Four students are trying to answer the following question:

Anna shoots at a target 15 times. The probability that she hits the target on any shot is 0.6, independently of the other shots. Find the probability that she hits the target at least 6 times.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$$\begin{aligned} \mathrm{P}(X \geqslant 6) &= 1 - \mathrm{P}(X \leqslant 6) \\ &= 1 - 0.095 \\ &= 0.905 \end{aligned}$$

Solution 2

$$\begin{aligned} \mathrm{P}(X > 6) &= 1 - \mathrm{P}(X \leqslant 6) \\ &= 1 - 0.095 \\ &= 0.905 \end{aligned}$$

Solution 3

$$\begin{aligned} \mathrm{P}(X\ > 6) &= 1 - \mathrm{P}(X \leqslant 5) \\ &= 1 - 0.034 \\ &= 0.966 \end{aligned}$$

Solution 4

$$\begin{aligned} \mathrm{P}(X \geqslant 6) &= 1 - \mathrm{P}(X \leqslant 5) \\ &= 1 - 0.034 \\ &= 0.966 \end{aligned}$$

The formula for binomial probabilities

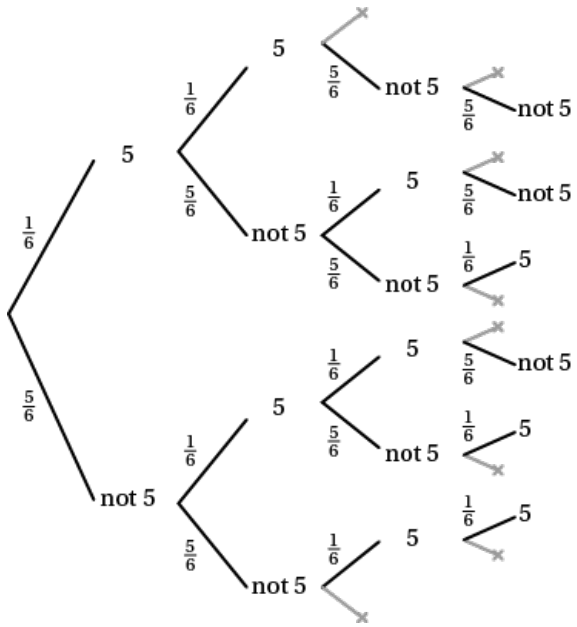
So far, you have used a calculator to find probabilities for the binomial distribution. But how are those probabilities calculated?

Consider the first example from the start of this section: a dice is rolled four times; what is the probability of getting exactly two fives?

There are four trials so $n = 4$. If you label a five as a 'success' then $p = \frac{1}{6}$. The probability of a 'failure' is therefore $\frac{5}{6}$. X stands for the number of fives, so you are interested in $\mathrm{P}\left(X = 2\right)$.

One way of getting exactly two fives is if on the first two rolls you get a five and on the last two rolls you get something else. The probability of this happening is $\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} = \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2$.

But this is not the only way in which two fives can occur. The two fives may be on the first and third, or second and fourth rolls. This can be illustrated on a tree diagram.



Each of the required paths has the same probability, $\left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2$ because two of the outcomes are a five

and two are something else. The number of paths leading to the outcome 'two fives' is 6 so $P(X=2)=6\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2$.

This reasoning can be generalised. Suppose that there are n trials, the probability of a success is p and you are interested in the probability of obtaining r successes. If you imagine representing this on a tree diagram, each relevant path will have probability $p^r(1-p)^{n-r}$, because r of the outcomes are successes (with probability p) and the remaining $n-r$ outcomes are failures (with probability $1-p$). It turns out that the number of paths that give r successes is given by the binomial coefficient $\binom{n}{r}$. This leads to the general formula for the probabilities of the binomial distribution.

Gateway to A Level

For a reminder of tree diagrams, see Gateway to A Level section W.

Key point 17.6

If X~B(n, p) then:

$$\mathrm{P}(X=x)=\binom{n}{x}p^x(1-p)^{n-x} \quad \text{for } x=0,\,1,\,2\ldots n$$

This will appear in your formula book.

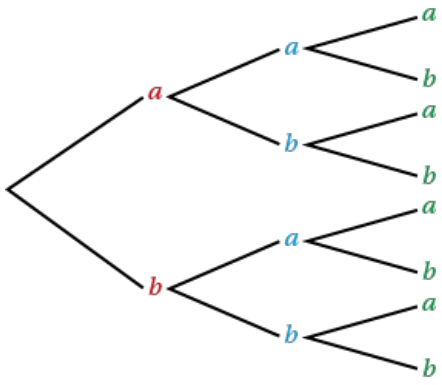
You may find it surprising that the same binomial coefficients appear in both binomial expansion and the binomial probabilities formula. To see why this is the case, consider multiplying out $(a+b)^n$. When you expand the brackets, each term contains either an 'a' or a 'b' from each bracket. For example:

Rewind

You met binomial coefficients in Chapter 9 when studying the binomial expansion. You can find them either on your calculator, from Pascal's triangle, or using the formula in Key point 9.3.

$(a+b)(a+b)(a+b)=aaa+aab+aba+abb+baa+bab+bba+bbb$

You can represent this on a tree diagram:



This is exactly the same tree diagram as you would use to show the possible outcomes of a binomial distribution with three trials, but you would replace a by p and b by $1-p$. Therefore, for example, the binomial probability $\mathrm{P}(X=2)$ has the same form as the term containing a2 b in the binomial expansion.

Now consider the binomial expansion of $(p+q)^n$, where $q=1-p$:

$$(p+q)^n=p^n+\binom{n}{1}p^{n-1}q+\binom{n}{2}p^{n-2}q^2+\ldots+q^n$$

Each term on the right is equal to one of the binomial probabilities, and the left-hand side equals 1 (since $p+q=1$). Hence this equation proves that the sum of all the probabilities in the binomial distribution equals 1 (as should be the case for any probability distribution).

When you know the values of n and p you can always use a calculator to find binomial probabilities. However, the formula from Key point 17.6 is useful when one of the parameters is unknown.

WORKED EXAMPLE 17.11

Ten students take a test. They all have a probability of p of passing, independent of the results of other students. X is the number of students passing the test.

If $\mathrm{P}(X=6)=4\mathrm{P}(X=4)$, find the value of p.

$\mathrm{P}(X=6)=\binom{10}{6}p^6(1-p)^4$
$=210p^6(1-p)^4$

Use the formula to write expressions for $\mathrm{P}(X=6)$ and $\mathrm{P}(X=4)$.

$\mathrm{P}(X=4)=\binom{10}{4}p^4(1-p)^6$
$=210p^4(1-p)^6$

So $210p^6(1-p)^4=840p^4(1-p)^6$

Write the equation given in terms of these expressions.

$p^2=4(1-p)^2$

You can divide by $210p^4(1-p)^4$ if $p\neq 0,1$.

$p=2(1-p)$
$p=2-2p$

You only need to consider the positive root, since both sides must be positive.

$3p=2$
$p=\frac{2}{3}$

EXERCISE 17C

1 In each example, decide whether the situation can be modelled using the binomial distribution. If not, give a reason why it cannot. If it can be, identify the distribution and the required probability (you don't need to calculate the probability).

a A fair coin is tossed 20 times. What is the probability of getting exactly 15 heads?

b Elsa enjoys answering quiz questions. On average she get 78% of the answers right. What is the probability that in a particular quiz, she gets 9 out of the first 10 questions right?

c A bag contains a large number of balls, with an equal proportion of red, blue and green balls. 20 balls are chosen at random. Find the probability that 5 are red and 8 are green.

d A drawer contains 5 black socks and 10 red socks. 6 socks are drawn at random without replacement. Find the probability that at least two black socks are drawn.

e It is known that 2% of a large population carry a gene for diabetes. If 100 people are chosen at random, what is the probability of getting at least 1 person with this gene?

Worksheet

For more practice finding cumulative binomial probabilities, see Support sheet 17.

2 Given that $X\sim\mathrm{B}(8,\ 0.2)$, calculate:

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a **i** $\mathrm{P}(X = 3)$

ii $\mathrm{P}(X = 4)$

b **i** $\mathrm{P}(X \leqslant 3)$

ii $\mathrm{P}(X \leqslant 2)$

c **i** $\mathrm{P}(X > 3)$

ii $\mathrm{P}(X > 4)$

d **i** $\mathrm{P}(X < 5)$

ii $\mathrm{P}(X < 3)$

e **i** $\mathrm{P}(X \geqslant 3)$

ii $\mathrm{P}(X \geqslant 1)$

f **i** $\mathrm{P}(3 < X \leqslant 6)$

ii $\mathrm{P}(1 \leqslant X \leqslant 4)$

g **i** $\mathrm{P}(1 < X < 5)$

ii $\mathrm{P}(2 < X < 7)$

h **i** $\mathrm{P}(3 \leqslant X \leqslant 7)$

ii $\mathrm{P}(2 \leqslant X \leqslant 5)$

3 In each question identify the distribution, write down the required probability and find it using your calculator.

a Jake beats Marco at chess in 70% of their games. Assuming that this probability is constant and that the results of games are independent of each other, what is the probability that Jake will beat Marco in at least 16 of their next $20\ \mathrm{g}$ ames?

b On a television channel, the news is shown at the same time each day. The probability that Salia watches the news on a given day is 0.35. Calculate the probability that on 5 consecutive days she watches the news on exactly 3 days.

c Sandy is playing a computer game and needs to accomplish a difficult task at least three times in five attempts in order to pass the level. There is a 1-in-2 chance that he will accomplish the task each time he tries, unaffected by how he has done before. What is the probability that he will pass to the next level?

4 Given that $Y \text{~} \mathrm{B}\left(5, \frac{1}{2}\right)$, use the formula to find the exact value of:

a **i** $\mathrm{P}(Y = 1)$

ii $\mathrm{P}(Y = 0)$

b **i** $\mathrm{P}(Y \geqslant 1)$

ii $\mathrm{P}(Y \leqslant 1)$

c **i** $\mathrm{P}(Y > 4)$

ii $\mathrm{P}(Y \leqslant 3)$

5 15% of students at a large school travel by bus. A random sample of 20 students is taken.

a Explain why the number of students in the sample who travel by bus is only approximately a binomial distribution.

b Use the binomial distribution to estimate the probability that exactly five of the students travel by bus.

6 A Biology test consists of eight multiple choice questions. Each question has four options: only one of which is correct. At least five correct answers are required to pass the test. Chen does not know the answers to any of the questions, so answers each question at random.

a What is the probability that Chen answers exactly five questions correctly?

b What is the probability that Chen manages to pass the test?

7 0.8% of people in the country have a particular cold virus at any time. On a single day, a doctor sees 80 patients.

a What is the probability that exactly two of them have the virus?

b What is the probability that three or more of them have the virus?

c State an assumption you have made in these calculations.

8 On a fair dice, which is more likely: rolling 3 sixes in 4 throws or rolling a five or a six in 5 out of 6 throws?

Did you know?



Question 8 is the problem which was posed to Pierre de Fermat in 1654 by a professional gambler who could not understand why he was losing. It inspired Fermat (with the assistance of Pascal) to set up probability as a rigorous mathematical discipline. Fermat was a French mathematician and lawyer.

9 This question is intended to help you understand the difference between 'constant probability' and 'independent trials', which are both required for the binomial distribution.

a A bag contains six red and four blue balls. A ball is taken at random and not replaced. This is repeated three times, so that in total three balls are selected.

i Find the probability that the second ball is red.

ii Find the probability that the third ball is red.

iii Find the probability that the first and the second balls are both red. Hence use the formula $\mathrm{P}(A \text{ and } B) = \mathrm{P}(A) \times \mathrm{P}(B)$ to show that the events 'the first ball is red' and 'the second ball is red' are not independent.

iv Find the probability that exactly one of the three balls is red. Compare this to $\mathrm{P}(X = 1)$ when $X \sim \mathrm{B}(3, 0.6)$.

b Repeat parts **i–iv** if the three balls are selected with replacement.

10 In each of the following situations discuss whether the trials are independent and whether the probability of success is constant (i.e. the probability of success on the n th trial is not dependent on n).

a Pulling socks from a drawer; success is pulling a red sock out.

b Consecutive rolls of a fair dice; success is rolling a 6.

c Each member of the class flips a fair coin. If 'heads', they write down the first letter of their first name. If 'tails', they write the last letter; success is writing a vowel.

11 A fair coin is tossed ten times. What is the probability that it shows heads at least six times?

12 A multiple choice test has 20 questions, each with five possible options, only one of which is right. A student guesses the answer to each question, with an equal probability of guessing any answer

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correctly.

a What is the probability that he gets fewer than five correct answers?

b State a further assumption you made in your calculation in part a.

13 On a particular day, 20 babies are born at a certain hospital. Denote by G the number of girls.

a State two assumptions you need to make so that G can be modelled by the distribution $\mathrm{B}(20, 0.5)$.

b Under these assumptions, find the probability that more than 15 of the babies are girls.

14 At a certain college, it is known that on average the probability that a student is late in the morning is 0.07. Let X denote the number of students in a class of 24 who are late on a particular morning.

a In order to model X by a binomial distribution, it is necessary to assume that the probability of being late is the same for every student. Is this a reasonable assumption?

b State a further condition required for X to follow a binomial distribution. Would it be reasonable to assume that this condition is met in this context?

15 X~$\mathrm{B}(8, p)$. If $\mathrm{P}(X = 5) = \mathrm{P}(X = 6)$ find the value of p.

16 X is a binomial random variable, where the number of trials is 4 and the probability of success of each trial is p. Find the possible values of p if $\mathrm{P}(X = 2) = 0.3456$.

17 Over a one-month period, Ava and Sven play a total of n games of tennis. The probability that Ava wins any game is 0.4. The result of each game played is independent of any other game played. Let X denote the number of games won by Ava over the one-month period.

a Show that $\mathrm{P}(X = 2)$ can be given as $\mathrm{P}(X = 2) = \dfrac{2n(n-1)}{9} \times 0.6^n$.

b If the probability that Ava wins two games is 0.121 correct to three decimal places, find the value of n.

18 X is a random variable following $\mathrm{B}(n, 0.8)$. If $\mathrm{P}(X = n) = 64\mathrm{P}(X = 0)$, find n

19 If X~$\mathrm{B}(n, p)$ and $\mathrm{P}(X = 4) = \mathrm{P}(X = 5)$ find n in terms of p.

Tip

To do question 17b you need to try some values of n or use tables on your calculator.

Checklist of learning and understanding

- **Mutually exclusive** events satisfy $\mathrm{P}(A \text{ and } B) = 0$. Their probabilities can be added: $\mathrm{P}(A \text{ or } B) = \mathrm{P}(\mathrm{A}) + \mathrm{P}(\mathrm{B})$
- An event and its **complement** are mutually exclusive, and $\mathrm{P}(A) + \mathrm{P}(A') = 1$. The probability of the complement is sometimes easier to find.
- For **independent** events, $\mathrm{P}(A \text{ and } B) = \mathrm{P}(\mathrm{A}) \times \mathrm{P}(B)$. If A and B are independent then so are A′ and B′, A′ and B, and A and B′.
- A **probability distribution** is a list of all possible outcomes and their probabilities.
- All probabilities in a probability distribution must add up to 1.
- **Cumulative probability** is the probability of obtaining up to and including a given outcome.
- The **binomial distribution** is a model for the number of successes when an experiment is repeated several times. If there are n trials and the probability of success is p, the distribution is denoted by $\mathrm{B}(n, p)$.

- For the binomial distribution to be a good model, the following conditions must be satisfied:
  - The number of trials is constant.
  - Each trial has two possible outcomes.
  - The probability of success is the same for each trial.
  - The trials are independent.
- The probabilities for the binomial distribution can be found using your calculator, or the formula:

$$\mathrm{P}\left(X = x\right) = \binom{n}{x} p^x (1-p)^{n-x} \text{for } x = 0,\ 1,\ 2 \ldots n$$

Mixed practice 17

1 A factory making bottles knows that, on average, 1.5 % of its bottles are defective. Find the probability that, in a randomly selected sample of 20 bottles, at least 1 bottle is defective.

2 The mark on a Physics test is an integer between 1 and 5 inclusive. The distribution of test grades is given in this table:

Mark	1	2	3	4	5
Probability	0.05	0.23	0.35	k	0.21

a Find the value of k.

b Find the probability that a randomly selected student scores at least a 3 in the test.

c Write down the most likely mark in this test.

3 A spinner has four equal sections with numbers 1, 2, 5 and 7 written on them. The spinner is spun twice. Find the probability distribution of the positive difference between the scores (larger minus smaller).

4 When a boy bats at baseball, the probability that he hits the ball is 0.4. In practice he gets pitched 12 balls. Let X denote the total number of balls he hits. Assuming that his attempts are independent, find:

a $\mathrm{P}(X = 3)$

b $\mathrm{P}(X > 5)$

5 A probability distribution of a variable X is shown in this table:

x	2	3	4	5	6
$\mathrm{P}(X = x)$	p	0.2	0.3	0.3	q

a Writ down the value of $p + q$.

b Give that $\mathrm{P}(X \geqslant 4) = 0.75$, find the values of p and q.

6 Given that $X \sim \mathrm{B}(8,\ 0.65)$, find:

a $P(X < 6)$

b $P(X = 8)$

c $(5 \leqslant X < 7)$

7 When Robyn shoots an arrow at a target, the probability that she hits the target is 0.6. In a competition she has eight attempts to hit the target. If she gets at least 7 hits on target she will qualify for the next round.

a Find the probability that she hits the target exactly 4 times.

b Find the probability that she fails to qualify for the next round.

c What assumption did you need to make in your answers in parts **a** and **b?**

8 A test is marked on the scale from 1 to 5. The cumulative distribution of test scores is shown in the table:

Grade (s)	1	2	3	4	5
P (grades $\leqslant$ s)	0.32	0.48	0.65	0.82	1

Find the percentage of candidates whose grade was:

a 2

b between 3 and 5 inclusive.

9 A student has the probability 0.7 of answering a question correctly, independently of any other questions.

Find the probability that, in a test containing 15 questions, the student gets more than seven but fewer than 12 correct answers.

10 A biased coin has probability p of showing heads. The coin is tossed 6 times. The probability that it shows no heads is 0.072. Find the value of p correct to two significant figures.

11 Sasha and Elijah both roll a fair six-sided dice.

a Find the probability distribution of the difference between the scores (Sasha's score – Elijah's score).

b Hence find the probability that Elijah gets a higher score than Sasha.

12

a A biased coin is thrown twice. The probability that it shows heads both times is 0.04. Find the probability that it shows tails both times.

b Another coin is biased so that the probability that it shows heads on any throw is p. The probability that the coin shows heads on exactly one in two throws is 0.42. Find the possible values of p.

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[Question part reference style adapted]

13 A company producing light bulbs knows that the probability that a new light bulb is defective is 0.5%.

a Find the probability that a pack of six light bulbs contains at least one defective one.

b Hamish buys 20 packs of six light bulbs. Find the probability that more than 4 of the boxes contain at least one defective light bulb.

14 A fair coin is tossed repeatedly until it shows tails.

a Find the probability that the first five tosses all show heads.

b Hence find the probability that the first tails appear on the sixth toss.

15 The probability that a student forgets to do homework is 5% independently of other students. If at least one student forgets to do homework, the whole class has to do a test.

a There are 12 students in a class. Find the probability that the class will have to do a test.

b For a class with n students, write down an expression for the probability that the class will have to do a test.

c Hence find the smallest number of students in the class such that the probability that the class will have to do a test is at least 80%.

16 Two fair dice are rolled and the difference between the two scores is recorded (larger − smaller).

a Find the probability distribution of the recorded number.

b This experiment is repeated ten times. Find the probability that the recorded number is zero on more than three occasions.

17 Four fair six-sided dice are rolled. Let X be the largest number rolled.

a Explain why $\mathrm{P}(X \leqslant k) = \left(\frac{k}{6}\right)^4$, for $k = 1, 2, \ldots 6$.

b Copy and complete the following probability distribution table.

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$		$\frac{15}{1296}$	$\frac{65}{1296}$			$\frac{671}{1296}$

18 A fair six-sided dice is rolled until the fourth 6 is obtained.

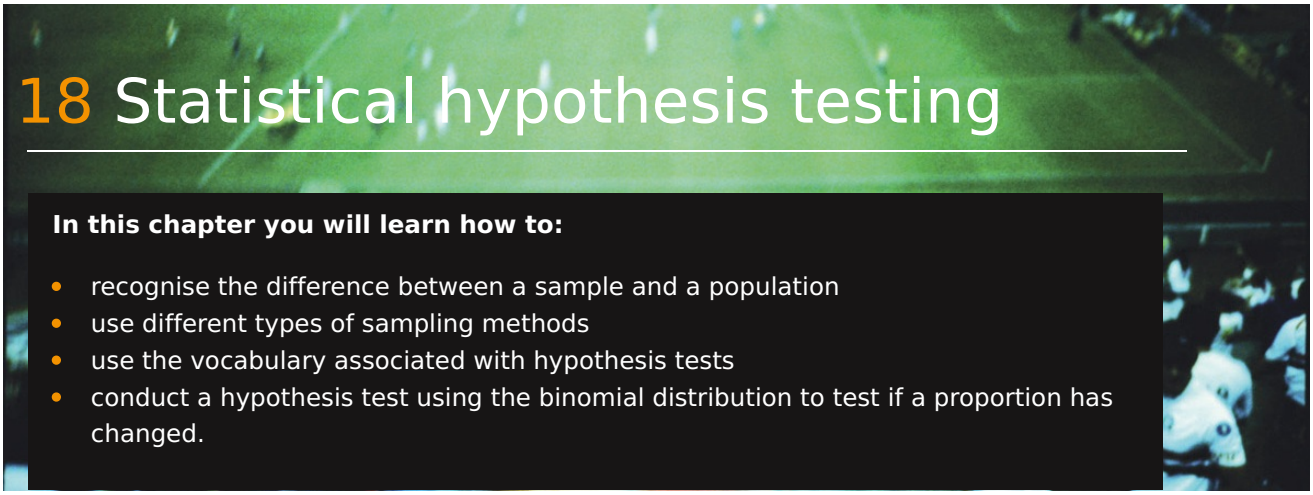
a Find the probability that there are exactly three 6s in the first seven rolls.

b Hence find the probability that the fourth 6 is obtained on the eighth roll.

Worksheet

For a selection of more challenging problems, see Extension sheet 17.

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18 Statistical hypothesis testing

In this chapter you will learn how to:

- recognise the difference between a sample and a population
- use different types of sampling methods
- use the vocabulary associated with hypothesis tests
- conduct a hypothesis test using the binomial distribution to test if a proportion has changed.

Before you start...

Chapter 17	You should know how to calculate cumulative probabilities for a binomial distribution.	1 Given that $X \sim B(25, 0.6)$, find: a $\mathrm{P}(X \leqslant 15)$ b $\mathrm{P}(X \geqslant 17)$
Chapter 17	You should be able to deduce parameters of a binomial distribution in context.	2 It is known that, on average, 35 out of 50 people like coffee. For a random sample of 12 people, let X be the number of those who like coffee. State the distribution of X, including any parameters.

What is hypothesis testing?

The aim of statistics is to find out various things about a population. Here are some examples of questions you might ask:

- What is the mean height of all 17-year-old boys in the UK?
- What is the range of ages of all professional football players?
- What proportion of the electorate intend to vote for a particular political party in the next election?

It is often impossible to collect all the required data, so you use samples to make inferences about the whole population. A sample value can only give you an estimate of the **population parameter** (this is some numerical characteristic of the population, such as mean or range). How good this estimate is depends on the size and quality of the sample. The first part of this chapter looks at issues around sampling and using samples. Once you have used a sample to find an estimate of the population parameter it would be useful to know how accurate this estimate is likely to be. This question is in general very difficult to answer and requires some advanced probability theory. The second part of this chapter looks at a slightly simpler question, asking whether the parameter value has changed from a previously known, or assumed one. This leads to the **hypothesis test**, a procedure for determining whether a given sample provides significant evidence that a population parameter (for example, mean/spread/proportion) has changed from a previously known or assumed value. The hypothesis test is one of the most commonly used statistical tools.

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Section 1: Populations and samples

Suppose you wanted to know the average height of all adults in the UK. One way to find out would be to measure everyone's height. This is called a **census**. It involves collecting information about the entire **population** – all the individuals of interest (in this example this would be all UK adults). A large organisation may be able to carry out such a survey; indeed, the UK government undertakes a census every ten years in order to plan public services.

For a small organisation or an individual, a census is mostly not an option, because of the time and costs involved. Census data sets also take a very long time to analyse (although modern technology has made this less of a problem). This means that it is not possible for you to find out the exact average height of all UK adults. But you can try to estimate it by taking a sample – measuring heights of part of the population. The average of the sample will most likely be different from the population average. However, if the sample is selected well, it may provide a reasonable estimate.

There are situations where it is impossible to carry out a census, even if a large amount of time and money is available. One example is if it is impossible to identify, or get access to, all members of the population. For example, suppose a zoologist wanted to find out the average mass of an ant. It is impossible to find all the ants in the world, or tell which ones you have not yet measured. Another situation is when the process of collecting data destroys the object being measured. For example, a manufacturer needs to know what maximum load can be placed on a shelf. Testing the whole population would mean breaking all the shelves.

Once you have selected a sample and collected the data you can apply to it any of the techniques you learnt in Chapter 16 – you can draw statistical diagrams, calculate averages, measures of spread or correlation. You then need to decide what this tells you about the whole population.

WORKED EXAMPLE 18.1

A clothing company carries out a survey to find out the average and the range of heights of 17-year-old girls in the UK. A sample of 200 girls from a large sixth form college has a mean height of $158\,\text{cm}$ and a range of $28\,\text{cm}$.

a Is the true population mean more likely to be larger or smaller than $158\,\text{cm}$?

b State one possible reason why taking a sample from a single college may not result in a good estimate for the mean height.

c Siobhan says that $28\,\text{cm}$ is a good estimate for the range. Priya says that the range is almost certainly larger than $28\,\text{cm}$. Explain who is right.

a The sample mean is equally likely to be larger or smaller than the true population mean.

This should be the case if the sample has been selected well.

b For example, the college might be located in an area populated by an ethnic group which is on average taller or shorter than the whole population.

The part of the population from which the sample was taken may not be typical of the entire country.

c It is unlikely that both the shortest and the tallest 17-year-old girl in the country go to this particular college. So Priya is right, the sample range will be smaller than

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the population range.

Explore

It turns out that all measures of spread tend to be underestimated by a sample. However, in the case of the variance there is a change to the formula you met in Chapter 16, Section 2, which provides a better estimate. Find out about 'unbiased estimates' – you will meet them if you study the Statistics option of Further Mathematics.

A sample is often used to find an estimate of a population **parameter** (some numerical characteristic of the population, such as its mean or variance). If you want a sample to provide a good estimate of a population parameter then the sample needs to be **representative** of the population. This means that the distribution of the values in the sample is roughly the same as in the whole population. This will not be the case if the sampling procedure is **biased**. For example, if you wanted to find out about people's attitudes to music and you selected a sample from those attending a particular concert, then this sample would contain an unusually large number of fans of a particular type of music.

WORKED EXAMPLE 18.2

Comment on possible sources of bias in each of these samples:

a The basketball team as a sample of students at a college used to estimate the average height of all students.

b A sample of people from a particular political party's conference used to find out about the UK population's attitudes to taxation.

c A sample taken from those in a waiting room at a doctor's surgery for a survey to find out how many days people in the country have had off sick this year.

a Basketball players are on average taller than the general population.

b Attitudes to taxation tend to be related to political affiliation, so members of a particular party will have attitudes not representative of the wider population.

c People at a doctor's surgery are likely to have poorer health than the general population.

There are several different methods for selecting samples. This section looks at which of them are most likely to lead to representative samples. It is important to remember, however, that even a good (unbiased) sampling method can lead to an unrepresentative sample. This is because the process of sampling is inherently random so there is always a possibility that, for example, only extreme values are selected. The aim of a good sampling method is to minimise the probability of this happening.

Simple random sample

This is the type of sample most people have in mind when they talk about random samples.

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Key point 18.1

Simple random sampling is a procedure where every possible sample (of a given size) has an equal chance of being selected.

Many common sampling techniques do not produce a simple random sample. For example, it is common to include equal numbers of each gender when selecting a sample for a social science study. This is not a simple random sample because a sample consisting of all females has probability zero of being selected. Another example would be selecting names from a list by choosing the starting point randomly and then taking every tenth name. There are perfectly good reasons for selecting samples in this way in certain situations, you just need to be aware that they are not simple random samples.

Common methods for generating a simple random sample include lottery machines and random number generators. When using these methods, it is a common practice to sample without replacement; this means that you do not include the same individual more than once. If the population is large, it is possible to do this for any reasonably sized sample.

WORKED EXAMPLE 18.3

A student wants to take a sample of students from his college. He has a list of all students, numbered 1 to 478. He uses a random number generator on his calculator, which can generate three-digit random numbers between 001 and 999, inclusive. The first ten numbers he obtains are:

$237, 155, 623, 078, 523, 078, 003, 381, 554, 263$

Suggest which could be the first four students in his sample.

$237, 155,$	Skip 623, as there is no student with this number.
$78,$	078 stands for student number 78.
3	Skip 523 because there is no student with this number, and skip 78 because it has already been included.

Opportunity sampling

Ensuring a simple random sample is remarkably difficult. A large part of the difficulty stems from the fact that it may not be possible to get the list of all the members of the population to which you can then apply the sampling procedure, or to obtain measurements from all the individuals you wish to sample. **Opportunity sampling** avoids these difficulties by sampling only from those individuals who are available and willing to take part.

Key point 18.2

Opportunity sampling involves choosing respondents based upon their availability and convenience.

This clearly does not produce a simple random sample, but it may be the only possible option and it may still produce a good estimate of the population parameters you are interested in. However, in some situations it can introduce bias if the group consists of very similar members. It might, therefore, not be generalisable. For example, if you ask your friends which subject they like most there is no guarantee that the majority of your school shares that opinion; it would therefore not be wise to draw conclusions about the whole school based on this sample.

WORKED EXAMPLE 18.4

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Del wishes to take a sample of residents from her neighbourhood. She decides to ask some people waiting at the bus stop.

a What sort of sample is this?

b Is the sample likely to be representative if her question is about

i attitudes to the environment

ii their favourite football team?

a Opportunity sample

b **i** The sample may not be representative because people who use public transport are arguably more likely to have 'green' attitudes.

ii The sample could be representative, as there is no obvious link between use of public transport and football.

You may be able to think of possible sources of bias here; for example, linked to age.

Systematic sampling

A simple random sample might just happen to include only people from London, or only people with the first name John. If these outcomes would be problematic, an alternative is **systematic sampling**. This requires a list of all participants ordered in some way.

Key point 18.3

Systematic sampling means taking participants at regular intervals from a list of the population, with the starting point chosen at random.

WORKED EXAMPLE 18.5

A sample is formed by taking a telephone book and calling the person at the top of each page.

a State the name given to this type of sampling.

b Explain why this is not simple random sampling.

The calls are made at between 10 a.m. and 2 p.m. on a Wednesday to enquire about the number of children in the household.

c Suggest a reason why the mean value calculated will be biased.

a Systematic sample

b Not all samples are equally likely – e.g. the sample with all the people at the bottom of each page has zero probability of being

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selected.

c People without children may be working at this time, so they may not answer. This would mean the calculated mean is higher than it should be.

There are many possible reasons biasing the mean in either direction. Any valid, well-reasoned argument would be acceptable.

Stratified sampling

A simple random sample might not be representative of the overall population. There may be more pensioners, or men, or people with Mathematics A level in your sample than the background population. One way to overcome this is to use a stratified sample. First you need to decide in advance which factors you think might be important. You separate the population by these factors and within each group you take a simple random sample. The size of each sample is in proportion to the size of the group.

Key point 18.4

Stratified sampling is splitting the population into groups based on factors relevant to the research, then random sampling from each group in proportion to the size of that group.

WORKED EXAMPLE 18.6

A school is made up of 250 girls and 150 boys. A sample of size 80 is to be chosen, stratified between boys and girls. How many girls must be included in the sample?

The proportion of the school which is girls is $\frac{250}{250+150} = \frac{5}{8}$

The total school population is $250 + 150 = 400$.

$\frac{5}{8}$ of 80 is 50.

Calculate this fraction of the sample of 80.

Quota sampling

Stratified sampling is excellent in principle, but it is often not practical. You need to have access to every member of the population to make a random sample. A common alternative is to use opportunity sampling instead of simple random sampling within each group.

Key point 18.5

Quota sampling is splitting the population into groups based on factors relevant to the research, then opportunity sampling from each group until a required number of participants are found.

WORKED EXAMPLE 18.7

A market researcher is required to sample 100 men and 100 women in a supermarket to find out how much they are spending on that day.

a State the name given to this type of sampling.

b Explain why this method is used rather than stratified sampling.

c State one disadvantage of this method.

a Quota sampling

b The researcher would have to know in advance who was going to be shopping on that day to create a random sample, and this is not feasible.

c The people who stop to talk to the researcher might not be representative.

There are several other disadvantages too!

Cluster sampling

One of the main concerns in real-world sampling is cost. Creating a list of all members of a population and travelling or contacting the sample may be very difficult and expensive. One method that tries to make the process more efficient is **cluster sampling**. Like stratified sampling, this involves splitting the population into groups, called clusters. Unlike stratified sampling, these clusters do not have to be based on factors relevant to the research – they may be based just on convenience.

Key point 18.6

Cluster sampling is splitting the population into clusters based on convenience, then randomly choosing some clusters to study further.

In stratified sampling, all groups are sampled in proportion to their size, but in cluster sampling only some of the clusters are chosen (at random) to be studied. This makes it less accurate than stratified sampling, as choosing an unrepresentative cluster can have a large effect on the outcome.

WORKED EXAMPLE 18.8

Jacob wants to estimate the percentage of people in the UK who travel to work by train. He selects five local authorities at random and uses their information to work out the mean.

a State the name given to this type of sampling.

b Describe the difference between this sampling and a stratified sample across local authorities.

a Cluster sample

b Only some local authorities are chosen in this sample. If it were a stratified sample, values would be chosen from all local authorities and combined in proportion to the size of the local authority.

Comparing sampling methods

Method	Advantage	Disadvantage
Random methods		
Simple random	Produces an unbiased sample.	Hard to do in practice. Needs a list of

		the entire population and everybody to respond. Time consuming and expensive.
Systematic	Avoids unwanted clustering of data. Practically easier than using random number generators.	Needs a list of the entire population. Less random than simple random as no longer independent.
Stratified	Produces a sample representative over the factors identified.	Needs a list of the entire population with additional information about each member. Time consuming and expensive. Determining which factors to consider is not always obvious.
Cluster	Cheaper and easier than other random methods.	Less accurate than other random methods – clusters may not be representative.
Non-random methods		
Opportunity sample	Cheap and convenient.	May introduce bias and not be generalisable.
Quota sample	Ensures the sample is representative over the factors identified.	May introduce bias and not be generalisable.

Worksheet

You will learn how to carry out different sampling methods, and explore how different samples can lead to different conclusions about the population, in the Large Data Set section on the Cambridge Elevate digital platform.

Did you know?

The issue of how big a sample to use is of vital importance to statisticians. A sample of size 200 is not twice as good as a sample of size 100. This is not just a rule of thumb – you can prove it using some advanced ideas in statistics that you will meet if you study the Statistics option of Further Mathematics.

Focus on ...

Obtaining a good sample can be very difficult in practice. Focus on ... Problem solving 3 looks at some famous examples of experimental design in statistics.

EXERCISE 18A

1 Comment on possible sources of bias in the following samples:

a Determining the attitude towards university tuition fees in the UK by asking sixth form students.

b Finding out about British people's perception of the cost of food by asking people in supermarkets to estimate the cost of a pint of milk.

c Measuring the average height of people in the UK using a sample taken from a school.

d Predicting the outcome of the 1948 US presidential election by a telephone survey of US citizens.

2 Name the type of sampling described in each of these situations:

a A zoologist is investigating different flies in a forest. She knows there are roughly equal numbers of male and female flies so she puts up fly paper that attracts female flies and fly

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paper that attracts male flies. She waits until each paper has 20 flies on it and then takes the paper down.

b A doctor believes that drug resistance may depend on the age of the patient. She knows that 20% of patients in the hospital are less than 18 years old and 30% are over 65. She makes a simple random sample to choose four aged under 18, ten people aged between 18 and 65 and six over 65.

c There are about 4000 professional footballers in England playing for 92 clubs. To estimate the fitness of professional footballers 10 clubs are chosen at random and fitness tests are conducted on a random sample of players at those clubs.

d The names of all students in a school are put into a hat and a sample is formed from the first 10 names taken out.

e To find the average population of countries a list of all countries is written in alphabetical order and the 3rd, 13th, 23rd … counties are chosen and investigated.

f To determine the outcome of the next election you ask everybody in your class how they will vote.

3 In order to find out the mean and standard deviation of masses of a particular breed of cat, Dougal measures a random sample of 20 cats. The results are summarised as follows:

$$\sum w = 51, \quad \sum w^2 = 138.32$$

a Find the mean mass and show that the standard deviation of this sample is 0.64 to 2 decimal places.

b Is the standard deviation of the whole population likely to be larger or smaller than 0.64 ?

c Dougal hopes to obtain a more accurate estimate for the mean by taking a sample of size 100. The mean of this sample is 2.63. Is this necessarily a better estimate of the population mean than the one found in part **a**? Explain your answer.

4 An ecologist wants to study the proportion of adult fish in the North Sea. She believes that 40% of fish in the North Sea are cod, 40% are haddock and 20% are of other varieties. She catches fish until she has 20 cod, 20 haddock and 10 of other varieties.

a Explain why a random sampling method is not feasible in this situation.

b State the name of sampling method used.

c Why might this method be better than an opportunity sample of the first 20 fish caught?

5 Larissa wants to find out how many students at her college travel by train. She decides to conduct a survey on a simple random sample of 70 students. Describe how she could obtain such a sample.

6 Dineth is investigating attitudes to healthy eating for his Biology project. He decides to interview ten students while waiting in a queue at a burger bar.

a State the name for this type of sampling.

b Identify one possible source of bias in Dineth's sample.

7 Lenka needs to take a simple random sample of five students at her college to take part in her psychology experiment. She obtains the list of all 847 students and uses a random number function on her calculator to generate random numbers. Her calculator produces three-digit random numbers from 001 to 999. It produces the following numbers:

$$016, 762, 938, 537, 016, 722, 886, 152, 721$$

Suggest which five students Lenka should select for her sample.

8 A psychologist is studying the reading age of people in a city. He wants to create a stratified sample. He thinks that gender and age are important. According to census data, 40% of the city is

female and the median age is 42 for both genders.

a Copy and complete this table to show the number of people in each category required in a sample of 80 people.

	Female	Male
Under 42		
42 or over		

b State one advantage of stratified sampling over quota sampling.

c Explain why quota sampling might be preferable in this situation.

9 A shop has 40 staff in each of 10 branches in different parts of the country. The owner wants to find out about staff wellbeing. She wants to interview a sample of 20 staff. The following suggestions are made as to how to choose the sample.

A Pick four branches at random and then interview five randomly chosen people at each branch.

B Use a random number generator to pick 20 staff from all staff.

C Get the manager of each branch to select two staff members to send for interview.

D Use a 20-sided dice to randomly select two members from each branch to send for interview.

a Name the method of sampling for each suggested method.

b Which method is likely to give the most accurate answer?

c Why might method A be used instead of method B?

Worksheet

Poor sampling methods can lead to misleading results. See Extension sheet 18 for some examples.

10 A polling firm wants to investigate the voting intentions of a London borough. They have access to details of all registered voters in the borough so make a numbered list and use a random number generator to select 100 participants. They send a questionnaire to each person selected and study the responses. Explain why this will not necessarily produce a simple random sample.

11 A school is attended by 500 girls and 500 boys. A simple random sample is obtained by selecting names from a box (without replacement) to get a sample of 10 students.

a Find the probability that the students are all boys.

b If names are put back in the box it is possible that the same student gets picked more than once.

i Find the probability of someone being picked more than once.

ii Find the probability of any one student being picked using this method.

c What is the percentage difference between the probability of all 10 students being boys in the situations described in **a** and **b**?

d Instead of simple random sampling an opportunity sample is taken by choosing the first 10 students a teacher sees on the playground. Without further calculation explain whether this will increase or decrease the probability of all 10 students being boys.

Tip

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Why might this sampling method not be completely random?

Focus on …

This problem is looked at in more detail Focus on … Problem solving 3.

Section 2: Introduction to hypothesis testing

This section looks at using a sample to make inferences about the population. Consider the following questions:

- A particular drug is known to cure a disease in 78% of cases. A new drug is trialled on 100 patients and 68 of them were cured. Does this mean that the new drug is less effective than the old one?
- A sociologist believes that more boys than girls are born during war time. In a sample of 200 babies born in countries at war, 126 were boys. Do these data support the sociologist's theory?
- In the last general election, Party Z won 36% of the vote. An opinion poll surveys 100 people and finds that 45 support Party Z. Does this imply that their share of the vote will change in the next election?

In all these questions you are trying to find out whether the proportion of the population with a certain characteristic is different from a previously known or assumed value, by calculating the corresponding proportion from a sample. You would not expect the sample proportion to be exactly the same as the population proportion. If you took a different sample you may well get a different proportion. This means that a sample, however large, cannot provide a definitive answer to the question; it can only suggest what the answer is likely to be.

One common procedure for answering this type of question is called a 'hypothesis test'. It requires the question to be phrased in a specific way.

Fast forward

A In the AS course you only look at hypothesis tests for the population proportion using the binomial distribution. In Student Book 2 you will also meet a hypothesis test for the population mean.

Key point 18.7

A **hypothesis test** is a procedure for answering a question of the following type:

- Does a sample provide significant evidence that a population parameter (mean/spread/proportion) has changed from a previously known or assumed value?

Look at the first example given in this section. The old drug cured 78% of patients. In a trial of a new drug, 68 out 100 patients in a sample were cured; this number is called the **test statistic**. The question is:

- Does this sample provide sufficient evidence that the new population proportion is smaller than 78%?

The key phrase here is **sufficient evidence**. The sample proportion of $\frac{68}{100} = 68\%$ seems significantly smaller than 78%, but it could happen even if the population proportion is still the same. So how likely would this be?

To calculate this probability you need to assume that the population proportion hasn't changed. This is our 'default position', or **null hypothesis**, which is tested against an **alternative hypothesis**, which represents the idea you have that there has been a change.

Key point 18.8

The null hypothesis, denoted H_0, specifies the previous or assumed population proportion.

The alternative hypothesis, denoted H_1, specifies how you think the proportion may have changed.

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So in the example, the null hypothesis is:

H_0: The proportion of cured patients is 78%.

and the alternative hypothesis is:

H_1: The proportion of cured patients is less than 78%.

Tip

You will normally state the hypotheses in terms of the parameter value. For example:

$$\mathrm{H}_0 : p = 0.78, \mathrm{H}_1 : p < 0.78,$$

where p is the proportion of cured patients.

The question now becomes:

- Does the sample provide sufficient evidence against the null hypothesis and in favour of the alternative hypothesis?

If the population proportion of cured patients is still 78%, then the probability of any particular patient being cured is 0.78. You have a sample of 100 patients, so the number of cured patients can be modelled by the binomial distribution $\mathrm{B}(100, 0.78)$. The number of 'successes' in the sample is 68. You are looking for evidence that the population proportion is smaller than 0.78; if this were the case, the number of cured patients would be smaller than expected under the null hypothesis. Therefore, you want to find the probability of 68 or fewer patients being cured:

If X~$\mathrm{B}(100, 0.78)$ then $\mathrm{P}(X \leqslant 68) = 0.0134$.

So if the new drug is as good as the old drug, there is a probability of around 1.3% that only 68 or fewer out of a sample of 100 patients get cured.

Is this sufficiently 'unlikely' to conclude that the population proportion for the new drug must be smaller than 78%? This is a matter of judgement, and may be different in different contexts. You need to decide on what probability is 'sufficiently small'. This is called the **significance level** of the test.

The significance level of a hypothesis test specifies the probability that is sufficiently small to be accepted as evidence against the null hypothesis.

If you assume that H_0 is correct, then the probability of the observed, or more extreme, sample value is called the **p-value**.

Tip

It is very important that you find $\mathrm{P}(X \leqslant 68)$ and not $\mathrm{P}(X = 68)$. You can remember this as 'the probability of the observed value, or more extreme'.

Key point 18.9

If the p-value is smaller than the significance level, you have sufficient evidence against H_0 and you can reject it in favour of H_1. Otherwise, the sample does not provide sufficient evidence against H_0 and you should not reject it.

Common significance levels used in practice are between 1% and 10%. If you conduct your test at the 1% significance level, then the p-value 0.0134 is greater than the significance level, so you don't have sufficient evidence that the population proportion has decreased. However, if you use a 5% significance level then you do have sufficient evidence that the new drug cures less than 78% of the patients.

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Key point 18.10

A hypothesis test for a population proportion includes the following steps:

1 State the null and alternative hypotheses, defining any parameters.

2 Decide on the significance level.

3 State the distribution of the test statistic, assuming the null hypothesis is true.

4 Using this distribution calculate the probability of observing the test statistic, or more extreme. This is the p-value.

5 Compare the p-value to the significance level. If the p-value is smaller than the significance level, there is sufficient evidence to reject the null hypothesis.

6 Interpret the conclusion in context, remembering to make it clear that the conclusion is not a statement of certainty, but of significance.

Notice that there are two possible conclusions you can reach from a hypothesis test:

- the sample provides sufficient evidence to reject H_0 in favour of H_1

or

- the sample does not provide sufficient evidence to reject H_0.

In the latter case, you cannot say that H_0 is correct, just that you have not found sufficient evidence to reject it.

Once you have reached a conclusion, it is important to interpret it in the context of the question. So in the previous example you need to write something like 'There is sufficient evidence that the new drug is less effective', rather than just 'There is sufficient evidence that the proportion is less than 78%'.

Tip

You should always use the word 'evidence' in your conclusion, to make it clear that it is not certain.

WORK IT OUT 18.1

Which of these conclusions to hypothesis tests are incorrectly written, and why?

1 Reject H_0. The proportion of red flowers has decreased.

2 There is sufficient evidence to reject H_0, and thus sufficient evidence that the proportion has increased.

3 There is insufficient evidence to reject H_0. The proportion of girls in the club is probably still 45%.

4 Accept H_0, as there is insufficient evidence that the percentage of A grades has increased.

5 There is evidence to accept H_0; the dice isn't biased.

WORKED EXAMPLE 18.9

A sociologist believes that more boys than girls are born during war time. In a sample of 200 babies born in countries at war, 116 were boys.

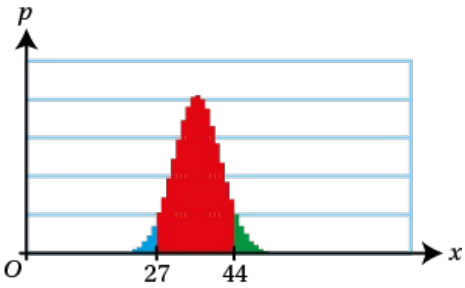
Do these data support the sociologist's theory at the 5% significance level?

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Let p be the proportion of boys in the population. $\mathrm{H}_0 : p = 0.5$ $\mathrm{H}_1 : p > 0.5$	You want to find out whether there is evidence that the proportion of boys is more than $\frac{1}{2}$. So the null hypothesis is that the proportion is $\frac{1}{2}$ and the alternative hypothesis is that it is greater.
Let X be the number of boys in a sample of 200 babies. If H_0 is correct then $X \sim B(200, 0.5)$. Test statistic: $X = 116$	The test statistic is the number of boys in the sample. Since you are looking for evidence that the population proportion is greater than 0.5, you need to find the probability of observing this number, or more, if H_0 is correct.
$\mathrm{P}(X \geqslant 116) = 1 - \mathrm{P}(X \leqslant 115)$ $= 0.0141$	Use your calculator to find the cumulative probability.
$0.0141 < 0.05$	Compare the probability to the significance level. Large probability means that the event is 'not unlikely'.
so there is sufficient evidence to reject H_0.	The conclusion is not definite – you must use the phrase 'sufficient/insufficient evidence'.
At the 5% significance level, there is evidence to support the sociologist's theory.	The conclusion needs to be interpreted in the context of the question.

In Worked example 18.9, the alternative hypothesis was that the population proportion has increased. This is called a **one-tail test**. Sometimes you might have a reason to believe that the population proportion has changed, but you can't predict in which direction. In this case you have to do a **two-tail test**. The only difference is that you need to compare the p-value to half the significance level, because both very small and very large values of the test statistic would provide evidence against H_0.

For example, the diagram shows the binomial distribution $\mathrm{B}(100, p)$ under H_0: $p = 0.36$. The blue and the green regions both correspond to probabilities of 5%. So observing the sample in either the blue or the green section would lead you to reject the null hypothesis at the 10% significance level.



Key point 18.11

- In a one-tail test the alternative hypothesis is of the form $p < \alpha$ or $p > \alpha$.
  You need to compare the p-value to the significance level.
- In a two-tail test the alternative hypothesis is of the form $p \neq \alpha$.
  You need to compare the p-value to half the significance level.

In the third question at the start of this section, we wanted to know whether the proportion of voters supporting a particular political party has **changed**. Without any further information you cannot anticipate whether the proportion has increased or decreased, so you have to do a two-tail test.

WORKED EXAMPLE 18.10

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In the last general election, Party Z won 36% of the vote. An opinion poll surveys 100 people and finds that 45 support Party Z.

Does this provide sufficient evidence at the 10% significance level that the proportion of voters who support Party Z has changed?

Let p be the proportion of voters supporting Party Z.

$\mathrm{H}_0 : p = 0.36$

$\mathrm{H}_1 : p \neq 0.36$

Start by stating the hypotheses. This is a two-tail test because you are looking for evidence of change, rather than just increase or decrease. Remember to define the meaning of the parameter p.

Let X be the number of supporters of Party Z in a sample of 100 people.

If H_0 is true, $X \sim \mathrm{B}(100, 0.36)$.

If H_0 is correct, the number of supporters in a sample will follow a binomial distribution with the probability given in H_0.

Test statistic: $\mathrm{P}(X = 45)$

$$\begin{aligned}\mathrm{P}(X \geqslant 45) &= 1 - \mathrm{P}(X \leqslant 44) \\ &= 0.0397\end{aligned}$$

Since 45 out of 100 is more than 36%, you need to calculate the probability in the 'right tail'.

$0.0397 < 0.05$ so there is sufficient evidence to reject H_0.

The significance level is 10%, so for a two-tail test you compare the p-value to half the significance level, which is 5%, or 0.05.

There is evidence, at the 10% significance level, that the proportion of voters who support Party Z has changed.

Remember to interpret the conclusion in the context of the question, making it clear that it is not certain.

Tip

Be sure to think carefully about whether you have a one-tail or two-tail test. Remember to halve the significance level in each tail with a two-tail test.

WORK IT OUT 18.2

A dice is rolled 5 times and 3 sixes are observed. Test at the 5% significance level if the dice is biased.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$\mathrm{H}_0 : p = \frac{1}{6}$

$\mathrm{H}_1 : p \neq \frac{1}{6}$

$X \sim \mathrm{B}\left(5, \frac{1}{6}\right)$

$\mathrm{P}(X = 3) = 0.032 < 0.05$

So reject H_0.

Solution 2

$\mathrm{H}_0 : p = \frac{1}{6}$

$\mathrm{H}_1 : p > \frac{1}{6}$

$X \sim \mathrm{B}\left(5, \frac{1}{6}\right)$

$\mathrm{P}(X \geqslant 3) = 0.032 < 0.05$

So reject H_0.

Solution 3

$\mathrm{H}_0 : p = \frac{1}{6}$

$\mathrm{H}_1 : p \neq \frac{1}{6}$

$X \sim \mathrm{B}\left(5, \frac{1}{6}\right)$

$\mathrm{P}(X \geqslant 3) = 0.035 > 0.025$

So there is not sufficient evidence to reject H_0.

EXERCISE 18B

1 Write down the null and alternative hypotheses for the following tests, defining the meaning of any parameters.

a i Daniel wants to test whether the proportion of children in his school who like football is higher than 60%.

ii Elsa wants to find out whether the proportion of households with a pet is higher than 1 in 3.

b i The proportion of faulty components produced by a machine was 6% and the manager wants to check whether this has decreased following a service.

ii Joseph thinks that fewer than half of all children eat 5 or more pieces of fruit a day and wants to confirm this by using a hypothesis test.

c i Sofia has a coin which she thinks is biased, and wants to use a hypothesis test to check this.

ii Max knows that, last year, 26% of entries in AS level Psychology were graded 'A' and wants to check whether this proportion has changed.

2 In each question you are given null and alternative hypotheses (where p stands for the population proportion), the significance level and the observed data. Decide whether or not there is sufficient evidence to reject the null hypothesis.

a i $\mathrm{H}_0 : p = 0.3$, $\mathrm{H}_1 : p > 0.3$, significance level 5%, observed 15 successes out of 40 trials

ii $\mathrm{H}_0 : p = 0.6$, $\mathrm{H}_1 : p > 0.6$, significance level 10%, observed 23 successes out of 45 trials

b i $\mathrm{H}_0 : p = 0.5$, $\mathrm{H}_1 : p < 0.5$, significance level 10%, observed 18 successes out of 40 trials

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ii $H_0 : p = 0.45$, $H_1 : p < 0.45$, significance level 3%, observed 23 successes out of 60 trials

c **i** $H_0 : p = 0.4$, $H_1 : p \neq 0.4$, significance level 8%, observed 35 successes out of 100 trials

ii $H_0 : p = 0.8$, $H_1 : p \neq 0.8$, significance level 5%, observed 17 successes out of 20 trials

3 It is known that in the UK, 63% of households own at least one car. David, who lives in a big city, believes that in his neighbourhood car ownership is lower than this. He uses a hypothesis test, based on the binomial distribution, to confirm this.

a State suitable null and alternative hypotheses for his test.

David surveys a random sample of 50 households in his neighbourhood and finds that 29 of them own at least one car.

b Use this data to test David's hypothesis at the 10% significance level. State your conclusion clearly.

4 The 2011 census found that 68% of 16- to 19-year-olds in a particular town attended a sixth form college. In 2015 a sample of 60 teenagers in this age range was surveyed and it was found that 46 of them attended a sixth form college.

Is there evidence, at the 5% significance level, that the proportion of 16- to 19-year-olds attending a sixth form college has increased?

5 Rahul has a six-sided dice that he believes is biased so that the probability of rolling a 3 is smaller than $\frac{1}{6}$. He rolls the dice 30 times and gets four $3s$.

Does this provide sufficient evidence to support Rahul's belief?

6 In a certain local authority, the proportion of workers who drive to work is known to be 33%. In a sample of 28 teachers from a particular college, 12 drive to work. Is there evidence, at the 5% significance level, that the teachers at this college are more likely to drive to work than the average for the local authority?

7 An established treatment for a particular disease is known to be effective in 82% of the cases. A doctor devises a new treatment that she believes is even more effective. She uses the treatment on a random sample of 50 patients and finds that the new treatment is effective in 43 cases.

Does this data support the doctor's belief at the 2% significance level?

Worksheet

For further examples and practice of hypothesis testing see Support sheet 18.

8 A teacher knows that in his old school, a third of all sixth-formers had a younger sibling at the school. He moves to a new school and wants to find out whether this proportion is different. He asks a sample of 60 sixth-formers, and finds that 27 of them have a younger sibling at the school.

Conduct a hypothesis test at the 5% significance level to decide whether there is evidence that the proportion of sixth-formers with a younger sibling at the new school is different from the old school.

9 Angela is playing a board game with her friends, but thinks the dice is biased and that a 6 is rolled too infrequently. In the subsequent 40 rolls of the dice she got only three 6s.

Test Angela's belief at the 10% significance level.

10 A large athletics club had the same running coach for several years. Records show that 28% of his athletes could run 100 metres in under 12 seconds. The club brings in a new coach and over the following year, 26 out of a sample of 75 athletes recorded 100-metre times under 12 seconds.

Do these data support the hypothesis that the proportion of athletes who can run 100 metres in under 12 seconds has changed? Use the 5% significance level for your test.

11 The proportion of students in Year 13 in favour of a new uniform is known to be 70%. Rhianna wants

to find out whether the proportion in Year 12 is the same. She proposes to test the null hypothesis $\mathrm{H}_0 : p = 0.7$ against two different alternative hypotheses, $\mathrm{H}_1 : p < 0.7$ and $\mathrm{H}_2 : p \neq 0.7$, using the 10% significance level.

Rhianna asks a sample of 25 Year 12 students. The data give her sufficient evidence to reject H_0 in favour of H_1, but not in favour of H_2.

How many students in Rhianna's sample were in favour of the new uniform?

12 A student tests the hypothesis $\mathrm{H}_0 : p = 0.4$ against $\mathrm{H}_1 : p > 0.4$, where p is the proportion of brown cats of a particular breed.

In a sample of 80 cats of this breed 40 were brown, and this leads him to reject the null hypothesis.

What can you say about the significance level he used for his test?

13 A doctor wants to find out whether the proportion of people suffering from a certain genetic condition has decreased from its previous value, q. She decides to conduct a hypothesis test at the 5% significance level, using a sample of 120 patients. However, after doing some calculations, she realises that, even if none of her sample had the condition, this would not provide sufficient evidence that the proportion has decreased from q.

Find the maximum possible value of q.

Section 3: Critical region for a hypothesis test

Suppose you toss a coin 200 times and get 86 heads. Does this provide evidence that the coin is biased and that the probability of heads is less than $\frac{1}{2}$?

You can now do a hypothesis test to address this question. If you conduct a hypothesis test using a 1% significance level, you will find that there is insufficient evidence that the coin is biased.

The next logical question to ask is: How many heads would provide sufficient evidence that the coin is biased?

WORKED EXAMPLE 18.11

In order to find out whether a coin is biased against heads, Roberto decides to test the hypotheses

$\mathrm{H}_0 : p = \frac{1}{2}$ against $\mathrm{H}_1 : p < \frac{1}{2}$

where p is the probability of the coin showing heads. He tosses the coin 200 times and uses the 1% level of significance.

What is the greatest number of heads he can observe and still have sufficient evidence that the coin is biased?

Let X be the number of heads out of 200 coin tosses.

State the distribution of the test statistic if the null hypothesis is true.

If H_0 is true, $X \sim \mathrm{B}\left(200, \frac{1}{2}\right)$

Significance level is 1%, so look for a number k such that $\mathrm{P}(X \leqslant k) < 0.01$

$\mathrm{P}(X \leqslant 85) = 0.020 > 1\%$

$\mathrm{P}(X \leqslant 84) = 0.014 > 1\%$

$\mathrm{P}(X \leqslant 83) = 0.0097 < 1\%$

You know from the previous example in the text that 86 heads does not provide sufficient evidence, so try values smaller than 86.

In order to have sufficient evidence against H_0 at the 1% significance level, Roberto would need to observe 83 or fewer heads out of 200 coin tosses.

The range of values of the test statistic found in Worked example 18.11 ($X \leqslant 83$) is called the **critical region** for the test.

Key point 18.12

The critical region (or **rejection region**) for a test is the set of values of the test statistic that provide sufficient evidence to reject the null hypothesis.

The value at the edge of the critical region is called the **critical value**.

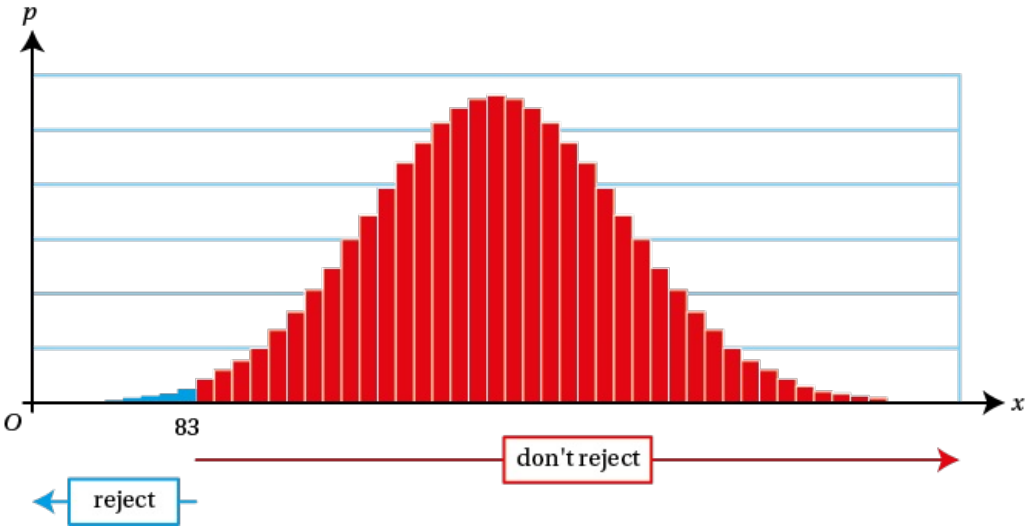
The **acceptance region** is the set of values of the test statistic that do not provide sufficient evidence to reject the null hypothesis.

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So in Worked example 18.11 the critical value is 83, the critical region is $X \leqslant 83$ and the acceptance region is $X \geqslant 84$.

Tip

The name 'acceptance region' is misleading: you should never say that you accept the null hypothesis, only that you don't have suffcient evidence to reject it.



Note that the form of the critical region depends on the form of the alternative hypothesis. If you were looking for evidence that the population proportion has increased, the critical region would be of the form $X \geqslant k$.

WORKED EXAMPLE 18.12

The proportion of students getting an A in AS Mathematics is currently 33%. A publisher produces a new textbook that they hope will lead to improved performance. They trial their textbook with a sample of 120 students and want to test their hypothesis at the 5% significance level.

Find the critical region for this test.

Let p be the proportion of A grades.

$H_0: p = 0.33$

You should always start by stating the hypotheses. Remember to define the meaning of p.

$H_1: p > 0.33$

The null hypothesis states the current, known proportion.

Let X be the number of A grades in the sample of 120 students.

If H_0 is true then $X \sim B(120, 0.33)$

Next state the distribution you are going to use.

$$P(X \geqslant 47) = 1 - P(X \leqslant 46) = 0.091 > 5\%$$
$$P(X \geqslant 48) = 1 - P(X \leqslant 47) = 0.064 > 5\%$$
$$P(X \geqslant 49) = 1 - P(X \leqslant 48) = 0.044 < 5\%$$

If H_0 were true you would expect around $120 \times 0.33 \approx 40$ A grades, so you only need to check values of X larger than 40. Use your calculator to find the probabilities.

The critical region is $X \geqslant 49$.

Look for the first probability smaller than the significance level.

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Tip

Some calculators can do 'inverse binomial distribution' which tells you where to look for the critical value. You still need to show the probability calculations, as illustrated in Worked example 18.12.

Once you have found the critical region you can easily tell whether a value of the test statistic provides sufficient evidence against the null hypothesis.

Key point 18.13

If the value of the test statistic is inside the critical region you have sufficient evidence to reject H_0.

So if, for example, 52 out of the 120 students got A grades, the publisher would have sufficient evidence to claim that their textbook leads to improved performance.

As with any conclusion from a statistical investigation, there is no proof that the improvement was actually caused by the textbook – there could be other factors. Perhaps more importantly, since you are only using a sample, there is no guarantee that the proportion of A grades for the whole population has increased. It is possible that the selected sample was unrepresentative, and that most other samples would result in a smaller proportion of A grades.

Tip

The critical value is included in the critical (rejection) region.

The meaning of the significance level

To judge how reliable your test is, it is useful to estimate the probability of drawing an incorrect conclusion. It is possible that your sample leads you to reject a correct null hypothesis. In Worked example 18.12, even if the proportion of A grades is still 33%, a sample could suggest that the proportion has increased. That would happen if the test statistic (the number of A grades in the sample) is in the critical region, in this case $X \geqslant 49$. But you have already found the probability of this happening: $\mathrm{P}(X \geqslant 49) = 0.044$; so there is a 4.4% chance that you reject the null hypothesis even though it is correct.

Key point 18.14

The probability of rejecting a correct null hypothesis is the same as the probability of the test statistic being in the critical region. It is always smaller than (or equal to) the significance level of the test.

Fast forward

A In Student Book 2 you will test hypotheses involving continuous distributions; in that case the probability of rejecting a correct null hypothesis will be exactly equal to the significance level.

WORKED EXAMPLE 18.13

A machine produces smartphone parts. Previous experience suggests that, on average, 7 in every 200 parts are faulty. After the machine was accidentally moved, a technician suspects that the proportion of faulty parts may have increased. She decides to test this hypothesis using a random

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sample of 85 parts.

a State suitable null and alternative hypotheses.

The technician decides that the critical region for the test should be $X \geqslant 5$. After checking her sample, she finds that 4 parts are faulty.

b State what conclusion she should draw and justify your answer.

c What is the probability of incorrectly rejecting the null hypothesis?

a $\mathrm{H}_0: p = 0.035$

$\mathrm{H}_1: p > 0.035$

(where p is the proportion of faulty parts)

The null hypothesis is that the proportion hasn't changed, so the probability of a part being faulty is $\frac{7}{200} = 0.035$.

b The test statistic $X = 4$ is not in the critical region, so there is not sufficient evidence to reject H_0.

Check whether the test statistic is in the critical region.

There is not sufficient evidence that the proportion of faulty parts has increased.

Interpret the conclusion in the context of the question.

c If $X \sim \mathrm{B}(85, 0.035)$ then the probability of rejecting H_0 is

$$\begin{aligned}\mathrm{P}(X \geqslant 5) &= 1 - \mathrm{P}(X \leqslant 4) \\ &= 0.177\end{aligned}$$

Incorrectly rejecting the null hypothesis' means that H_0 is correct (so $p = 0.035$) but the test statistic happens to be in the critical region (so $X \geqslant 5$).

Explore

It is also possible to reach a different type of incorrect conclusion: your sample may not provide sufficient evidence against H_0 even if it is incorrect. This is especially likely to happen if the proportion has changed by just a little bit. Find out about Type I and Type II errors.

EXERCISE 18C

1 Find the critical regions for the following hypothesis tests. You are given the null and alternative hypotheses, the significance level (SL) and the sample size (n).

a i $\mathrm{H}_0 : p = 0.6, \mathrm{H}_1 : p < 0.6, \mathrm{SL} = 5\%, n = 50$

ii $\mathrm{H}_0 : p = 0.2, \mathrm{H}_1 : p < 0.2, \mathrm{SL} = 10\%, n = 40$

b i $\mathrm{H}_0 : p = 0.5, \mathrm{H}_1 : p < 0.5, \mathrm{SL} = 7\%, n = 120$

ii $\mathrm{H}_0 : p = 0.8, \mathrm{H}_1 : p < 0.8, \mathrm{SL} = 1\%, n = 75$

c i $\mathrm{H}_0 : p = 0.6, \mathrm{H}_1 : p > 0.6, \mathrm{SL} = 5\%, n = 50$

ii $\mathrm{H}_0 : p = 0.2, \mathrm{H}_1 : p > 0.2, \mathrm{SL} = 10\%, n = 40$

d i $\mathrm{H}_0 : p = 0.5, \mathrm{H}_1 : p > 0.5, \mathrm{SL} = 7\%, n = 120$

ii $\mathrm{H}_0 : p = 0.8, \mathrm{H}_1 : p > 0.8, \mathrm{SL} = 1\%, n = 75$

2 For each of the tests in question 1, find the probability of incorrectly rejecting the null hypothesis.

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3 Ayesha is trying to find out whether it is true that students studying A level Mathematics are more likely to be boys than girls. She sets up the following hypotheses:

$\mathrm{H}_0 : p = 0.5$, $\mathrm{H}_1 : p < 0.5$, where p is the proportion of girls studying A level Mathematics.

She uses a sample of 30 A level students from her college, and decides to test her hypotheses at the 10% significance level.

Find the critical region for her test.

4 A company is testing a new drug. They want to find out whether the drug cures a certain disease in more than 85% of cases.

a State suitable null and alternative hypotheses, defining any parameters.

The company decide to conduct their test at the 5% significance level, using a sample of 180 patients.

b Let X be the number of patients who are cured after using the drug. Find the critical region for the test.

5 A manufacturer knows that, in the past, 12% of the population have purchased their products. After a new advertising campaign, they believe that this proportion has increased. The marketing manager wants to test this belief using a random sample of 200 people.

a Write down suitable null and alternative hypotheses.

The marketing manager decides that the rejection for the test should be $X \geqslant 32$, where X is the number of people in the sample who have purchased their products.

b After collecting the data, it is found that 30 people have purchased the manufacturer's products. What should the marketing manager conclude?

c Find the probability of incorrectly rejecting the null hypothesis.

6 Sean has an eight-sided dice and wants to check whether it is biased by looking at the probability, p, of rolling a 4. He sets up the following hypotheses:

$$\mathrm{H}_0 : p = \frac{1}{8} \quad \mathrm{H}_1 : p \neq \frac{1}{8}$$

To test them he decides to roll the dice 80 times and reject the null hypothesis if the number of 4s is greater than 15 or fewer than five.

a Let X be the number of 4s observed out of the 80 rolls. State the name given to the region $5 \leqslant X \leqslant 15$.

b What is the probability that John incorrectly rejects the null hypothesis?

7 A hypothesis test is proposed to decide whether the proportion of children in a certain country who walk to school is greater than 30%.

a State suitable hypotheses, defining any parameters.

The test is to be carried out at the 5% significance level using a random sample of 170 children.

b Find the critical region for the test.

After the data was collected, it was found that 64 of the 170 children walked to school.

c State the conclusion of the test.

8 A driving school has records showing that, over a long period, 72% of its students passed their test on the first attempt. Under a new management, some of the procedures have changed and they want to find out whether the pass rate has changed.

a Defining any parameters, state suitable hypotheses.

The hypotheses are tested using a random sample of 50 students. It is decided that the null hypothesis should be accepted if between 30 and 40 (inclusive) of the students pass the test on the first attempt.

b State the critical region for the test.

c Find the probability of incorrectly rejecting the null hypothesis. What can be said about the significance level of the test?

Checklist of learning and understanding

- A **population** is the set of all individuals or items of interest in a statistical investigation. A **population parameter** is some numerical characteristic of the population (such as the mean or the range).
- A **sample** is a subset of the population. A **statistic** is some numerical characteristic of the sample that is used to estimate the population parameter of interest.
- A good sampling procedure avoids **bias**, so the sample is more likely to be **representative** of the population. However, even an unbiased sampling procedure may accidentally produce an unrepresentative sample.
- Some common sampling procedures include:
  - **Simple random sampling**: Each possible sample of a given size has equal probability of being selected. This can be achieved by using a random number generator.
  - **Opportunity sampling**: A sample is made up of individuals who are available and willing to take part.
  - **Systematic sampling**: Taking participants at regular intervals from a list of the population.
  - **Stratified sampling**: The proportion of members of the sample with certain characteristics is fixed to be the same as in the whole population; the individuals for each group are then selected randomly.
  - **Quota sampling**: Groups are determined as for stratified sampling, but the individuals for each group are then selected by opportunity sampling.
  - **Cluster sampling**: A number of subsets of the population (clusters) are selected, and then a simple random sample is taken from each cluster.
- A **hypothesis test** determines whether there is evidence that the value of a population parameter has changed from a previously known, or assumed one. The conclusion from a hypothesis test is never certain; two possible types of conclusion are:
  - There is insufficient evidence that the value of the population parameter has changed.
  - There is sufficient evidence that the value of the population parameter has changed.
- To carry out a hypothesis test, with a given **significance level**, for the proportion of the binomial distribution, you must follow these steps:
  1. State the null and alternative hypotheses, defining the meaning of any parameters.
  2. State the binomial distribution which the test statistic would follow if the null hypothesis was correct.
  3. Calculate the probability of observing the given value of the test statistic, or more extreme (the ***p*-value**).
  4. Compare this probability to the significance level. Reject the null hypothesis if the p-value is smaller than the significance level.

     In a **two-tail test**, compare the probability to half the significance level.
  5. State the conclusion of the test, interpreting it in context and making it clear that there is some uncertainty (using the word '**evidence**').
- The **critical region** (or **rejection region**) for a hypothesis test is the set of values of the test statistic that lead us to reject the null hypothesis. The remaining values form the **acceptance region**.

- For a two-tail test, the critical region is made up of two parts.
- The probability of incorrectly rejecting the null hypothesis is equal to the probability of the critical region. This is always less than or equal to the significance level of the test.

Mixed practice 18

1 A market researcher is asked to conduct a survey outside a library. He is asked to sample 100 male library users, 100 female library users, 50 males who have not been into the library and 50 females who have not been into the library.

What type of sampling method is this?

2 The organisers of the school concert want to find out how many of the students are planning to attend the concert. The school has 48 different tutor groups, and they decide to select a sample of students in the following way:

- They choose five tutor groups randomly.
- From each tutor group, they select a random sample of ten students.

a What name is given to this type of sampling procedure?

b Explain why this procedure might not give a representative sample in this case.

The organisers later decide that they should take a simple random sample of 50 students instead.

c Describe how they might obtain such a sample.

3 Gavin has a six-sided dice that he thinks is biased and shows more 5s than it should. He wants to conduct a hypothesis test to test his belief.

a State suitable null and alternative hypotheses.

Gavin rolls the dice 75 times and obtains $185s$.

b Conduct the test at the 5% significance level, stating your conclusion clearly.

4 Lisa, who takes the bus to school, is late for school on average once in every eight days. She has recently moved closer to the school and now walks. In the last 30 days she was late only twice.

Is there evidence, at the 10% significance level, that the probability of Lisa being late for school has decreased? State your hypotheses and your conclusion clearly.

5 A village has a population of 600 people. A sample of 12 people is obtained as follows. A list of all 600 people is obtained and a three-digit number, between 001 and 600 inclusive, is allocated to each name in alphabetical order. Twelve three-digit random numbers, between 001 and 600 inclusive, are obtained and the people whose names correspond to those numbers are chosen.

a Find the probability that the first number chosen is 500 or less.

b When the selection has been made, it is found that all of the numbers chosen are 500 or less. One of the people in the village says: ‘The sampling method must have been biased’. Comment on this statement.

6 The head teacher of a school asks for volunteers from among the pupils to take part in a survey on political interests.

a Explain why a sample consisting of all the volunteers is unlikely to give a true picture of the political interests of all pupils in the school.

b Describe a better method of obtaining the sample.

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[Question part reference style adapted]

7 A doctor knows that 20% of people suffer from side effects when treated with a certain drug. He wants to see if the proportion of people suffering from side effects is lower with a new

drug. He looks at a random sample of 30 people treated with the new drug.

What is the largest number of people who could suffer from side effects and still conclude at 5% significance that the new drug has a lower proportion of side effects?

8 In the UK, the proportion of families who own their home (as opposed to renting) is 64%. Sabina wants to find out whether this proportion is different in Germany. She surveys a random sample of 180 families in Germany and finds that 98 of them own their homes.

Conduct a hypothesis test at the 5% significance level to test whether the proportion of families in Germany who own their home is different from that in the UK.

9 **a** Define a simple random sample.

Aneka is investigating attitudes to sport among students at her school. She decides to carry out a survey using a sample of 70 students. There are the same number of boys and girls at the school, so Aneka randomly chooses 35 boys and 35 girls.

b **i** State the name for this type of sample.

ii Explain why in this case, this type of sample is better than a simple random sample.

One of Aneka's questions is about participation in school sports teams. She wants to find out whether more than 40% of students play for a school team. She sets up the following hypotheses: $\mathrm{H}_0 : p = 0.4$,$\mathrm{H}_1 : p > 0.4$, where p is the proportion of students who play for a school sports team.

c Find the critical region for the hypothesis test at the 10% significance level, using a sample of 70 students.

d What is the probability of incorrectly rejecting the null hypothesis?

e In Aneka's sample, 32 students play for a school team. State the conclusion of the test.

10 In a rearrangement code, the letters of a message are rearranged so that the frequency with which any particular letter appears is the same as in the original message. In ordinary German the letter e appears 19% of the time. A certain encoded message of 20 letters contains one letter e.

a Using an exact binomial distribution, test at the 10% significance level whether there is evidence that the proportion of the letter e in the language from which this message is a sample is less than in German, i.e., less than 19%.

b Give a reason why a binomial distribution might not be an appropriate model in this context.

OCR, GCE Mathematics, Paper 4733, June 2007
[Question part reference style adapted]

11 A test is constructed to see if a coin is biased. It is tossed 10 times and if there are 10 heads, 9 heads, 1 head or 0 heads it is declared to be biased.

For each of the following, explain whether or not it could be the significance level for this test:

a 1%

b 2%

c 10%

d 20%

FOCUS ON … PROOF 3

Using mathematical notation

In Chapter 16, Section 2, you saw that there are two different formulae for standard deviation:

$$\sqrt{\sum \frac{(x-\bar{x})^2}{n}} \quad and \quad \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$$

In these formulae, x represents the individual data items, n is the number of items, $\bar{x}$ is the mean.

Fast forward

You will learn more about the $\sum$ symbol (upper case sigma) if you study the Statistics option of Further Mathematics; for now, you just need to know that it means 'add up'.

Here is a proof that the two formulae are equivalent. It works with the expression under the square root (this is called the variance).

Prove: $\frac{\sum (x-\bar{x})^2}{n} = \frac{\sum x^2}{n} - \bar{x}^2$

Rewind

Remember from Chapter 16, Section 2, that the symbol σ (lower case sigma) is used for standard deviation.

PROOF 9

$$LHS = \frac{\sum (x-\bar{x})^2}{n}$$

$$= \frac{\sum (x^2 - 2x\bar{x} + \bar{x}^2)}{n}$$

Start by expanding the brackets on the left and see if you can simplify it to look like the expression on the right.

$$= \frac{\sum x^2}{n} - \frac{\sum 2x\bar{x}}{n} + \frac{\sum \bar{x}^2}{n}$$

There are three terms added up in the numerator. The first one looks like the first term in the expression you are trying to prove, so it seems sensible to split the sum into three parts.

$$= \frac{\sum x^2}{n} - 2\bar{x}\frac{\sum x}{n} + \frac{n\bar{x}^2}{n}$$

Now remember that you are adding up the terms corresponding to different values of x. There are n things to add in each sum. The mean, $\bar{x}$, is a constant. The sum in the last term just says 'add $\bar{x}^2$ n times, so it equals $n\bar{x}^2$.
In the middle sum, $2\bar{x}$ is a constant multiplying each 'x' , so it can be taken outside the sum.

$$= \frac{\sum x^2}{n} - 2\bar{x}\bar{x} + \bar{x}^2$$

$$= \frac{\sum x^2}{n} - 2\bar{x}^2 + \bar{x}^2$$

	The first term is exactly what you want. The last term equals $\bar{x}^2$, but it has a $+$ instead of $-$. But look at the middle term: $\frac{\sum x}{n}$ means 'add up all the values and divide by n'; this gives the mean, $\bar{x}$.
$= \frac{\sum x^2}{n} - \bar{x}^2 = RHS,\ as\ required.$	You have reached the expression on the right, so the proof is complete.

Questions

1. Was the use of $\sum$ notation helpful, or did you find it confusing? Would it be easier to write something like $a + b + c...$ instead? Try rewriting the proof using different notation.

2. Somewhat surprisingly, the equivalence of the two standard deviation formulae can be used to prove a pure mathematical result about inequalities:

   For any real numbers $x_1, x_2, \ldots, x_n$:

   $$\sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}} \geqslant \frac{x_1 + x_2 + \ldots + x_n}{n}$$

   Can you prove this result?

3. You know that skewness is a measure of how asymmetrical a distribution is. One measure of skewness is called the Pearson's coefficient of skewness, and has the formula:

   $\frac{1}{\sigma^3} \frac{\sum (x - \bar{x})^3}{n}$ where σ is the standard deviation.

   Prove that this formula is equivalent to $\frac{1}{\sigma^3} (\frac{\sum x^3}{n} - 3\bar{x}\sigma^2 - \bar{x}^3)$.

Explore

The two expressions in the inequality are called the 'root mean squared' (RMS) and the 'arithmetic mean' (AM) of the set of n numbers. There are other types of mean as well; for example 'geometric mean' (GM) and 'harmonic mean' (HM). When all the numbers are positive, the four means always come in the same order: $\mathrm{HM} \leqslant \mathrm{GM} \leqslant \mathrm{AM} \leqslant \mathrm{RMS}$. Can you find some proofs of this result?

FOCUS ON … PROBLEM SOLVING 3

Experimental design in statistics

Many questions about populations require extremely creative methods to get to an answer. This section looks at two examples.

1 Persuading people to reveal embarrassing traits

Psychologists used to find that when people were asked directly about traits they found embarrassing (such as their drug use, sexuality, criminal history or infidelity) they would not be truthful. Even anonymous surveys underpredicted the rates compared to a method called ‘randomised response’ designed by S.L Warner in 1965. In this method people were sent to a private booth and asked to flip a coin. If they got a head on the coin flip they were asked to write ‘Yes’, otherwise they were asked to give an honest answer to the question. The researchers could then explain that they could estimate the overall proportion of the population answering yes, without knowing the result for any individual who answered yes.

2 How many adult cod are in the North Sea?

It is not practical to count all of the fish, so a method called ‘mark and recapture’ is used. A sample of fish is captured and marked then some time later another sample is taken. If we assume that the proportion of marked fish in the sample is the same as the proportion in the population, we can estimate the population size.

Did you know?

You might like to use the Internet to research this question – it turns out that the answer varies greatly depending upon definitions of ‘adult cod’, something which caused some controversy in the media.

Questions

1 A sample of 1000 people were asked if they had ever stolen goods from a shop. Randomised response was used.

a If 612 people responded yes estimate the proportion of the whole population who have stolen goods from a shop.

b Is this likely to be an underestimate or overestimate?

2 A sample of $100\,00$ adult cod is caught in the North Sea and marked using microchips. Some time later another sample of $100\,00$ adult cod is caught and one of the fish is found to have a microchip.

a Use this information to estimate the number of adult cod.

b What assumptions do you have to make in this calculation?

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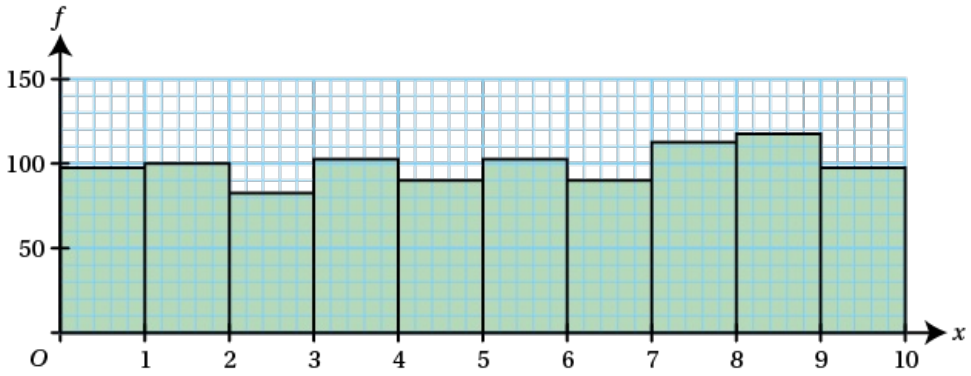
Using simulation to test statistical models

Once you have created a probability model you often want to use it to make predictions. This is useful to both check whether the outcomes predicted fit in with known facts and to use the model to influence decisions about future events. Even when the maths is very complicated you can use technology to produce a simulation.

For example, the length of a side of a square is chosen at random to be a number between 0 and 10 (not necessarily a whole number) with all values equally likely. This is called a uniform distribution.
A spreadsheet can be used to do this using a command such as:

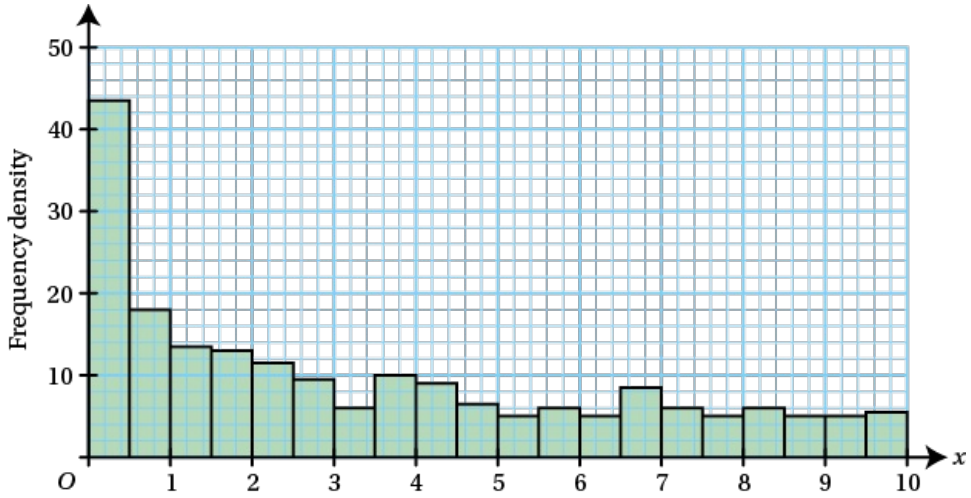
$$= 10*\text{RAND}()$$

If this is done over 1000 cells of the spreadsheet you can get a sample of random numbers between 0 and 10. This histogram illustrates one such sample.



Notice that it is not a perfect uniform distribution. There are some random variations. If you repeat the experiment you will get a slightly different distribution.

Many people's intuition says that the area should also be a uniform distribution. However, you can square all of your random numbers and form a new histogram:



This shows the initially surprising result that the probability of small values of area is significantly higher than large values. The probability of being less than 5 is about 70.3% and the mean is about 3.4. It is possible to predict this from theory, but most modern day statistics is done using this type of simulation method – often called 'Monte Carlo simulation' because it uses random numbers.

Questions

Try to answer the following questions using Monte Carlo simulation.

1. If points are picked at random inside a square of side 1, what is the probability that they fall within a circle of diameter one centred at the centre of the square? Use the result to estimate a value of pi.
2. Every time a man moves he either takes a step left or a step right, each with probability 0.5. Find the average distance away from his starting point after 100 steps.
3. What is the distribution of an average of four values taken from the same uniform distribution?



Did you know?

This problem is called a ‘drunken walk’ and it is of huge importance in many areas, from physics (where it is used to model the movement of atoms in a gas) to finance (where it is used to model the movement of share prices).

CROSS-TOPIC REVIEW EXERCISE 3

1 What is the probability of getting an average of 3 on two rolls of a fair dice?

2 The discrete random variable x has the probability distribution:

$\mathrm{P}(X = x = \ln kx$ for $x = 1, 2, 3, 4$. Find the exact value of k.

3 A biased six-sided dice follows this probability distribution:

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	0.12	0.18	p	0.27	0.22	0.08

a Find the probability that the dice lands on either 2 or 3.

b The dice is rolled 12 times. Find the probability that it lands on either 2 or 3 at least five times.

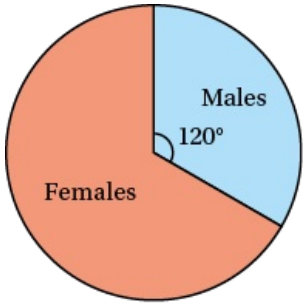
4 A student wants to find out how many people in her town work from home, and compare this to the national average of 11%. She thinks that this may vary according to the gender. She decides to interview people in the street until she has a sample of 20 men and 20 women.

a i State the name for this type of sampling.

ii How is it different from a stratified sample?

b In the sample of 40 people, she finds that 6 work from home. Does this provide sufficient evidence, at the 10% significance level, that the proportion of people in her town who works from home is greater than the national average?

5 **a** The number of males and females in Year 12 at a school are illustrated in the pie chart. The number of males in Year 12 is 128.

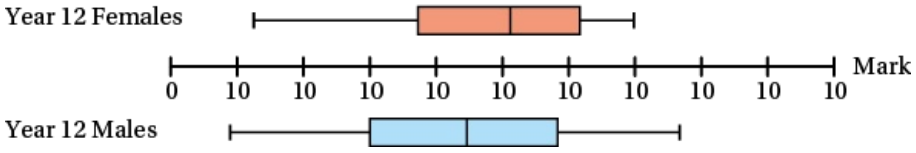


Year 12

i Find the number of females in Year 12.

ii On a corresponding pie chart for Year 13, the angle of the sector representing males is $150°$. Explain why this does not necessarily mean that the number of males in Year 13 is more than 128.

b All the Year 12 students took a General Studies examination. The results are illustrated in the box-and-whisker plots.



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i One student said “The Year 12 pie chart shows that there are more females than males, but the box-and-whisker plots shows that there are more males than females”.

Comment on this statement.

ii Give two comparisons between the overall performance of the females and the males in the General Studies examination.

iii Give one advantage and one disadvantage of using box-and-whisker plots rather than histograms to display the results.

c The mean mark for 102 of the male students was 51. The mean mark for the remaining 26 male students was 59. Calculate the mean mark for all 128 male students.

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[Question part reference style adapted]

6 All students in a class recorded how long, in minutes, it took them to travel to school that morning. The results are summarised in a cumulative frequency table:

Time in minutes (t)	Cumulative frequency
$0 < t \leqslant 6$	0
$0 < t \leqslant 10$	6
$0 < t \leqslant 15$	12
$0 < t \leqslant 20$	22
$0 < t \leqslant 30$	38
$0 < t \leqslant 6$	45

a Fill in the following frequency table:

Time in minutes (t)	Frequency
$0 < t \leqslant 6$	0
$6 < t \leqslant 10$	6
$10 < t \leqslant 15$	
$15 < t \leqslant 20$	
$20 < t \leqslant 30$	

$30 < t \leqslant 45$	

b Calculate an estimate for the mean and variance of the data.

c Explain why your answer is only an estimate.

7 Three data items are collected: $3, x^2$ and x. Find the minimum possible value of the mean.

8 Three data items are collected: $3, 7, x$. Find the smallest possible value of the variance.

9 In 2005, a city council implemented an advertising campaign to encourage more people to use the tram system. According to their surveys, in 2003 there was an average of 4560 tram journeys per day, and this rose to 4830 in 2006.

a Give two reasons why these numbers alone do not provide evidence that a larger proportion of people are using the tram system.

b A 2003 census found that 18% of workers in the city were using the tram to get to work. A 2005 survey of 230 workers found that 52 from them used the tram. Test, at the 5% significance level, whether the proportion of workers in the city who use the tram has changed.

10 At a building site the probability, $\mathrm{P}(A)$, that all materials arrive on time is 0.85. The probability, $\mathrm{P}(B)$, that the building will be completed on time is 0.60. The probability that the materials arrive on time and that the building is completed on time is 0.55.

a Show that events A and B are not independent.

b The same company builds 25 buildings.

i Calculate the probability that 20 of them are completed on time.

ii What assumptions did you need to make in your calculation?

c The company made some improvements to their procedures. After this, 20 out of the next 25 buildings were completed on time. Test at the 5% significance level whether the probability of a building being completed on time has increased.

11 Bryn has a coin with a probability p of returning 'heads' when flipped.

a Show that the probability of flipping two or three 'heads' out of three coin tosses is $f(p) = 3p^2 - 2p^3$

b Sketch the graph of $y = f(p)$ for $0 \leqslant p \leqslant 1$. Indicate the coordinates of any stationary points.

c The probability of flipping two or three 'heads' out of three coin tosses is $\frac{5}{32}$.
Show that $p = \frac{1}{4}$, and explain whether there are any other solutions.

12 The mean test score of a group of eight students is 34.5 and the variance of the scores is 5.75. Another student with the score of 38 joins the group. Find the new mean and variance of the scores.

13 Aseem records his monthly expenditure over one whole year in order to help him plan his budget. He finds that his average monthly expenditure for the eight months from January to August is $£620$ and that his average monthly expenditure over the whole year is $£586$. What was his average monthly expenditure for the four months from September to December?

14 The random variable x has the probability distribution given in the table:

x	1	2	3	4
$\mathrm{P}(X = x)$	0.26	e^{-k}	e^{-2k}	0.50

a Calculate the exact value of k.

b 70 independent observations of x are made. Find the probability that at least 30 of them are $X = 1$.

15 At a factory that makes crockery the quality control department has found that 10% of plates have minor faults. These are classed as 'seconds'. Plates are stored in batches of 12. The number of seconds in a batch is denoted by X

a State an appropriate distribution with which to model X Give the value(s) of any parameter(s) and state any assumptions required for the model to be valid.

Assume now that the model is valid.

b Find:

i $\mathrm{P}(X = 3)$

ii $\mathrm{P}(X \geqslant 1)$

c A random sample of 4 batches is selected. Find the probability that the number of these batches that contain at least 1 second is fewer than 3.

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[Question part reference style adapted]

16 The random variable x has the distribution $B(6, p)$.

a Given that $\mathrm{P}(X = 4) = 3\mathrm{P}(X = 3)$ find the value of p.

b Given instead that $\mathrm{P}(X = 4) = 3\mathrm{P}(X = 2)$ find the value of p.

17 The probability of an event occurring is found to be $\frac{1}{7}\left(x^2 - 14x + 38\right)$ where x is known to be an integer parameter. Find all possible values of x.

18 A village has a population of 726 people. A sample of 10 people is obtained as follows. A list of all 726 people is obtained and a three-digit number, between 001 and 726 inclusive, is allocated to each name in alphabetical order. Ten three-digit random numbers, between 001 and 726 inclusive, are obtained and the people whose names correspond to those numbers are chosen.

One of the people in the village wants to estimate the probability that all 10 of the numbers chosen are 500 or less.

a Explain why this probability cannot be calculated exactly using a binomial distribution. Explain also why it is possible to get a good estimate of this probability by assuming that the selection is done with replacement.

b Hence estimate the probability that all 10 of the numbers chosen are 500 or less.

When the selection has been made, it is found that all of the numbers chosen are 500 or less. Another one of the people in the village says, 'The sampling method must have been biased.'

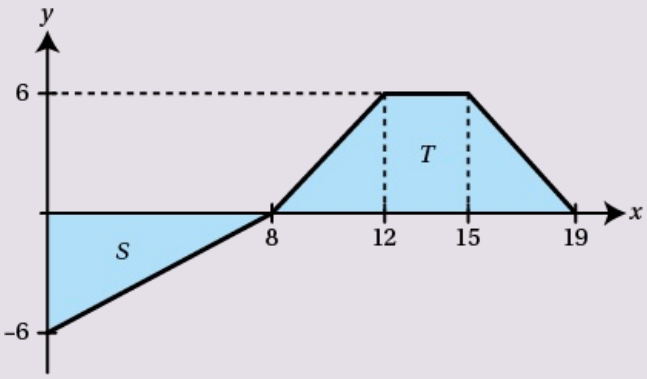
c Comment on this statement.

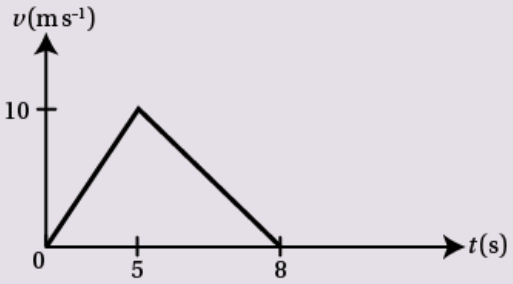
19 Introduction to kinematics

In this chapter you will learn how to:

- use the basic concepts in kinematics: displacement, distance, velocity, speed and acceleration
- use differentiation and integration to relate displacement, velocity and acceleration
- represent motion on a travel graph
- solve more complicated problems in kinematics: for example, involving two objects or several stages of motion.

Before you start...

GCSE	You should know how to find the gradient of a straight line connecting two points.	1 Consider the points $A(2, 5)$, $B(-1, 3)$ and $C(7, -2)$. Find the gradient of the straight line connecting: a A and C b B and A
Chapters 13 and 14	You should know how to differentiate polynomials.	2 a Given that $y = 3x^2 - 4x + \frac{5}{x}$: i find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ii find the gradient of the curve when $x = -1$. b Find the coordinates of the maximum point on the graph of $y = -x^3 + 12x + 5$.
GCSE	You should know how to find areas of triangles and trapeziums.	3 Find the areas marked S and T. 
Chapter 15	You should know how to use integration to find the area	4 Find the area enclosed by the graph of $y = 6x - 3x^2$ and the x-axis. (You may find it

	under a graph.	helpful to sketch the graph first.)
Chapter 15	You should know how to find the value of a constant of integration.	5 A curve has gradient $\frac{\mathrm{d}y}{\mathrm{d}x} = 5 - 6x^2$ and passes through the point $(1, 2)$. Find the equation of the curve.
GCSE	You should know how to interpret displacement–time and velocity–time graphs.	6 Use this velocity–time graph to find: a the acceleration of the object during the first 5 seconds b the distance travelled during the first 8 seconds. 

What is kinematics?

Kinematics describes the motion of objects: how their position, velocity and acceleration depend on time, as well as on how they are related to each other. It is **not** concerned with what causes the motion; this comes into dynamics, which you will meet in Chapter 21. Kinematics and dynamics together form the branch of applied mathematics called mechanics.

You may have studied some mechanics before, probably within Physics. In this course, the focus is on applying mathematical techniques from previous chapters to analyse problems in mechanics. You will use vectors, trigonometry, differentiation and integration, and represent information in several different forms, mainly graphs and equations.

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Section 1: Mathematical models in mechanics

The theory developed in this course is based on **mathematical models** of real-life situations. This means that some assumptions had to be made to simplify the situation, so that it can be described mathematically. It is important to consider how realistic these assumptions are, and whether improving the model would lead to substantially different results.

If you consider a car, for example, there are several different types of motion going on. The car might be travelling in a straight line, while the wheels are rotating around the axles and the wipers are moving left and right. Even if you are just interested in the position of the car, the front and the back aren't in exactly the same place. However, if all you need to know is how long it takes to drive from Newcastle to Manchester, then you can ignore all those details and consider the car as a single object, occupying a single point in space. We say that you are considering the car to be a **particle**.

The **particle model** is a mathematical model which assumes that an object occupies a single point in space and moves as one.

This does not mean that the object is very small. You could consider an aeroplane as a particle if all you were interested in was its distance from its destination – the length of the plane is negligible compared to the length of its journey. However, if you wanted to look at how different wing flaps move during turbulence, then the particle model would not be appropriate. In this course we are mainly concerned with the motion of particles.

Fast forward

A In Student Book 2 you will encounter situations in which an object cannot be modelled as a particle. For example, you will study **moments**, which affect how objects rotate.

Another assumption often made is that the object moves in a straight line. This means that its position can be described by a single number, such as its distance from its starting point. If the object is allowed to move in two or three dimensions then you need vectors to describe its position.

Fast forward

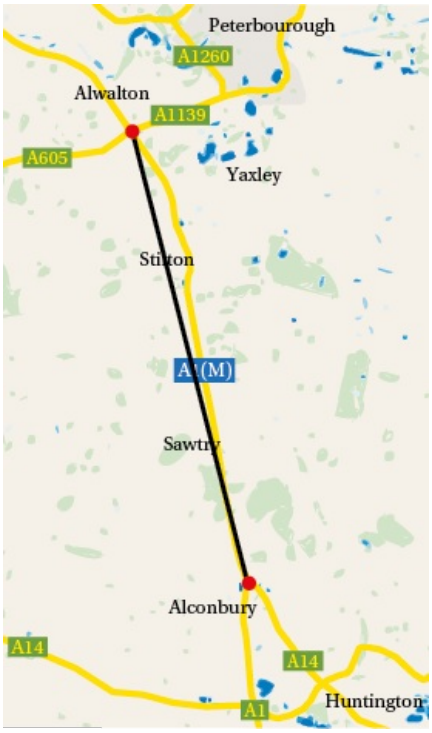
A Vector equations of motion are covered in Student Book 2.

WORKED EXAMPLE 19.1

The stretch of the A1(M) motorway between Junction 14 (Alconbury) and Junction 17 (Peterborough) is often cited as the longest straight stretch of UK motorway. The distance along the motorway between the two junctions is $16.8\ \text{km}$, while the straight line distance is $16.6\ \text{km}$.

A car travels along the motorway at an average speed of $110\ \text{km h}^{-1}$. Find the percentage difference between the times taken to travel the actual distance and the straight line distance. Hence comment whether the straight line model is appropriate for this stretch of motorway.

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Actual time:

$t_1 = \dfrac{16.8}{110} = 0.153 \text{ hours}$

Straight line model:

$t_2 = \dfrac{16.6}{110} = 0.151 \text{ hours}$

Percentage difference:

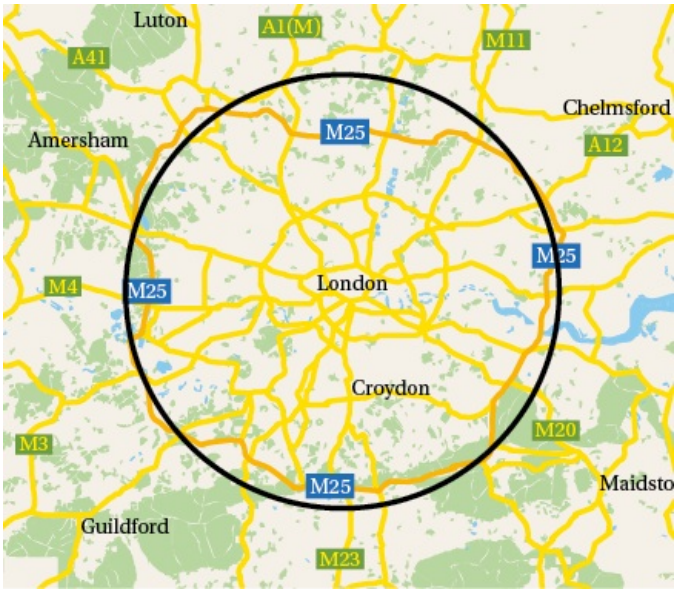
$\dfrac{0.153 - 0.151}{0.153} \times 100 \approx 1.3\%$

This is a small difference, so the straight line model is appropriate.

Use $\text{time} = \dfrac{\text{distance}}{\text{average speed}}$.

1.2% seems an acceptable difference compared to the benefits gained from the simplicity of the straight line model.

The straight line model isn't always appropriate. For example, the M25 motorway would be better modelled as a circle.



EXERCISE 19A

1 Discuss whether the particle model is appropriate in each of the following situations.

a You want to calculate how long it would take a car to complete the journey from Bristol to Birmingham.

b You are designing a car park.

c You want to predict the motion of a large box on a smooth floor when a force is applied at one corner.

d You want to decide how a football should be kicked so that it curves towards the goal.

2 For each of these questions, state some factors that have been ignored in the described model. Discuss how including each factor would affect the answer to the question.

a A box is modelled as a particle. The box falls from the top of a building. The only force acting on the box is gravity. How long does it take to reach the ground?

b A snooker ball is hit towards a cushion with the cue making a 30° angle with the cushion. The ball is modelled as a particle. Will it hit another ball (lying in a specified position)?

c A bus (of a given mass, modelled as a particle) travels between two cities. You are given how its speed varies with time and the fuel consumption at various speeds. How much fuel does it need?

d An aeroplane (modelled as a particle) flies between London and Tokyo. The Earth is modelled as a sphere. Assume that the aeroplane flies in a straight line, at a constant height and a constant speed. How long does it take?

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Section 2: Displacement, velocity and acceleration

If the position of a particle changes with time, you can measure its **displacement** from one point to another. Displacement is a vector, because you need to specify both the distance and **direction** away from the starting point. If a particle can **only** move in one dimension, its displacement can be described by a single number. However, this number can be either positive or negative. You should always draw a diagram and make clear which direction is positive. In the rest of this chapter we will take the positive direction to be to the right unless stated otherwise.

If a particle changes direction during the motion, then the **total distance travelled** is not necessarily the same as the final distance from the starting point.

Tip

Remember that distance is a **scalar**.

WORKED EXAMPLE 19.2

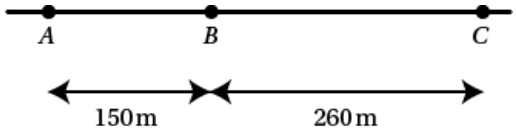
Points A, B and C lie, in that order, in a straight line, with $AB = 150\,\text{m}$ and $BC = 260\,\text{m}$. The direction of positive displacement is from A towards B. A particle travels from A to C and then from C to B. Find:

a the displacement from C to B

b the final displacement from A

c the total distance travelled by the particle.

a



Always draw a diagram to illustrate the situation.

C to $B = -260\,m$

B is to the left of C, so the displacement is negative.

b $AB = 150\,m$

The particle ends up at B. This is to the right of A, so the displacement is positive.

c $AC = 150 + 260 = 410\,m$

$CB = 260\,m$

total distance $= AC + CB = 670\,m$

To find the total distance, you need to add the distance from A to C to the distance from C to B.

If a particle travels a distance of $100\,\text{m}$ in 20 seconds, then **on average** it travels at 5 metres per second; its **average speed** is 5ms^{-1}. This does not mean that its speed equals 5ms^{-1} during the whole 20-second period; you can't tell what its speed is at any particular point in time.

If you take into account the **direction** of motion as well as the speed, you get the **velocity**, which is a vector. If the particle changes direction during the motion, this means that the direction of the velocity vector is changing.

The average velocity is equal to overall displacement divided by time.

Key point 19.1

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{time}}$$

$$\text{average velocity} = \frac{\text{final displacement} - \text{initial displacement}}{\text{time}}$$

The rate of change of velocity is called **acceleration**. Just like velocity, acceleration is a vector. For motion in one dimension, you represent vectors by a single number, but this number can also be positive or negative.

Tip

Speed is a scalar and velocity is a vector.

For motion in one dimension, velocity is usually denoted by the letter v and acceleration by the letter a. The units of velocity are ms^{-1}, because you divide distance (in metres) by time (in seconds). For acceleration, you divide velocity by time, so the units are $\frac{\text{m s}^{-1}}{\text{s}} = \text{m s}^{-2}$.

The basic quantities needed to describe motion of a particle are summarised in Key point 19.2.

Key point 19.2

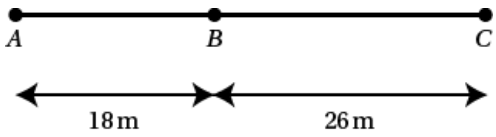
Scalar quantities (units)	Vector quantities (units)
time $t(\text{s})$	
distance $s(\text{m})$	displacement $\mathbf{x}(\text{m})$
speed u and $v(\text{ms}^{-1})$	velocity $\mathbf{u}$ and $\mathbf{v}$ (m s^{-1})
magnitude of acceleration $a(\text{m s}^{-2})$	acceleration $\mathbf{a}(\text{m s}^{-2})$

Tip

Although displacement, velocity and acceleration are vectors, for motion in a straight line you can describe them using a single number. You will therefore denote them using italics, s (or x), v and a. This means that, for example, you use v to denote both velocity and speed – don't get confused by this! However, when you study motion in two or three dimensions you will use vector notation: $\mathbf{x}$, $\mathbf{v}$ and $\mathbf{a}$.

WORKED EXAMPLE 19.3

Three points, A, B and C, lie in a straight line, as shown in the diagram.



A particle starts at A, moving towards B with speed 2.5m s^{-1}. Six seconds later, it passes B with speed $3.1\,\text{m s}^{-1}$. After a further 7.5 seconds it reaches C, where it stops and moves back towards B, which it passes 8.5 seconds later with speed $4.1\,\text{m s}^{-1}$. Find:

a the average velocity of the particle moving from A to B

b the average speed of the particle for the whole journey.

a $v = \frac{18}{6} = 3\,\text{m s}^{-1}$

$$\text{average velocity} = \frac{\text{final displacement-initial displacement}}{\text{time}}$$

b Total distance $= AB + BC + CB$

$= 18 + 26 + 26 = 70\,\text{m}$

$\text{Total time} = 6 + 7.5 + 8.5 = 22\,\text{s}$

$\therefore \text{average speed} = \dfrac{70}{22} = 3.2\,\text{m s}^{-1}\ (2\,\text{s. f.})$

$\text{average speed} = \dfrac{\text{total distance}}{\text{total time}}$

The numbers in the question are given to $2\,\text{s.f.}$, so round the final answer to the same accuracy.

In all the examples so far, you have measured distance in metres and time in seconds. These are the fundamental units of distance and time in the S.I. system. However, in real-life applications it is sometimes more convenient to use other units, such as kilometres and hours. You need to be able to convert the derived units for velocity and acceleration.

WORKED EXAMPLE 19.4

Convert:

a $2.6\ \text{ms}^{-1}$ into kilometres per hour

b $170\,\text{km h}^{-2}$ into m s^{-2}.

a $2.6\,\text{m s}^{-1} = \dfrac{0.0026}{\frac{1}{3600}\text{h}} = 9.36\,\text{km h}^{-1}$

$2.6\,\text{m} = 0.0026\,\text{km}$

$1\text{s} = \dfrac{1}{3600}\text{h}$

b $170\,\text{km h}^{-2} = \dfrac{170\,000\,\text{m}}{(3600\,\text{s})^2} = 0.0131\,\text{m s}^{-2}$

$170\,\text{km} = 170000\,\text{m}$

$1\,\text{h} = 3600\,\text{s}$

The hours are squared.

Did you know?



S.I. stands for 'Système internationale', the international system of units. It is the most widely used metric system, in which all units can be expressed in terms of seven fundamental (base) units. It was established in the mid-20th century. The photograph shows the kilogram reference masses.

It is important to realise that you can only convert between units that measure the same type of quantity. For example, you can convert metres per second into kilometres per hour but not into kilograms per metre. Similarly, you can only add units that represent the same quantity. For example, $3\text{m} + 42\,\text{cm} = 342\,\text{cm}$ or $3.42\,\text{m}$. But you cannot add $42\,\text{cm}$ to $3\,\text{kg}$.

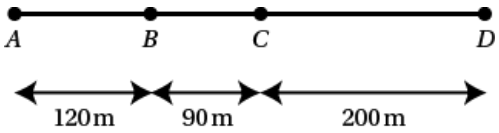
Fast forward

If you study the Mechanics option in Further Mathematics you will learn about dimensional analysis, which is a way of predicting formulae based on units.

EXERCISE 19B

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Questions 1 and 2 refer to the four points, A, B, C and D, which lie in a straight line with distances between them shown in the diagram. The displacement is measured from left to right.



1 Find:

a i the displacement from D to A

ii the displacement from D to B

b i the distance from D to B

ii the distance from C to A

c i the total displacement when a particle travels from B to C and then to A

ii the total displacement when a particle travels from C to D and then to A

2 **a i** A particle travels from A to C in 23 seconds and then from C to B in 18 seconds. Find its average speed and average velocity.

ii A particle travels from B to D in 38 seconds and then from D to A in 43 seconds. Find its average speed and average velocity.

b i A particle travels from B to D in 16 seconds and then back to B in 22 seconds. Find its average speed and average velocity.

ii A particle travels from A to C in 26 seconds and then back to A in 18 seconds. Find its average speed and average velocity.

3 Write the following quantities in the specified units, giving your answers to $3\,\text{s.f.}$

a i $3.6\,\text{km h}^{-1}$ in m s^{-1}

ii $62\,\text{km h}^{-1}$ in m s^{-1}

b i $5.2\,\text{m s}^{-1}$ in km h^{-1}

ii $0.26\,\text{m s}^{-1}$ in km h^{-1}

c i $120\,\text{km h}^{-2}$ in m s^{-1}

ii $450\,\text{km h}^{-2}$ in m s^{-1}

d i $0.82\,\text{m s}^{-2}$ in km h^{-2}

ii $2.7\,\text{m s}^{-2}$ in km h^{-2}

4 A particle moves with the speed of $12.3\,\text{m s}^{-1}$. After 5 seconds its speed is $4.3\,\text{m s}^{-1}$. Which of these could be its average acceleration during the 5 seconds?

A $1.6\,\text{m s}^{-2}$

B $3.32\,\text{m s}^{-2}$

C -1.6m s^{-2}

D -3.32m s^{-2}

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Section 3: Kinematics and calculus

In general, as a particle moves, its displacement, velocity and acceleration are constantly changing. **Equations of motion** describe how they vary with time.

$x(t)$ is normally used to denote the displacement from some reference point. The **average velocity** during the time period from $t = t_1$ to $t = t_2$ is:

$$\frac{\text{final displacement} - \text{initial displacement}}{\text{time}} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

If you draw a displacement–time graph, this equation says that the average velocity is the gradient of the chord between two points on the graph.

The **instantaneous velocity** at any particular point in time is given by the gradient of the tangent to the displacement graph at that point. You can think about taking the average velocity over a very small time period. This is the same process we went through when finding the gradient of a function; instantaneous velocity is the derivative of the displacement. This instantaneous velocity is itself a function of time, usually denoted by $v(t)$.

Similarly, acceleration is the rate of change of velocity with respect to time. The average acceleration is $\frac{\text{change of velocity}}{\text{time}}$, and the **instantaneous acceleration** is the derivative of the velocity function (and the gradient function of the velocity–time graph). Instantaneous acceleration is usually denoted by $a(t)$.

Rewind

See Chapter 13, Section 2, if you need a reminder of how to differentiate from first principles.

Key point 19.3

If the displacement is given by the function $x(t)$:

- the velocity is $v(t) = \frac{\mathrm{d}x}{\mathrm{d}t}$
- the acceleration is $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$

Once you have got the equations for velocity and acceleration in terms of t, you can find the values of the velocity or the acceleration at a specific point by substituting in the relevant value of t.

WORKED EXAMPLE 19.5

A toy car moves so that its displacement from a flag x metres after t seconds is given by the equation $x = 2.1 + 1.7t^2 - 0.2t^3$.

a Find the instantaneous velocity and acceleration of the car after 5 seconds.

b Find the initial displacement of the car from the flag.

c Find the average velocity during the first 5 seconds of the motion.

a $v = \frac{\mathrm{d}x}{\mathrm{d}t} = 3.4t - 0.6t^2$

When $t = 5$, $v = 2\,\mathrm{m\,s^{-1}}$

To find velocity, differentiate the displacement equation and substitute in the value of t.

$a = \frac{d^2x}{dt^2} = 3.4 - 1.2t$

To find the acceleration, differentiate again and substitute in the value of t.

When $t = 5$, $a = -2.6\text{m}\,\text{s}^{-2}$

b When $t = 0$, $x = 2.1\text{m}$

The initial displacement is when $t = 0$.

c When $t = 5$,

$x = 2.1 + 1.7 \times 5^2 - 0.2 \times 5^3 = 19.6\text{m}$

To find average velocity, you need the final displacement from the starting point. This is the value of $x(t)$ when $t = 5$.

The average velocity is

$$v = \frac{19.6 - 2.1}{5} = 3.5\,\text{m}\,\text{s}^{-1}$$

The average velocity is

$$\frac{\text{final displacement-initial displacement}}{\text{total time}}$$

Tip

You should not assume that the initial displacement is zero.

Sometimes you know the equation for the velocity and want to find how the displacement changes with time. This is done by integration. Since the velocity is the derivative of the displacement, it follows that the displacement is the integral of the velocity.

There are also many situations where you know the equation for acceleration in terms of time. You can integrate this equation to find velocity. You can then integrate it again to find the equation for the displacement.

Fast forward

You will see in Chapter 21 that the acceleration is related to the force. This is why in many models you can predict the equation for acceleration.

Note that these integrals are **indefinite** integrals. This means that the result will involve a constant of integration. This can be found by using the values of v or x that are given to you (usually, but not always, when $t = 0$).

Key point 19.4

$$v(t) = \int a(t)\,\mathrm{d}t$$
$$x(t) = \int v(t)\,\mathrm{d}t$$

WORKED EXAMPLE 19.6

An object moves in a straight line, with the acceleration given by the equation $a = \left(3t^2 - 6t\right)\text{m}\,\text{s}^{-2}$ where t is measured in seconds. The initial velocity of the object is $1.6\,\text{m}\,\text{s}^{-1}$.

a Find the equation for the velocity of the particle.

b Find the equation for the displacement from the initial position at time t.

c Find the displacement and the velocity of the object after 2 seconds.

a $v(t) = \int 3t^2 - 6t\,\mathrm{d}t = t^3 - 3t^2 + c$

To find the velocity, integrate the acceleration equation. Don't forget the $+ c$!

When $t = 0$, $v = 1.6$:

$1.6 = 0 - 0 + c \Rightarrow c = 1.6$

Find the constant of integration by using the initial values of t and v.

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$\therefore v(t) = t^3 - 3t^2 + 1.6$

b $x(t) = \int t^3 - 3t^2 + 1.6\,\mathrm{d}t$

$= \frac{1}{4}t^4 - t^3 + 1.6t + d$

To find the displacement, integrate the velocity equation. There is another unknown constant to add (you should use a different letter for each new constant in a question).

When $t = 0$, $x = 0$:

$0 = 0 + d \Rightarrow d = 0$

$\therefore x(t) = \frac{1}{4}t^4 - t^3 + 1.6t$

When $t = 0$ the object is at the starting point, so its displacement is zero.

c When $t = 2$:

$x(2) = -0.8\,\mathrm{m}$

$v(2) = -2.4\,\mathrm{m\,s^{-1}}$

To find the values at a specific point in time, substitute in the given value of t.

Tip

If the displacement is measured from left to right, this means that the object is to the left of its initial position, and it is moving to the left.

If you are only interested in the change in displacement between two times (rather than finding the displacement as a function of t), you can use a definite integral.

Key point 19.5

The change in displacement between times t_1 and t_2 is $\int_{t_2}^{t_2} v(t)\mathrm{d}t$.

Tip

You will see in the next section that this is not always the same as distance travelled.

WORKED EXAMPLE 19.7

A particle moves in a straight line so that its velocity at time t is given by $v = 3t - t^2$. Find the change in the displacement of the particle between $t = 2$ and $t = 5$.

$\int_2^5 (3t - t^2)\,\mathrm{d}t = \left[\frac{3}{2}t^2 - \frac{1}{3}t^3\right]_2^5$

You could find an equation for $x(t)$ and then evaluate the difference between $x(2)$ and $x(5)$. This is the same as evaluating a definite integral.

$= -\frac{25}{6} - \frac{10}{3}$

$= -7.5$

The particle has moved 7.5 units to the left. You should state the direction in your answer.

Taking the direction to the right as positive, the change in displacement is -7.5.

EXERCISE 19C

In questions 1–5, x is measured in metres, t in seconds, v in $\mathrm{m\,s^{-1}}$ and a in $\mathrm{m\,s^{-2}}$.

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1 A particle moves in a straight line. Its displacement from the point A is given by $x(t)$. Find the equations for the velocity and acceleration in terms of t if:

a **i** $x(t) = 3t^3 - 4t + 2$

ii $x(t) = t^4 - 3t^2 + 4t$

b **i** $x(t) = \frac{t^2}{5} - 4.2t$

ii $x(t) = -0.5t^3 + \frac{t^2}{3}$

2 For each part of question 1, find the values of the displacement, velocity and acceleration initially and after 3 seconds.

3 A particle moves in a straight line, with its velocity given by $v(t)$. The initial displacement from point A is $x(0)$. Find the equation for the displacement from A if:

a **i** $v(t) = 3t - 4,\ x(0) = 2$

ii $v(t) = 1 - 2t, x(0) = 1$

b **i** $v(t) = 5 - 3t^2,\ x(0) = 0$

ii $v(t) = t^2 - 2t,\ x(0) = 0$

4 For each part of question 3, find the displacement from A after 5 seconds.

5 A particle moves in a straight line with given acceleration $a(t)$. The initial velocity and displacement from point A are also given. Find the equations for the velocity and displacement in terms of t.

a **i** $a = 1 - 2t,\ v(0) = 3,\ x(0) = 0$

ii $a = 4t + 2,\ v(0) = -2,\ x(0) = 0$

b **i** $a = -5,\ v(0) = 3,\ x(0) = 5$

ii $a = 3,\ v(0) = -2,\ x(0) = 7$

6 A particle moves in a straight line. Its velocity, $v\,\mathrm{m\,s^{-1}}$, at time t seconds is given by $v = 2.4t - 1.5t^2$.

a Find the acceleration of the particle after 2 seconds.

b Find the velocity at the point when the acceleration is zero.

c The displacement of the particle from its initial position is x m. Find an expression for x in terms of t.

7 An object moves in a straight line. t seconds after it passes point O its displacement, $x\,\mathrm{m}$, from O is given by $x = 0.1t^3 - 1.2t^2 + 3.5t$.

a Find the speed and the magnitude of the acceleration of the object 6 seconds after passing O.

b What is the speed of the object when it first returns to O?

c Find the first time when the speed is zero, and the object's distance from O at this time.

8 A car starts from rest and moves in a straight line. Its acceleration, $a\,\mathrm{m\,s^{-2}}$, is given by $a(t) = 0.12t^2 - 1.44t + 4.32$.

a Find the equations for the car's velocity and its displacement from the starting point.

b Find the velocity and the displacement at the point when the acceleration is zero.

9 A particle moves in a straight line. Its displacement from the point P is x metres and its acceleration is $a = (1 - 0.6t)\,\mathrm{m\,s^{-2}}$. The particle is initially 25 m from P and moving away from P with velocity $7.5\,\mathrm{m\,s^{-1}}$.

a Find an expression for the velocity in terms of t.

b Find the particle's displacement from P after 10 seconds.

c Find the particle's displacement from P at the time when its acceleration is -2m s^{-2}.

10 A particle moves in a straight line with acceleration $a = (2 - 6t)\ \text{m s}^{-2}$, where the time is measured in seconds. When $t = 2$ its velocity is $-8\ \text{m s}^{-1}$. Find the average velocity of the particle between $t = 5$ and $t = 8$.

Section 4: Using travel graphs

The information on how displacement and velocity change with time can be described using equations, as you did in Section 3, or represented on a graph. A visual representation gives you a good idea of the motion even without doing any detailed calculations. Moreover, in some cases the graph can be drawn even without finding the equations, and some calculations can be done straight from the graph.

Gateway to A Level

For a reminder and more practice of travel graphs, see Gateway to A Level section X.

Displacement–time graphs

On a displacement–time graph (s–t graph), time is shown on the horizontal axis and displacement (measured from some specified reference point) on the vertical axis. Remember that the displacement can be negative – this means that the particle is on the other side of the reference point.

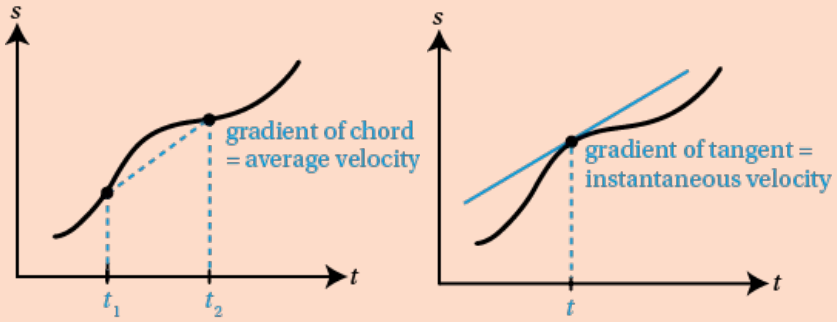
Tip

s is used as an alternative to x for displacement. You will usually see it on travel graphs and in constant acceleration formulae.

Key point 19.6

On a displacement–time graph:

- average velocity is the gradient of the chord between two points
- instantaneous velocity is the gradient of the tangent to the graph.

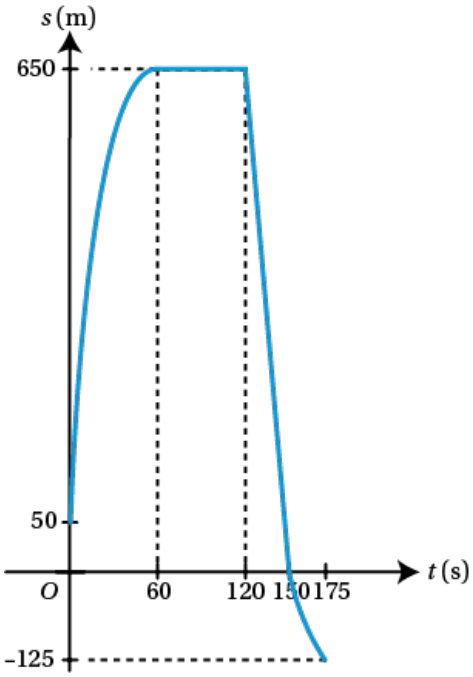


Focus on …

The concept of average velocity has some unexpected applications in pure mathematics, as explored in Focus on … Proof 4.

WORKED EXAMPLE 19.8

A small boat moves in a straight line. The graph shows its displacement, in metres, from a lighthouse.



a How far from the lighthouse does the boat start?

b Describe the motion of the boat during the first 175 seconds.

c Find the velocity of the boat between 120 and 150 seconds.

d Find the average velocity for the first 175 seconds.

a The boat starts 50 m from the lighthouse.

This is the displacement when $t = 0$.

b For the first 60 seconds, the boat moves away from the lighthouse, slowing down.

The displacement is increasing.

The gradient of the graph is positive but decreasing.

For the next 60 seconds, the boat is stationary. Its velocity is zero.

The displacement is not changing.

From 120 seconds the boat moves back towards the lighthouse with constant velocity.

The displacement is decreasing.

The graph is straight, so the velocity is constant.

It passes the lighthouse at 150 seconds and continues to move away from it, slowing down.

When $t = 150$ the displacement is zero.

Negative displacement means that the boat has passed the lighthouse.

The gradient of the graph is negative and increasing (getting closer to zero) so the speed is actually decreasing.

c $\dfrac{0-650}{30} = -21.7\text{ m s}^{-1}\,(3\text{ s. f.})$

Velocity is the gradient between the points $(120, 650)$ and $(150, 0)$.

The answer should be negative.

d $\text{initial displacement } = 50\text{ m}$

$\text{final displacement } = -125\text{ m}$

The average velocity is

$$\text{average velocity} = \frac{\text{final displacement} - \text{initial displacement}}{\text{time}}$$

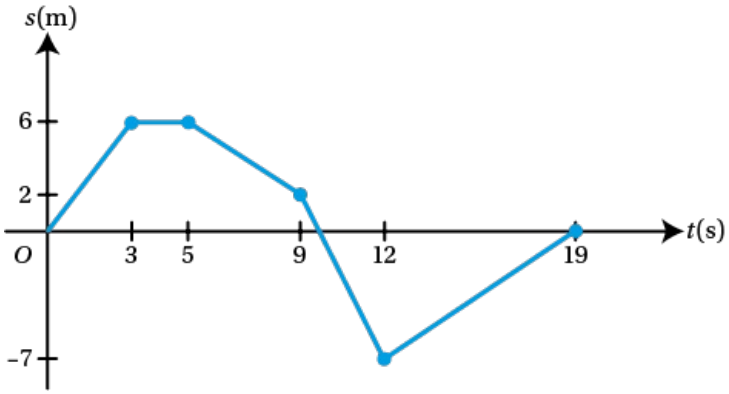
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$$\frac{-125-50}{175} = -1\,\text{m s}^{-1}$$

Remember that the gradient of the displacement–time graph represents the velocity. If you want to know the speed, you need to take the magnitude of the velocity. For motion in one dimension, this simply means taking the modulus of the number, so that negative numbers become positive.

WORKED EXAMPLE 19.9

The motion of a particle is represented on this displacement–time graph.



Find:

a the maximum velocity of the particle

b the maximum speed of the particle.

a The velocities are:

$\frac{6}{3} = 2$,
$\frac{0}{2} = 0$,
$\frac{-4}{4} = -1$,
$\frac{-9}{3} = -3$,
$\frac{7}{7} = 1$

So the maximum velocity is $2\,\text{m s}^{-1}$.

Find the velocity on each segment using $\frac{\text{change in displacement}}{\text{time}}$

b The maximum speed is $3\,\text{m s}^{-1}$.

Speed is the magnitude of the velocity, for example $|-3| = 3$.

Velocity–time graphs

Motion can also be shown on a velocity–time graph ($v - t$ graph). This shows time on the horizontal axis and velocity on the vertical axis.

Key point 19.7

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On a velocity–time graph:

- the gradient is the acceleration
- the total area between the graph and the horizontal axis is the distance travelled.

If the graph crosses the horizontal axis, the parts of the area above and below the axes need to be added to find the total distance travelled. The difference between the area above and the area below the axis gives the displacment from the starting point (this is because the displacement equals the distance travelled to the right minus the distance travelled to the left).

Gateway to A Level

You need to be able to find areas of triangles and trapeziums; see Gateway to A Level section R if you need a reminder.

WORKED EXAMPLE 19.10

The diagram shows the velocity–time graph for a particle moving in a straight line.

a Find the acceleration of the particle:

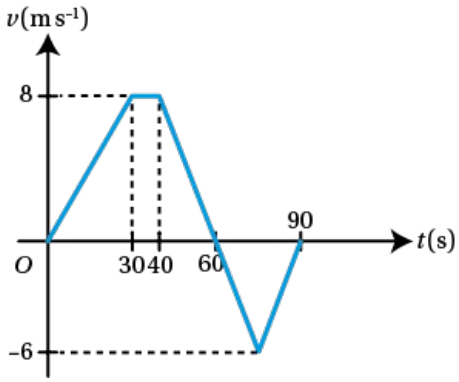
i during the first 30 seconds

ii between $t = 40$ and $t = 60$.

b What happens when $t = 60$?

c Find the distance travelled by the particle in the first 90 seconds.

d Find the change of the displacement of the particle during the first 90 seconds.



a i $a = \frac{8}{30} \approx 0.267 \text{ m s}^{-2}$

The acceleration is the gradient of the graph.

ii $a = -\frac{8}{20} = -0.4 \text{ m s}^{-2}$

Between $t = 40$ and $t = 60$ the gradient is negative.

b The particle changes direction.

The velocity changes from positive to negative.

c $\text{Total distance} = \left(\frac{60 + 10}{2} \times 8\right) + \frac{30 \times 6}{2}$,

$280 + 90 = 370\text{m}$

The distance is the total area between the x-axis and the graph, so you add the areas of the trapezium and the triangle.

d $\text{Change of displacement} = 280 - 90\ 190 \text{ m}$

The change of displacement is the difference between the two areas. (The particle travels 280 m to the right and then 90 m to the left, so its displacement from the starting point is 190 m.

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You need to be able to draw a velocity–time graph from a given description of motion. You can also write equations to find missing information.

WORKED EXAMPLE 19.11

A car moves in a straight line. When $t = 0$, it passes point P with a velocity of $12\,\text{m s}^{-1}$. It accelerates for 5 seconds with acceleration of $1.2\,\text{m s}^{-2}$ until it reaches a velocity of $V\,\text{m s}^{-1}$. It then moves with constant velocity V for T seconds and then decelerates at $0.8\,\text{m s}^{-2}$ until it comes to rest.

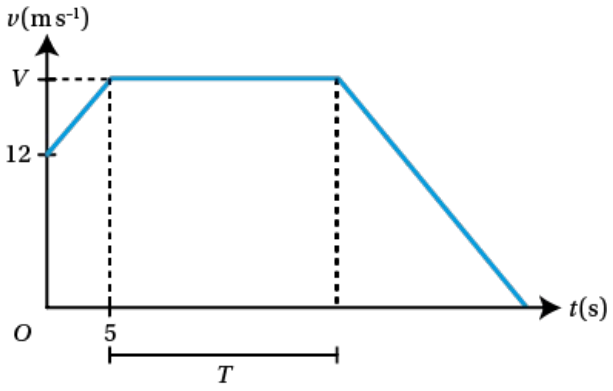
a Draw a velocity–time graph for the car's journey.

b Find the value of V.

c Find the time that the car spends decelerating.

d The total distance travelled by the car is $600\,\text{m}$. Find the value of T.

a



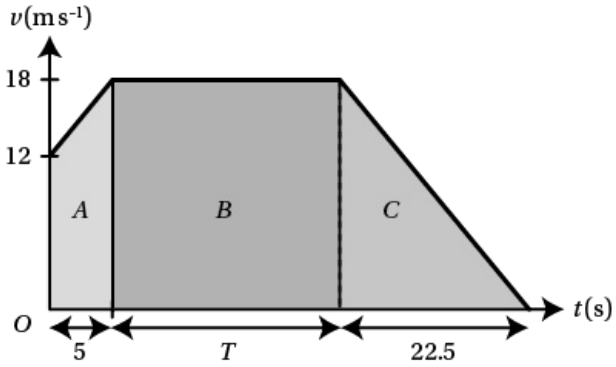
b $\dfrac{V-12}{5} = 1.2 \Rightarrow V = 18\ \text{ms}^{-1}$

$\text{acceleration} = \dfrac{\text{change in velocity}}{\text{time}}$

c $\dfrac{0-18}{t} = -0.8 \Rightarrow \quad t = 22.5\,\text{s}$

The velocity changes from 18 to 0. The acceleration is negative.

d



The distance travelled is the area under the graph. You can split it into several parts; for example, as shown in the diagram.

$\dfrac{18+12}{2} \times 5 + 18T + \dfrac{22.5 \times 18}{2} = 600$

$\Rightarrow 18T = 600 - 277.5 = 322.5$

$\Rightarrow T \approx 17.9\,\text{s}$

A is a trapezium with parallel sides 12 and 18 and height 5.

B is a rectangle with base T and height 18.

C is a triangle with base 22.5 and height 18.

Since the displacement and the velocity are related to each other, you should be able to draw the

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velocity–time graph from the displacement–time graph, and vice versa.

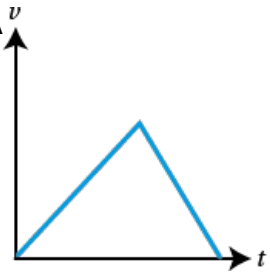
Rewind

This is the same as the relationship between the graph of a function and its derivative, which you met in Chapter 13, Section 1.

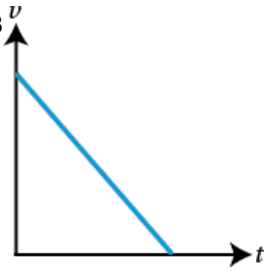
WORKED EXAMPLE 19.12

Match each velocity–time graph with the corresponding displacement–time graph.

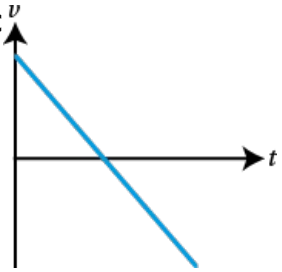
A



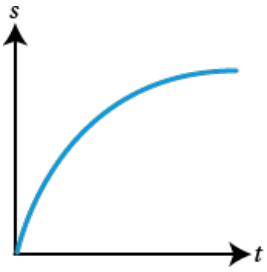
B



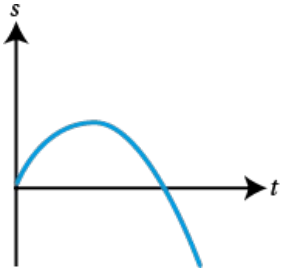
C



a

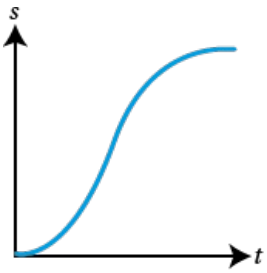


b



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c



Graph A corresponds to graph c.

In graph A the velocity increases from zero and then decreases back to zero. This means that the displacement–time graph starts and ends with a zero gradient.

Graph B corresponds to graph a.

The velocity starts positive and decreases towards zero; so the gradient of the displacement graph decreases towards zero but never becomes negative.

Graph C corresponds to graph b.

The velocity starts positive but decreases to zero and then becomes negative. This means that the particle stops and turns around, so the displacement starts decreasing and eventually becomes negative (the particle goes back past the starting point).

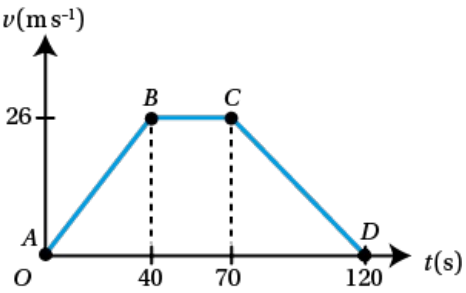
EXERCISE 19D

1 For each velocity–time graph, find:

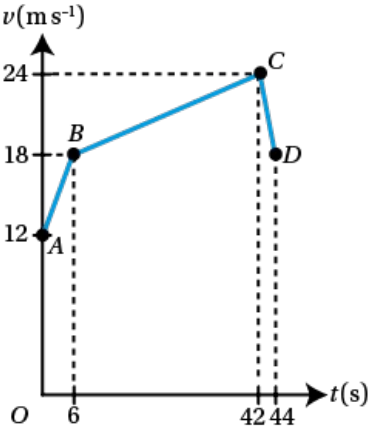
a the acceleration from A to B and from C to D

b the total distance travelled.

i

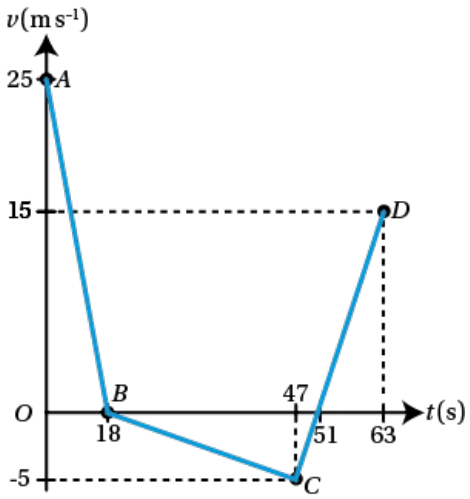


ii



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iii



2 For each of these descriptions of motion, draw the velocity–time graph and find the total distance travelled.

a A particle accelerates uniformly from $20\,\mathrm{m\,s^{-1}}$ to $32\,\mathrm{m\,s^{-1}}$ in 15 seconds, then moves with constant speed for 25 seconds and finally decelerates uniformly and comes to rest in another 10 seconds.

b An object starts from rest and accelerates at $2.5\,\mathrm{m\,s^{-2}}$ for 12 seconds. It then moves with a constant velocity for 8 seconds and finally decelerates at $6\,\mathrm{m\,s^{-2}}$ until it comes to rest.

c A particle accelerates uniformly from $11\,\mathrm{m\,s^{-1}}$ to $26\,\mathrm{m\,s^{-1}}$ with acceleration $0.4\,\mathrm{m\,s^{-2}}$. It then decelerates at $2\,\mathrm{m\,s^{-2}}$ until it comes to rest.

3 A particle moves in a straight line. Its displacement from point P is shown on the displacement–time graph.

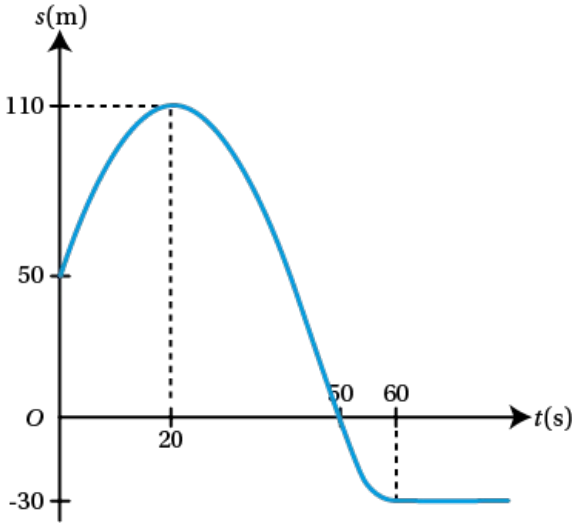
a How far from P does the particle start?

b In the first 20 seconds, is the particle moving towards P or away from it?

c What happens when $t = 20$ seconds?

d What happens after 60 seconds?

e At what time does the particle pass P?



f Is the particle's speed increasing or decreasing during the first 20 seconds?

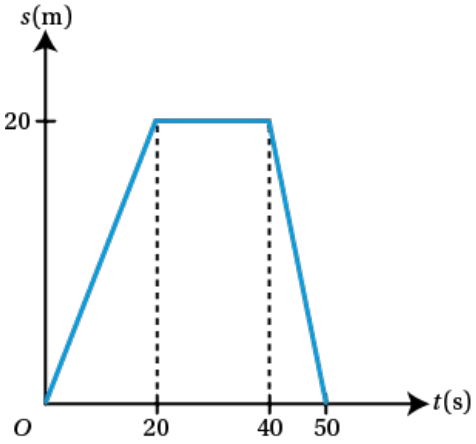
g Is the particle's speed increasing or decreasing between 50 and 60 seconds? What about its velocity?

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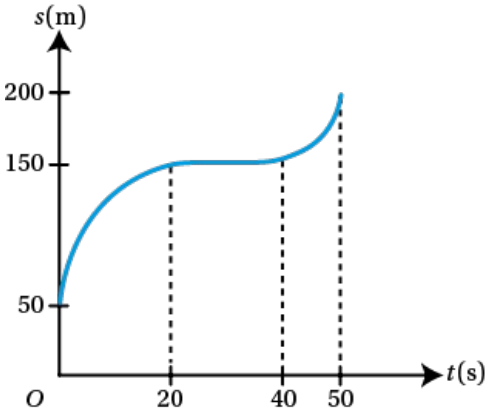
h Find the total distance travelled by the particle in the first 60 seconds.

4 For each displacement–time graph, draw the corresponding straight line velocity–time graph:

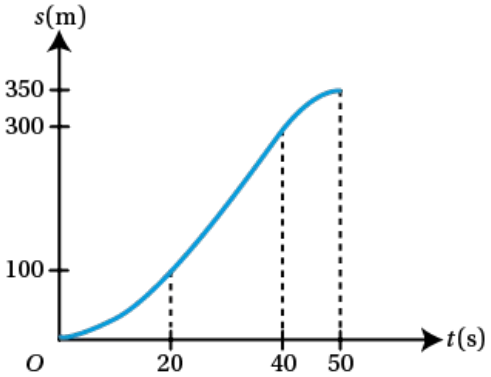
a



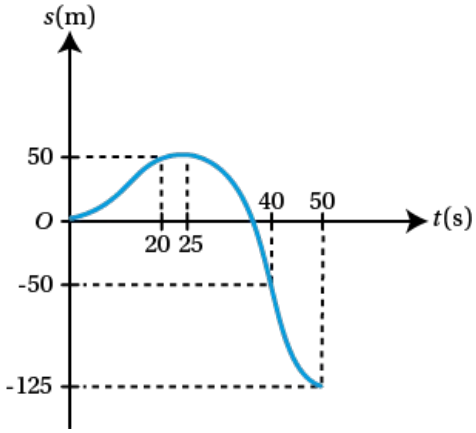
b



c



d



5 A car moves in a straight line. It passes the point P with a velocity of $16\,\mathrm{m\,s^{-1}}$. It continues to move with constant velocity for 20 seconds and then decelerates at a constant rate of $0.8\,\mathrm{m\,s^{-2}}$ until it comes to rest.

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a Represent the car's journey on a velocity–time graph.

b How far from P does the car stop?

6 A particle moves in a straight line. It starts from point A when $t = 0$. Its motion during 45 seconds is represented on this velocity–time graph.

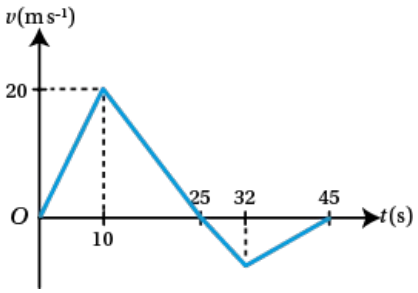
a Find the acceleration of the particle during the first 10 seconds.

b At what time does the particle change direction?

c The total distance travelled by the particle is $325\,\text{m}$.

d Find the velocity of the particle when $t = 32$.

e How far from A is the particle at the end of the 45 seconds?

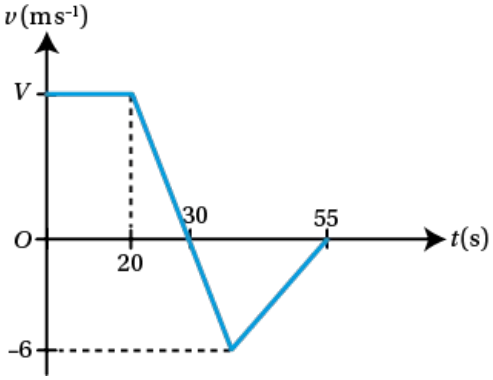


7 Sarah runs in a straight line at a constant speed of $6.2\,\text{m s}^{-1}$. When $t = 0$ she passes Helen who immediately starts running with constant acceleration of $1.3\,\text{m s}^{-2}$. Helen accelerates for 6 seconds and then continues to run at a constant speed until she catches up with Sarah.

a Draw a velocity–time graph to illustrate the motion of both girls.

b How long does it take for Helen to catch up with Sarah?

8 A particle moves in a straight line. Its velocity–time graph is shown. The total distance travelled during the 55 seconds is $275\,\text{m}$.

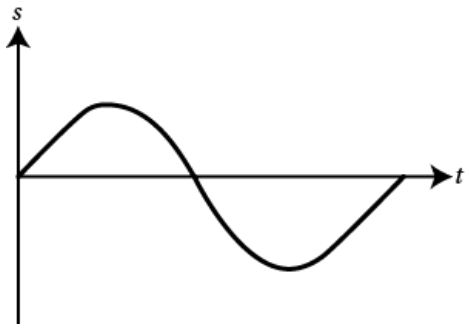


a Find the value of V.

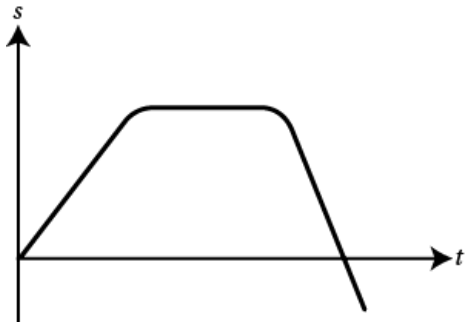
b Find the deceleration of the particle when $t = 30$.

c Which of these graphs could be the displacement–time graph for the same particle?

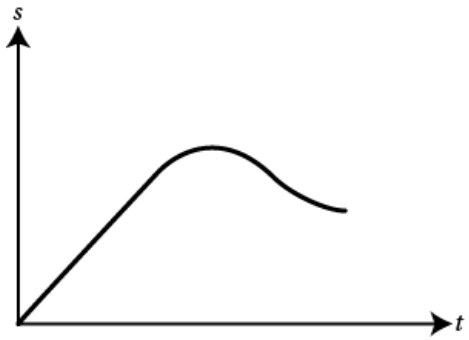
A



B



C



9 Peter and Sanjit are running in a race. They both start from rest.

Peter accelerates uniformly, then moves at a constant speed V for 5 seconds and then decelerates uniformly, coming to rest at the finish line.

Sanjit accelerates uniformly, at the same rate as Peter, to the same speed V and then decelerates immediately, coming to rest at the finish line. He finishes the race x seconds after Peter.

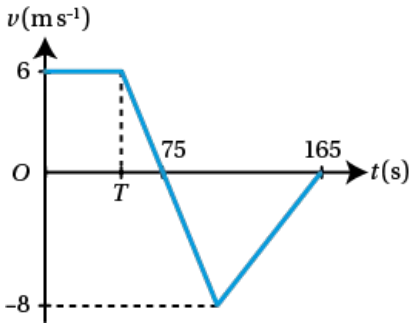
Find the value of x.

10 A particle moves in a straight line. Its velocity–time graph is shown.

a After 165 seconds the particle returns to the starting point. Find the value of T.

b At what time does the particle have maximum speed?

c Draw the displacement–time graph for this particle.



Focus on …

When solving problems in Mechanics, it is important to be able to represent information in several different ways. Focus on … Problem solving 3 looks at using alternative representations.

Worksheet

See Extension sheet 19 for further examples of alternative representations.

Average speed and average velocity

At any point in time, speed is the magnitude of velocity. However, average speed and average velocity are not necessarily related in the same way.

If an object moves without changing direction then distance travelled can be calculated from knowing the final and initial displacements. However, finding the distance becomes more complicated if the object changes direction part-way through the motion. For example, if you move 5 m to the right and then 2 m to the left then you have travelled a distance of 7 m but your displacement has changed by $+3\text{ m}$.

Rewind

Average speed and average velocity were defined in Key point 19.2.

Tip

Here we take the positive direction to the right. Remember that distance is a scalar quantity, and displacement is a vector.

WORKED EXAMPLE 19.13

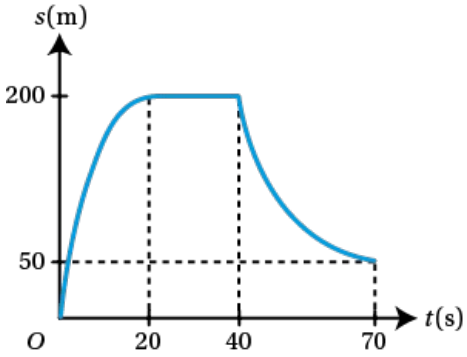
The diagram shows the displacement–time graph for a particle moving in a straight line. The displacement is measured from point A, with the positive displacement to the right.

a Describe how the displacement and the speed of the particle are changing.

b Find:

i the average velocity

ii the average speed for the whole journey.



a In the first 20 seconds the particle moves

200 m away from the starting point. Its speed is decreasing.

For the next 20 seconds the particle is stationary.

For the final 30 seconds the particle moves back towards the starting point. Its speed is decreasing. It ends up 50 m from the starting point.

In the first 20 seconds, the displacement changes from 0 to 200.

The gradient of the graph represents the speed. From 0 to 20 seconds the gradient is getting smaller.

From 40 to 70 seconds the velocity is negative and increasing, but the speed (the magnitude of the velocity) is decreasing.

b i $$\text{average velocity} = \frac{\text{total displacement}}{\text{time}}$$
$$= \frac{50}{70} \approx 0.714\,\text{m s}^{-1}\ (3\,\text{s. f.})$$

The final displacement is 50 m from the starting point.

ii $$\text{total distance} = 200 + 150 = 350\,\text{m}$$
$$\text{average speed} = \frac{\text{total distance}}{\text{total time}}$$
$$= \frac{350}{70} = 5\,\text{m s}^{-1}$$

The particle moves 200 m away from the starting point and then 150 m back towards it.

Remember that you can also find the change in displacement and the distance travelled from a velocity–time graph. The change in displacement is $\int_{t_2}^{t_2} v(t)dt$ while the distance is the area between the $v - t$ graph and the axis.

Rewind

See Key point 19.6. This difference between areas and integrals was discussed in Chapter 15, Section 5.

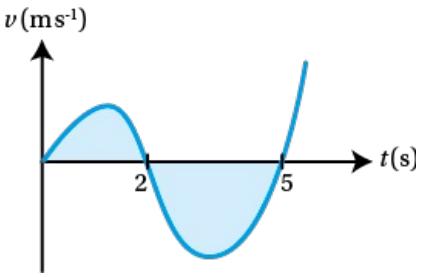
WORKED EXAMPLE 19.14

A particle moves with velocity $v(t) = t^3 - 7t^2 + 10t$ where t is measured in seconds and v in m s^{-1}. The diagram show the velocity-time graph of the motion. Find:

a the displacement from the starting point after 5 seconds

b the distance travelled by the particle during the first 5 seconds.

c Hence find the average velocity and the average speed of the particle.



The velocity-time graph shows that the particle changes direction, so the distance is not the same as the change in displacement.

a $\int_0^5 t^3 - 7t^2 + 10t \, \mathrm{d}t = \left[\frac{t^4}{4} - \frac{7t^3}{3} + 5t^2\right]_0^5$

$= \left(-\frac{125}{12}\right) - (0) = -10.4\,\mathrm{m}$

The change in displacement is the integral between $t = 0$ and $t = 5$. This equals the final displacement from the starting point.

The negative sign means that the particle is to the left of the starting point.

$\int_0^2 t^3 - 7t^2 + 10t \, \mathrm{d}t = \left[\frac{t^4}{4} - \frac{7t^3}{3} + 5t^2\right]_0^2$

$= \left(\frac{16}{3}\right) - (0) = \frac{16}{3}$

From $t = 0$ to $t = 2$ the particle moves to the right. The integral gives the change of displacement, which in this case is equal to the distance.

b $\int_2^5 t^3 - 7t^2 + 10t \, \mathrm{d}t = \left[\frac{t^4}{4} - \frac{7t^3}{3} + 5t^2\right]_2^5$

$= \left(-\frac{125}{12}\right) - \left(\frac{16}{3}\right) = -15.75$

From $t = 2$ to $t = 5$ the particle moves to the left. The change of displacement is negative. The particle has moved $15.75\,\mathrm{m}$ to the left.

The distance is

$d = \frac{16}{3} + 15.75 = 21.1$ (3s.f.)

Now add the distance travelled to the right and the distance travelled to the left.

c The average velocity is

$\frac{-10.4}{5} = -2.08\,\mathrm{m\,s^{-1}}$

To find the average velocity, use the change of displacement.

The average speed is $\frac{21.1}{5} \approx 4.22\,\mathrm{m\,s^{-1}}$

To find the average speed, use the total distance.

Fast forward

A If you have already studied modulus transformations (in Student Book 2, Chapter 3) you can write that the distance travelled is $\int_2^5 |t^3 - 7t^2 + 10t| \, \mathrm{d}t$.

Worksheet

For more practice finding distance from a velocity–time graph, see Support sheet 19.

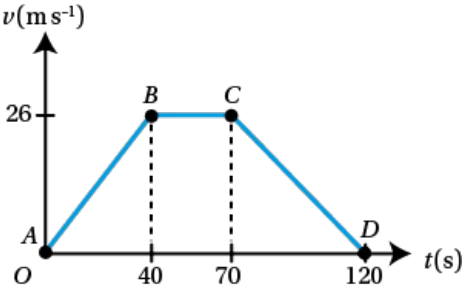
EXERCISE 19E

1 For each velocity–time graph, find:

a the average speed

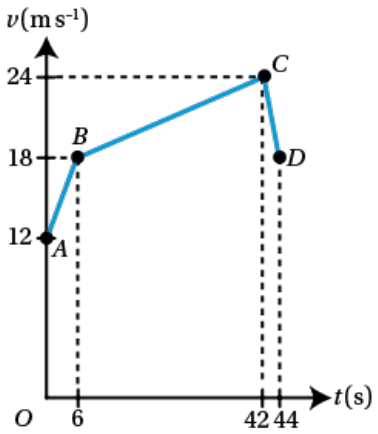
b the average velocity.

i

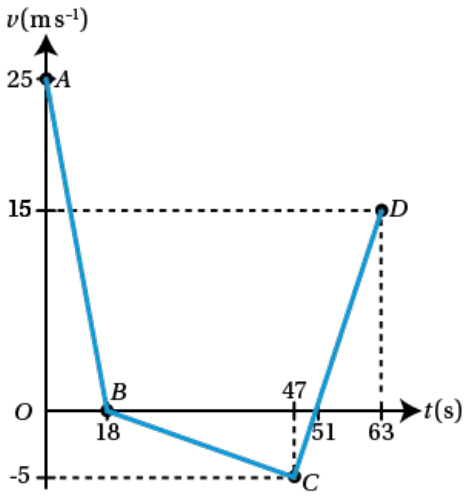


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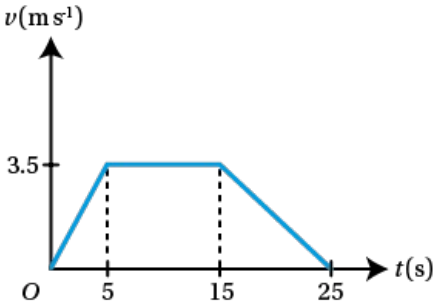
ii



iii



2 A particle moves in a straight line. Its motion, for the first 25 seconds, is represented on the following velocity–time graph:



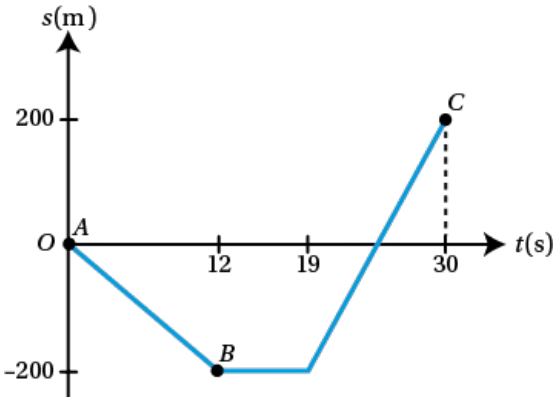
Find the average speed of the particle during the first 25 seconds.

3 The velocity of an object is given by $v(t) = 18t - 6t^2$ where time is measured in seconds and velocity in m s^{-1}.

a Sketch the velocity–time graph for $0 \leqslant t \leqslant 3$.

b Hence find the average speed of the particle during the first 3 seconds.

4 This displacement–time graph represents the motion of a particle moving in a straight line. The particle passes point A when $t = 0$.



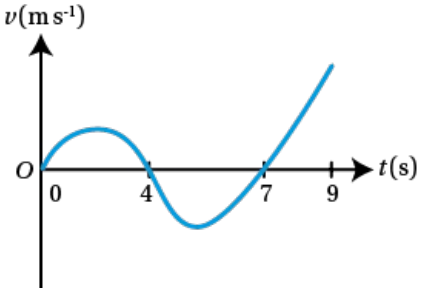
The particle is at point B when $t = 12$ and at point C when $t = 30$.

a Describe what happens between $t = 12$ and $t = 19$.

b Write down the displacement of C from A. Hence find the average velocity of the particle during the 30 seconds.

c Find the average speed of the particle during the 30 seconds.

5 The diagram shows the velocity–time graph for an object moving in a straight line. When $t = 0$ the object is at point O. The equation for the velocity is $v(t) = 3t(t-4)(t-7)$ where time is measured in seconds and velocity in metres per second.



a Find the two times when the particle changes direction.

b Find the displacement of the object from O when $t = 7$.

c Find the average velocity and average speed of the object during the first 7 seconds.

6 A particle moves in a straight line, starting from rest at point P. It accelerates for 5 seconds, until it reaches a speed of $16\,\mathrm{m\,s^{-1}}$. It maintains this speed for T seconds and then decelerates at $2\,\mathrm{m\,s^{-2}}$ until it comes to rest at point Q.

a Sketch the velocity–time graph to represent the motion of the particle.

b Given that the average speed of the particle on the journey from P to Q is $12\,\mathrm{m\,s^{-1}}$, find the value of T.

7 A particle moves in a straight line, with its velocity given by $v = \left(0.05t^3 - t^2 + 4.8t\right)\mathrm{m\,s^{-1}}$. The particle passes point A when $t = 0$.

a Explain how you can tell that the particle is initially at rest, and find two other times when the velocity is zero.

b Find the velocity and the displacement of the particle from A when $t = 6\,\mathrm{s}$.

c Find the average speed during the first 12 seconds.

8 A particle moves in a straight line. Its velocity, $v\,\mathrm{m\,s^{-1}}$, at time t seconds is given by $v(t) = (t+3)(t-2)(t-7)$. Find the average speed of the particle during the first 7 seconds.

9 The velocity of a particle is given by:

$$v(t) = \begin{cases} 5t - \frac{1}{2}t^2 \text{for} 0 \leqslant t \leqslant 5 \\ 7.5 - \frac{1}{2}t \text{for} 5 < t \leqslant 15 \end{cases}$$

The average speed of the particle during the first T seconds is $4\,\text{m s}^{-1}$. Find the value of T, where $T < 15$.

10 A particle moves in a straight line. Its displacement, $x\,\text{m}$, from point O is given by $x(t) = 24t - 3t^2$, where t is measured in seconds. The average velocity of the particle during the first T seconds is $9\,\text{m s}^{-1}$. Find its average speed during this time.

Tip

In question 9, the velocity suddenly jumps when $t = 5$. This can happen, for example, in a collision. The displacement still needs to be continuous.

Section 5: Solving problems in kinematics

You now have the main tools you need to solve problems in kinematics: You can use differentiation and integration to find equations for displacement, velocity and acceleration. You can also represent those quantities on travel graphs. We will now look at more complicated problems where you need to interpret the question and extract relevant information from the given context. You may also need to combine information from graphs and equations.

WORKED EXAMPLE 19.15

A boat moves in a straight line. At $t = 0$ it passes a rock and it is moving with velocity $4\,\text{m s}^{-1}$. Its acceleration, $a\,\text{m s}^{-2}$, is given by $a = 12 - 6t$ for $0 \leqslant t \leqslant 8$ seconds.

a Find the time when the boat changes direction.

b Find the maximum velocity of the boat.

c At what time does the boat pass the rock again?

a $v = \int 12 - 6t\,\text{d}t = 12t - 3t^2 + c$

When $t = 0$, $v = 4$ so $c = 4$

The boat changes direction when the velocity changes from positive to negative. So you are looking for the time when $v = 0$.

$v = 12t - 3t^2 + 4 = 0$

$\Rightarrow\ t = -0.309 \text{ or } 4.31$

But you are only looking for positive times,

$\therefore\ t = 4.31\,s\,(3\text{s. f.})$

Use the quadratic formula or the equation solver on your calculator.

b $\dfrac{\text{d}v}{\text{d}t} = 0$

$\Rightarrow\ 12 - 6t = 0$

$\Rightarrow t = 2\,s$

When the velocity is maximum, $\dfrac{\text{d}v}{\text{d}t} = 0$.

Note that this means the same as $a = 0$.

To check this is a maximum:

$\dfrac{d^2v}{dt^2} = -6 < 0$

so the stationary point is a maximum.

Check that this is a maximum by using the second derivative.

The maximum velocity is

$v\,(2) = 24 - 12 + 4 = 16\,\text{m s}^{-1}$

Notice that $16\,\text{ms}^{-1}$ is larger than the initial velocity of $4\,\text{ms}^{-1}$. When $t = 8$ the velocity is negative. So $16\,\text{ms}^{-1}$ is the maximum value in this interval.

c $x = \int 12t - 3t^2 + 4\,\text{d}t$

$= 6t^2 - t^3 + 4t + d$

If x is the displacement from the rock, you are looking for the time when $x = 0$.

When $t = 0$, $x = 0$ so $d = 0$

When $t = 0$, the boat passes the rock, so $x = 0$.

$x = 6t^2 - t^3 + 4t = 0$

$\Rightarrow\ t\left(-t^2 + 6t + 4\right) = 0$

$\Rightarrow\ t = 0\,or - t^2 + 6t + 4 = 0$

$\Rightarrow\ t = 0 \text{ or } -0.61 \text{ or } 6.61\,(2\,d.\,p.)$

You are looking for a positive value of t. (You already know that the boat passes the rock when $t = 0$).

The boat passes the rock again after 6.61 seconds.

Worked example 19.15 uses some common phrases you need to understand:

- When $v = 0$, the object is **instantaneously at rest** and may be changing direction.
- It reaches maximum/minimum velocity when $a = 0$ (or at either end of its motion).
- It returns to the starting point when $x = x(0)$.

You also need to be able to deal with two-stage problems, where the equation for velocity changes after a certain time.

Tip

Be careful: $x(0)$ is the initial position, and it is not always 0.

WORKED EXAMPLE 19.16

A car accelerates from its parking space and moves in a straight line. Its velocity, $v\,\mathrm{m\,s^{-1}}$, at time t seconds, satisfies:

$$v(t) \begin{cases} 14t - 5t^2 \text{ for } 0 \leqslant t \leqslant 2 \\ 13 - \dfrac{20}{t^2} \text{ for } t > 2 \end{cases}$$

a Find the displacement of the car from the parking space:

i when $t = 2$

ii when $t = 5$.

b At what time is the car $35\,\mathrm{m}$ from the parking space?

a i $x = \displaystyle\int_0^2 14t - 5t^2 \,\mathrm{d}t$

$= \left[7t^2 - \dfrac{5}{3}t^3\right]_0^2$

$\approx 14.7\,\mathrm{m}$

From $t = 0$ to $t = 2$, the first velocity equation applies.

You can find the displacement between those two times using Key point 19.5.

ii $\displaystyle\int_2^5 13 - \frac{20}{t^2}\,\mathrm{d}t = \left[13t + \frac{20}{t}\right]_2^5$

$= 33$

$x = 33 + 14.7\mathrm{m} \approx 47.7\mathrm{m}$

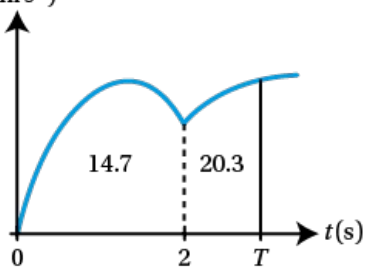
From $t = 2$ to $t = 5$ you need to use the second velocity equation.

The change of displacement is $\displaystyle\int_2^5 v\,\mathrm{d}t$, but this is measured from the point where the car was at $t = 2$, which is 14.7 m (from part **i**).

In the first 2 seconds, the car travels $14.7\,\mathrm{m}$ from the parking space. Therefore its gets to $35\,\mathrm{m}$ after more than 2 seconds, so you need to use the second equation.

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b $v(\text{m s}^{-1})$



$14.7 + \int_2^T 13 - \frac{20}{t^2}\,\mathrm{d}t = 35$

The total displacement of $35\ \text{m}$ is made up of the 14.7 m travelled in the first 2 seconds, plus the displacement between $t = 2$ and $t = T$.

$\Leftrightarrow \left[13t + \frac{20}{t}\right]_2^T = 20.3$

$\Leftrightarrow \left(13T + \frac{20}{T}\right) - (36) = 20.3$

$\Leftrightarrow 13T^2 - 56.3T + 20 = 0$

This can be rearranged into a quadratic equation.

$\Leftrightarrow 0.390 \text{ or } 3.94$

$\therefore T = 3.94\,\text{s}\,(3\,\text{s.f.})$

You know that you are looking for a value of T above 2.

Tip

Note that in part b above you should first check that the car doesn't change direction; otherwise it could return to the same place more than once. You can check that the velocity is always positive for the given values of t.

Tip

Remember that 14.7 and 20.3 are rounded values. You should use the full values in your calculator (saved in the calculator memory from part a).

Many problems can be solved either by deriving equations or by using a travel graph. It is useful to be able to decide which method would be easier. You saw in Worked example 19.16 that multi-stage problems can be quite complicated. If the velocity–time graph is made up of straight line segments, it is easy to answer questions about displacement by using areas.

WORKED EXAMPLE 19.17

A particle moving in a straight line passes point P when $t = 0$. Its velocity, $v\,\text{ms}^{-1}$, satisfies:

$$v = \begin{cases} 6.3 + 2.1t & \text{for } 0 \leqslant t \leqslant 6 \\ 29.7 - 1.8t & \text{for } t > 6 \end{cases}$$

where t is measured in seconds.

a Find the particle's displacement from P when $t = 10$.

b Find the maximum displacement from P.

c Point Q has the displacement of $90\ \text{m}$ from P.

i Find the first time when the particle passes Q.

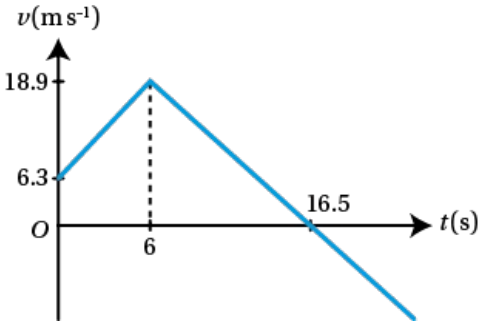
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ii How long does it take for the particle to return to Q?

When $t = 0, v = 6.3$

When $t = 6, v = 18.9$

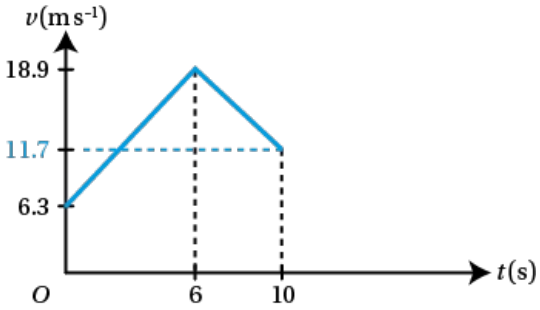
When: $v = 0: \ 29.7 - 1.8t = 0 \ \Rightarrow \quad t = 16.5$



Since the velocity–time graph is made up of straight line segments, you don't need to use integration to find areas. So start by sketching the graph, labelling all the relevant coordinates.

a When $t = 10$,

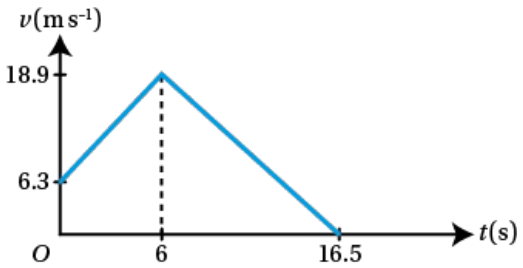
$v = 29.7 - 1.8 \times 10 = 11.7 \text{ms}^{-1}$



$\frac{6}{2}(6.3 + 18.9) + \frac{4}{2}(18.9 + 11.7) = 136.8 \text{ m } (4 \text{ s.f.})$

The displacement is the area under the graph between $t = 0$ and $t = 10$. This is made up of two trapezia.

b



$\frac{6}{2}(6.3 + 18.9) + \frac{10.5 \times 18.9}{2} = 175 \text{ m } (3 \text{ s.f.})$

The particle moves away from P from $t = 0$ to $t = 16.5$, because that's when the velocity is positive. For $t > 16.5$ it moves back towards P. So the maximum displacement is when $t = 16.5$. The corresponding area is made up of a trapezium and a triangle.

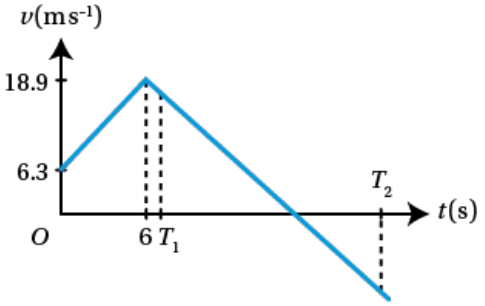
c **i** The displacement in the first 6 seconds:

$\frac{6}{2}(6.3 + 18.9) = 75.6\text{m}$

The particle is at Q when $t = T_1$.

The displacement from $t = 6$ to $t = T_1$:

$90 - 75.6 = 14.4\text{m}$



You can see from part b that the particle moves 175 m away from P and then comes back. This means that it will pass through Q twice: once in the first 16.5 seconds (when $t = T_1$) and once on the way back (when $t = T_2$).

$$\int_6^{T_1} 29.7 - 1.8t \, \mathrm{d}t = 14.4$$

$$\Leftrightarrow \left[29.7t - 0.9t^2\right]_6^{T_1} = 14.4$$

$$\Leftrightarrow \left(29.7T_1 - 0.9T_1^2\right) - (145.8) = 14.4$$

$$\Leftrightarrow 0.9T_1^2 - 29.7T_1 + 160.2 = 0$$

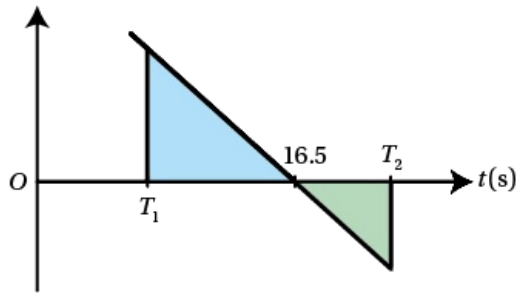
$$\Leftrightarrow T_1 = 6.79 \text{ or } 26.2$$

$\therefore$ The particle is at Q after 6.79 seconds.

You need to find T_1 when the area of the trapezium is 14.4. But you don't know either the base or the height. Instead, you can integrate the equation for v.

You are looking for T_1 between 6 and 16.5.

ii $v(\text{m s}^{-1})$



Between T_1 and T_1, the particle stops, changes direction and returns to Q. This means that the areas of the blue and the green triangles are equal, so 16.5 is the midpoint between T_1 and T_2.

$$T_2 = 16.5 + (16.5 - 6.79)$$

$$= 16.5 + 9.71$$

$$\Rightarrow \quad T_2 = 26.2$$

The particle returns to Q when $t = 26.2$ seconds.

EXERCISE 19F

1 A particle moving in a straight line passes point P when $t = 0$. Its displacement from P satisfies the equation $x = 3.6t - 1.2t^2$ where x is measured in metres and t in seconds. Find the time when the particle changes direction.

2 Ellie runs in a straight line. At time t seconds, her displacement, x metres, from the starting point satisfies the equation $x = 1.8t^2 - 0.2t^3$.

a Show that that Ellie starts running from rest.

b Find her maximum velocity.

3 A toy car's velocity, $v\,\text{m s}^{-1}$, depends on time, ts, according to the equation $v = -0.8t^2 + 1.44t + 9.72$.

Find the displacement from the starting point at the moment when the car comes to instantaneous rest, and the deceleration of the car at this point.

4 A particle moves in a straight line. Its velocity at time t seconds is $v = \left(11.2t - 3t^2\right) \text{m s}^{-1}$.

The particle is at A when $t = 0$. How long does it take for it to return to A?

5 The velocity of a car, $v\text{m s}^{-1}$ at time t seconds, satisfies:

$$v = \begin{cases} 4t - 0.5t^2 \text{ for } 0 \leqslant t \leqslant 4 \\ 0.4t^2 - 6.4t + 27.2 \text{ for } t > 4 \end{cases}$$

a Find the acceleration of the car after 5 seconds.

b Find the car's displacement from the starting point when:

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i $t = 3$

ii $t = 10$.

6 A dog runs past a tree when $t = 0$ with the speed of 3.7ms^{-1}. It accelerates for 5 seconds so that its speed satisfies $v = (u + 0.4t)\ \text{m s}^{-1}$.

a Write down the value of u.

For the next 5 seconds, the dog decelerates and its speed satisfies:

$$v = \left(\frac{142.5}{t^2}\right)\ \text{m s}^{-1}$$

b Find the dog's final speed.

c Find the dog's final displacement from the tree.

d After how long is the dog $32\ \text{m}$ from the tree?

7 A particle moving in a straight line accelerates from rest. For the first 7 seconds its acceleration at time t seconds satisfies $a = (6.5 - 1.3t)\ \text{m s}^{-2}$. Subsequently, the particle moves with constant acceleration. Find the equation for the velocity of the particle in terms of t when:

a $0 \leqslant t \leqslant 7$

b $t > 7$

8 The acceleration of a particle moving in a straight line, $a\text{m s}^{-2}$, satisfies $a = 0.1(t - 5)^2$ for $0 \leqslant t \leqslant 5$ seconds. The particle is initially at rest.

a Explain why the velocity of the particle is never negative between $t = 0$ and $t = 5$.

b Find the average speed of the particle between $t = 0$ and $t = 5$.

c By sketching the graph of $v(t)$, show that there is a value of t where the instantaneous speed equals the average speed.

9 A particle moves in a straight line so that its acceleration at time t seconds is given by $a = 3t^2 - 14t + 10$ (measured in m s^{-2}). The particle starts from rest when $t = 0$.

a Find the equation for the velocity of the particle in terms of t.

b Find:

i the maximum velocity

ii the maximum speed of the particle in the first 5 seconds.

10 A particle is moving in a straight line, starting from point A when $t = 0$. Its velocity, in m s^{-1}, is given by:

$$v = \begin{cases} 0.01\left(12t^2 - t^3\right) & \text{for } 0 \leqslant t \leqslant 10 \\ 10 - 0.8t & \text{for } 10 < t < 12.5 \end{cases}$$

where t is measured in seconds.

a Find the distance travelled by the particle in the first 10 seconds.

b After how long is the particle $17\ \text{m}$ from A?

11 An object moves in a straight line with velocity $v = 20t^2 - t^3$. The average velocity over the first T seconds equals the instantaneous velocity after $\frac{T}{2}$ seconds. Find the value of T.

12 Cars can go over speed bumps at $5\ \text{km h}^{-1}$. For an average car, the maximum acceleration is $1.4\ \text{m s}^{-2}$ and the maximum deceleration is $3.2\ \text{m s}^{-2}$. How far apart should the speed bumps be placed to restrict the maximum speed to $30\ \text{km h}^{-1}$?

Checklist of learning and understanding

- The **displacement**, **velocity** and **acceleration** of a particle moving in a straight line are vectors: they can have a positive or a negative value.
- **Distance** (from a certain point) is the magnitude of the displacement (from that point). This is not necessarily the same as **distance travelled**.
- The **instantaneous** velocity and **instantaneous acceleration** can be **found** by differentiating the displacement equation:
  $v = \frac{\mathrm{d}x}{\mathrm{d}t}, \qquad a = \frac{\mathrm{d}v}{\mathrm{d}t}$
- Integrating the velocity equation gives the displacement equation; integrating the acceleration equation gives the velocity equation:
  $x = \int v \,\mathrm{d}t, \qquad v = \int a \,\mathrm{d}t$
- The constant of integration can be found from the initial displacement or velocity.
- The change of displacement between time t_1 and t_2 can be found using the definite integral:
  $s = \int_{t_1}^{t_2} v \,\mathrm{d}t$
- If the particle doesn't change direction then this integral gives the total distance travelled.
- On a velocity–time graph:
  - the acceleration is the gradient
  - the distance travelled equals the area between the graph and the t-axis.
  - The change of displacement can be found by subtracting the area below the axis from the area above the axis.
- The **average velocity** equals change of displacement divided by time. The **average speed** equals total distance divided by time.

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Mixed practice 19

1 An object moves in a straight line so that its velocity, $v\,\mathrm{m\,s^{-1}}$, is given by the equation $v = 3t^2 - 8t$.

a Find the acceleration of the object after 2 seconds.

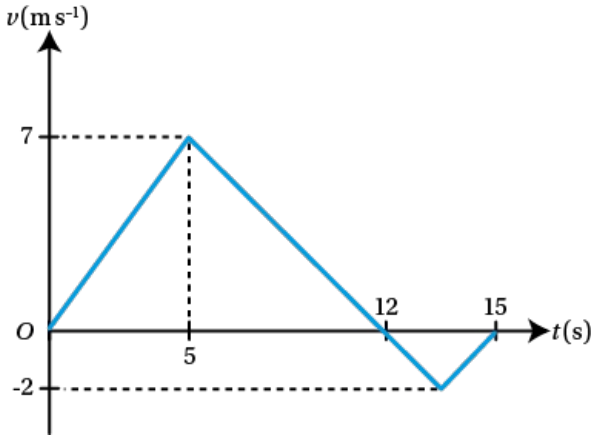
b Find the equation for the displacement from the initial position after t seconds.

2 A particle moves in a straight line with velocity $v = \left(3 - t^{-2}\right)\mathrm{m\,s^{-1}}$ for $t \geqslant 1$.

a Find the distance travelled between $t = 1$ and $t = 5$.

b Hence find the average speed of the particle between $t = 1$ and $t = 5$.

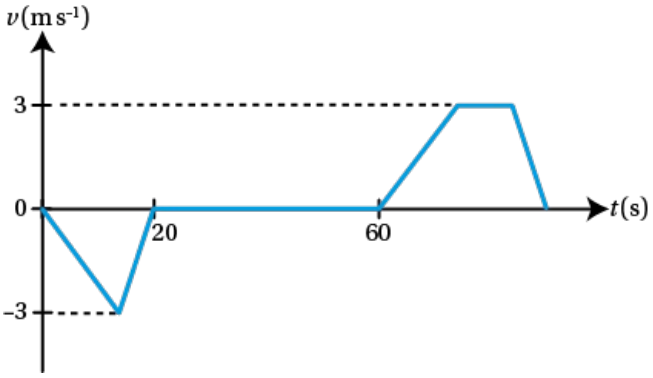
3 The motion of a particle moving in a straight line is represented on the following velocity–time graph:



a Find the acceleration of the particle between $t = 0$ and $t = 5$.

b State the times when the particle is instantaneously at rest.

c Find the average speed of the particle for the first 15 seconds.

4 

The diagram shows the (t, v) graph for a lorry delivering waste to a recycling centre. The graph consists of six straight line segments. The lorry reverses in a straight line from a stationary position on a weighbridge before coming to rest. It deposits its waste and then moves forwards in a straight line accelerating to a maximum speed of $3\,\mathrm{m\,s^{-1}}$. It maintains this speed for $4\,\mathrm{s}$ and then decelerates, coming to rest at the weighbridge.

a Calculate the distance from the weighbridge to the point where the lorry deposits the waste

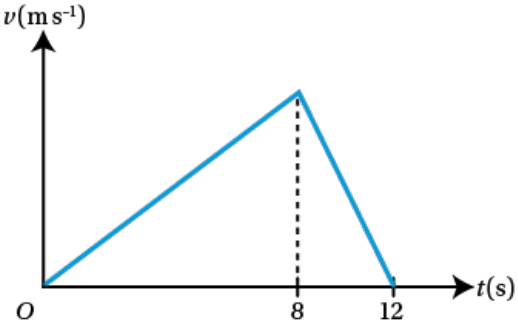
b Calculate the time which elapses between the lorry leaving the weighbridge and returning to it.

c Given that the acceleration of the lorry when it is moving forwards is $0.4\,\mathrm{ms^{-2}}$, calculate its

final deceleration.

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5 This velocity-time graph shows the motion of a particle moving in a straight line. The total distance travelled during the 12 seconds is $360\,\mathrm{m}$.



Find the acceleration of the particle during the final 4 seconds.

6 A car travels along a straight road. Its velocity, in kilometres per hour, is given by $v = 40 + 10t - 0.5t^2$ (for $0 \leqslant t \leqslant 20$), where time is measured in seconds. It passes point A when $t = 0$.

a Write an equation for the velocity in metres per second.

b Find the acceleration of the car in terms of t. Hence find the time when the car has maximum velocity.

c Find the distance of the car from A when $t = 12$.

d The car is modelled as a particle. Explain whether this is a suitable modelling assumption in the following two questions.

i How long does the car take to overtake a stationary van of length $6.2\,\mathrm{m}$?

ii How long does the car take to pass through a tunnel of length $380\,\mathrm{m}$?

7 A bus travels in a straight line. When it passes a man its speed is $8.5\,\mathrm{ms}^{-1}$. It decelerates uniformly until it comes to rest at the bus stop $44.2\,\mathrm{m}$ away.

As the bus passes the man, the man starts running at a constant velocity, $V\,\mathrm{ms}^{-1}$. He arrives at the bus stop at the same time as the bus.

Find the value of V.

8 A model train travels along a straight track. At time t seconds after setting out from station A, the train has velocity $v\,\mathrm{ms}^{-1}$ and displacement x metres from A. It is given that for $0 \leqslant t \leqslant 7$.

$x = 0.01t^4 - 0.16t^3 + 0.72t^2$

After leaving A the train comes to instantaneous rest at station B.

a Express v in terms of t. Verify that when $t = 2$, the velocity of the train is $1.28\,\mathrm{ms}^{-1}$.

b Express the acceleration of the train in terms of t and hence show that when the acceleration of the train is zero $t^2 - 8t + 12 = 0$.

c Calculate the minimum value of v.

d Sketch the (t, v) graph for the train, and state the direction of motion after the train leaves B.

e Calculate the distance AB.

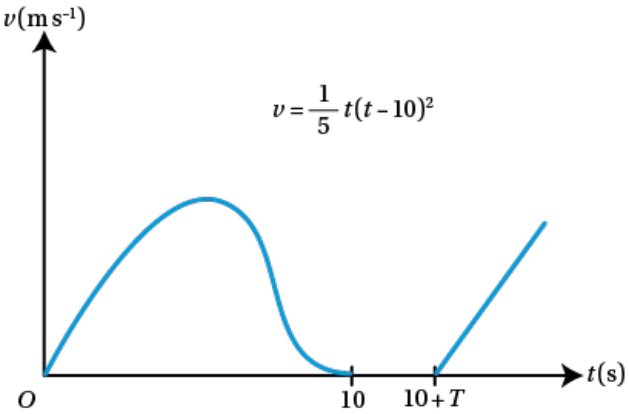
9 A particle moves with velocity $v\,\mathrm{ms}^{-1}$, where:

$$v(t) = \begin{cases} 0.16t^3 - 0.12t + 10.6 & \text{for } 0 \leqslant t \leqslant 5 \\ 40 - 2t & \text{for } t > 5 \end{cases}$$

Find the two times when the particle is $200\,\mathrm{m}$ away from the starting point.

10 A car is travelling along a road that has a speed limit of $90\,\mathrm{km\,h}^{-1}$. The speed of cars on the road is monitored via average speed check cameras, which calculate the average speed of a car by measuring how long it takes to travel a specified distance.

The car starts from rest next to one of the cameras. Its velocity in ms^{-1} is given by $v(t) = \frac{1}{5}t(t-10)^2$, and it comes to rest after 10 seconds. It stays stationary for T seconds and then starts moving again with a constant acceleration of $3.5\,\mathrm{ms}^{-2}$. The velocity–time graph of the car's motion is shown in the diagram.



The second camera is positioned $300\,\mathrm{m}$ away from the first one.

a Find the time the car takes to reach the second camera after it has started from rest the second time.

b Show that the car's speed exceeded $90\,\mathrm{km\,h}^{-1}$ during both stages of motion.

c The cameras did not detect the car breaking the speed limit. Find the smallest possible value of T.

11 A particle is moving in a straight line so that its displacement from the starting point, x metres, is given by $x = 0.8t^3 - 0.12t^4$.

Find the maximum speed of the particle during the first 6 seconds.

20 Motion with constant acceleration

In this chapter you will learn how to:

- derive equations for motion with constant acceleration
- use constant acceleration equations
- apply constant acceleration formulae to vertical motion under gravity
- solve multi-stage problems.

Before you start...

Chapter 19	You should know how to use integration to find velocity and displacement from acceleration.	1 A particle moves in a straight line. Its acceleration is given by $a = 2 - 3t^2$. a Given that its velocity at $t = 0$ is $4.2\ \text{ms}^{-1}$ find the equation for the velocity at time t. b Given that its initial displacement from point A is $14m$, find the equation for the displacement of A at time t.
Chapter 3	You should know how to solve quadratic equations.	2 Solve the following quadratic equations: a $4.2t^2 - 11.5t + 2.6 = 0$ b $12t - 4.9t^2 = 5.2$
Chapter 3	You should know how to find the vertex of a parabola.	3 Find the coordinates of the vertex of the parabola with equation: a $y = 12.2x - 36.1x^2$ b $y = 2.1x^2 - 6.3x + 7$

Why do you need the constant acceleration formulae?

There are many situations in mechanics where the acceleration can be modelled as constant: for example, vertical motion under gravity or an object slowing down due to a constant friction force. You can use techniques from Chapter 19 to derive special equations for acceleration, velocity and displacement which apply **only** when the acceleration is constant.

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Section 1: Deriving the constant acceleration formulae

If you know the acceleration of a particle, you can integrate its equation to find equations for velocity and displacement. When the acceleration is constant, you get the following formulae.

Key point 20.1

For a particle moving with constant (or uniform) acceleration a and initial velocity u:

- the velocity at time t is $v = u + \text{at}$
- the displacement from the starting point is $s = ut + \frac{1}{2}at^2$.

Tip

In the constant acceleration formulae, s rather than x is generally used for displacement.

WORKED EXAMPLE 20.1

Prove that if the acceleration, a, is constant and the initial velocity is u then:

a the velocity at time t is given by $v = u + at$

b the displacement from the starting point is $s = ut + \frac{1}{2}at^2$.

a

$v = \int a\,dt$

Link a and v using Key point 19.5.

$= at + c$

a is a constant so we are effectively doing $a\int 1\ \mathrm{d}t$.

When $t = 0$, $v = u$:
$u = 0 + c$
$c = u$
So $v = at + u$
i.e. $v = u + at$

Use the initial condition.

b

$s = \int v\,dt$

Link s and v using Key point 19.5.

$= \int u + at\ dt$

$= ut + \frac{1}{2}at^2 + d$

When $t = 0$, $s = 0$ so $d = 0$.
So $s = ut + \frac{1}{2}at^2$

Since s is measured relative to the starting point.

In these equations, the acceleration can be either positive or negative. Negative acceleration is called **deceleration**.

You can combine the two equations in Key point 20.1 to form another useful equation.

Key point 20.2

$s = vt - \frac{1}{2}at^2$

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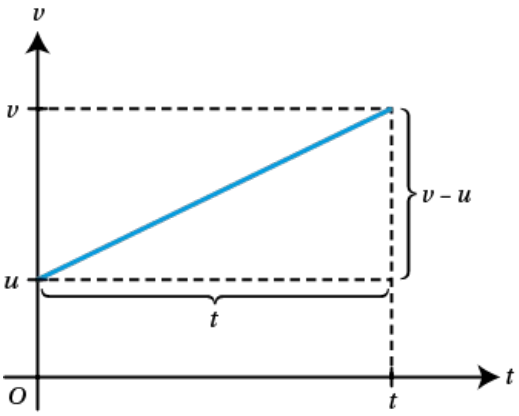
This is proved in question 3 in Exercise 20A.

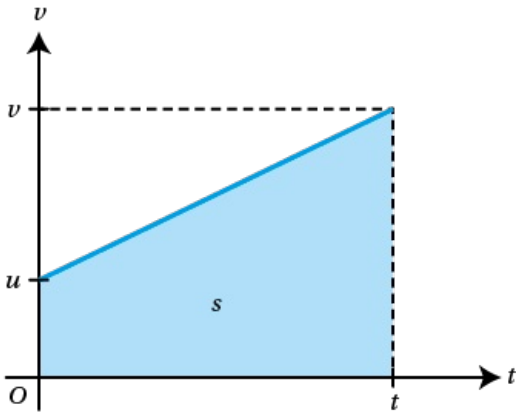
Tip

A deceleration of $3\ \text{ms}^{-2}$ is the same as an acceleration of $-3\ \text{ms}^{-2}$. You would use $a = -3$ in the equation(s) in Key points 20.1 to 20.3.

You can also derive these formulae by looking at a velocity–time graph. If the acceleration is constant, then the graph is a straight line with gradient a.

The method is most easily seen when considering the situation with positive acceleration and positive initial velocity.





At time t, the velocity is V. Since the gradient of the graph is a,

$$a = \frac{v - u}{t}$$
$$\Leftrightarrow v = u + at$$

The area under the graph represents the distance travelled. On the graph shown, the velocity is positive so this is the same as the displacement. You can find it by using the formula for the area of a trapezium.

Key point 20.3

If the initial velocity is u and the velocity at time t is v, then $s = \frac{1}{2}(u + v)\,t$.

You can combine this equation with $v = u + at$ to derive $s = ut + \frac{1}{2}at^2$ and $s = vt - \frac{1}{2}at^2$.

Although in the derivation the velocity and acceleration were assumed to be positive, this formula for the displacement still applies when part of the v–t graph is below the t–axis. You can prove this in question 8 in Exercise 20A.

All the formulae you have derived so far are examples of equations of motion: they tell you how the velocity and the displacement vary with time.

To find an equation not involving time you need to eliminate t from some of the equations.

Tip

You can interpret this formula as saying that the displacement equals average velocity multiplied by time.

WORKED EXAMPLE 20.2

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Use the equations $v = u + at$ and $s = \frac{1}{2}(u + v)\,t$ to derive an equation for v^2 in terms of u, a and s.

$v = u + at$

$t = \dfrac{v - u}{a}$

Make t the subject of the first equation...

$s = \dfrac{1}{2}(u + v)\,t$

$= \dfrac{1}{2}(u + v)\left(\dfrac{v - u}{a}\right)$

and then substitute t into the second to eliminate it.

$= \dfrac{v^2 - u^2}{2a}$

Notice the difference of two squares.

$v^2 = u^2 + 2as$

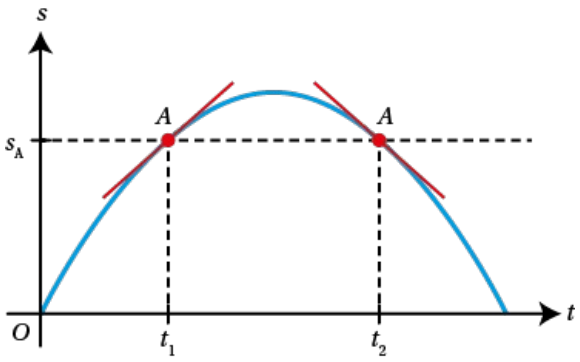
Rearrange to make v^2 the subject.

Key point 20.4

If the initial velocity of a particle is u and it is subject to a constant acceleration a, when its displacement from the starting point is s, its velocity satisfies $v^2 = u^2 + 2as$.

Notice that this equation gives two possible values of v since you can take either the positive or the negative square root. In some situations only one of the values is relevant, but sometimes both are possible.

For example, If the particle moves away from the starting point and then back again, it will pass through a point A twice (at times labelled t_1 and t_2 on the diagram). You can see that the gradients at those two points are the same size (magnitude) but have opposite signs. This means that the particle passes A with equal speeds both times, but moving in opposite directions.



WORKED EXAMPLE 20.3

A particle moves in a straight line with uniform deceleration of $5.1\,\mathrm{m\,s^{-2}}$. When $t = 0$ the particle is at the point A and its velocity is $14.6\,\mathrm{m\,s^{-1}}$.

a Find the possible values of the velocity when its displacement from A is 15 m.

b Find the two times when the particle's displacement from A is 15 m.

a $v^2 = u^2 + 2as$

$= 14.6^2 + 2(-5.1)(15)$

$= 60.16$

You want v:
$u = 14.6$
$a = -5.1$ (remember acceleration is negative)
$s = 15$

$\therefore v = \pm 7.76 ms^{-1}\,(3s.f.)$

v can be either positive or negative.

b $s = ut + \frac{1}{2}at^2$

$15 = 14.6t + \frac{1}{2}(-5.1)\,t^2$

You can use the equation for s in terms of t with the given values of u and a. (Or you could use the values of v you have just calculated and the equation $v = u + at$.) This second equation is easier, but you would need to use value of v which is not exact.

$2.55t^2 - 14.6t + 15 = 0$

$t = 1.34$ or 4.38 seconds (3 s.f.)

This is a quadratic equation, so rearrange before using the formula (or your calculator). You expect to find two possible values of t.

EXERCISE 20A

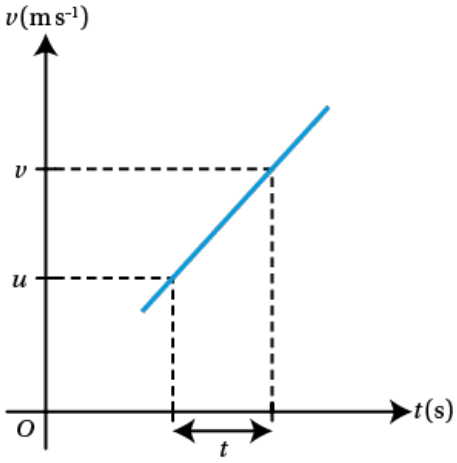
In this exercise avoid early rounding of numerical answers in your working out. If you can, use the memory function on your calculator to save intermediate values, or, alternatively, work to at least 4 s.f. in your working before rounding to 3 s.f. in your final answer.

1 Use the formulae $v = u + at$ and $s = \frac{1}{2}(v + u)\,t$ to prove that $s = ut + \frac{1}{2}at^2$.

2 The diagram shows a velocity–time graph for a particle moving with constant acceleration a. Its speed increases from u to v in time t.

a Use the graph to explain why $a = \frac{v - u}{t}$.

b By splitting the area under the graph into a rectangle and a triangle, show that the distance travelled during time t is given by $s = ut + \frac{1}{2}at^2$.



3 **a** Use the formulae $s = ut + \frac{1}{2}at^2$ and $v = u + at$ to derive the formula $s = vt - \frac{1}{2}at^2$.

b A particle moves with constant acceleration $3.1\,\text{m s}^{-2}$. It travels $300\,\text{m}$ in the first 8 seconds. Find its speed at the end of the 8 seconds.

4 A particle starts with initial velocity u and moves with constant deceleration. After time t its velocity is V. Both u and v are positive.

a Draw a velocity–time graph for the particle's motion during the first t seconds.

b Hence prove that the distance travelled by the particle is $s = \frac{(u + v)\,T}{2}$.

5 **a** If the acceleration is proportional to time, so that $a = kt$, find an expression for v in terms of u, a, k and t.

b Hence prove that:

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i $s = ut + \frac{1}{6}kt^3$

ii $s = \frac{1}{3}(2u + v)t$

iii $2(v - u)(2u + v)^2 = 9ks^2$.

6 Use the formulae $v = u + at$ and $s = ut + \frac{1}{2}at^2$ to derive the formula $v^2 = u^2 + 2as$.

7 The diagram shows velocity–time graphs for two particles. The initial speed of each particle is u.

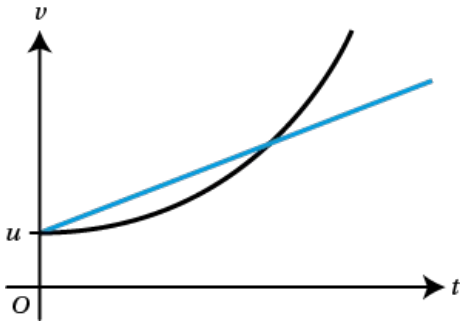
One particle moves with constant acceleration a.

a For this particle, write down an expression for the velocity at time T.

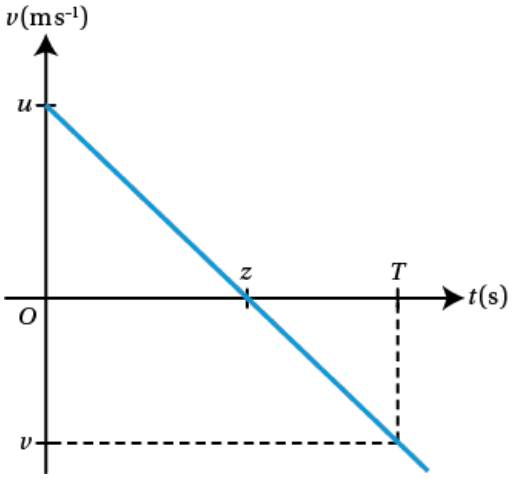
The other particle has velocity given by $v = u + at^2$.

b Find the time when the two particles have the same velocity.

c Given that the two particles travel the same distance in time T, find the value of T.



8 The diagram shows a velocity–time graph for a particle moving with constant acceleration a and initial velocity u. The velocity at time t is –v and the velocity at time z is zero.



Let s denote the displacement of the particle from the starting point at time t.

Show that $s = \frac{1}{2}uz + \frac{1}{2}v(T - z)$.

Section 2: Using the constant acceleration formulae

You now have five different equations that can be used to solve problems involving motion with constant acceleration.

Selecting which one to use depends on which of the five quantities a, u, v, s and t are involved in the question.

Quantities involved	Equation
$u, v, a,$ t	$v = u + at$
$a, u, s,$ t	$s = ut + \frac{1}{2}at^2$
$s, u, v,$ t	$s = \frac{1}{2}(u + v)\,t$
$a, u, v,$ s	$v^2 = u^2 + 2as$
$a, v, s,$ t	$s = vt - \frac{1}{2}at^2$

All five constant acceleration formulae will appear in your formula book.

If you write down what you are given in the question and what you are trying to find, you can then select the most useful equation.

WORKED EXAMPLE 20.4

A car moves with constant acceleration. When $t = 0$, it passes a junction with velocity $8.2\,\mathrm{m\,s^{-1}}$. It passes the next junction, $320\,\mathrm{m}$ away, 24 seconds later. Find the car's velocity as it passes the second junction.

$s = 320$
$u = 8.2$
$t = 24$
$v = ?$

Write down what you are given and what you are trying to find. You don't need to write units in the calculations, but remember to add them to the final answer.

$$s = \frac{1}{2}(u + v)\,t$$
$$320 = \frac{1}{2}(8.2 + v)\,(24)$$

Looking at the table, s, u, v and t feature in the third equation: $s = \frac{1}{2}(u + v)\,t$.

$$8.2 + v = \frac{640}{24}$$
$$8.2 + v = 26.7$$
$$v = 18.5\,\mathrm{m\,s^{-1}}$$

Now solve the equation to find V.

Remember to give correct units with your answer.

WORKED EXAMPLE 20.5

An object decelerates uniformly from $26.3\,\mathrm{m\,s^{-1}}$ to $16.2\,\mathrm{m\,s^{-1}}$, while covering a distance of 240 m. Find its deceleration.

$s = 240$
$u = 26.3$

Write down what you are given and what you are trying to find.

$v = 16.2$
$a = ?$

Make sure you get u and v the right way round.

$v^2 = u^2 + 2as$

s, u, v and a feature in the fourth equation: $v^2 = u^2 + 2as$

$16.2^2 = 26.3^2 + 2a(240)$
$480a = 16.2^2 - 26.3^2$
$480a = -429.25$

Solve for a.

$a = -0.894 ms^{-2} (3\,s.f.)$

Give your answer as a positive **deceleration**, remembering to include units.

So the deceleration is $0.894\ \mathrm{m\ s^{-2}}$.

WORKED EXAMPLE 20.6

Imogen is walking down the street when she sees a bus at the bus stop $25\ \mathrm{m}$ away. She starts accelerating uniformly at $0.9\,\mathrm{m\,s^{-2}}$ and reaches the bus stop 4 seconds later. Find her velocity when she arrives at the bus stop.

$s = 25$
$a = 0.9$
$t = 4$
$v = ?$

Write down what you are given and what you are trying to find.

$s = vt - \frac{1}{2}at^2$

Choose the equation that links s, a, t and V.

$25 = 4v - \frac{1}{2} \times 0.9 \times 4^2$
$= 4v - 7.2$
$32.2 = 4v$
$v = 8.05\ \mathrm{ms^{-1}}$

Put in the numbers and rearrange.

EXERCISE 20B

1 Choose an appropriate formula to answer each question.

a Find the values of u and s when:

i $a = 2.4\,\mathrm{m\,s^{-2}}$, $v = 18\,\mathrm{m\,s^{-1}}$, $t = 6\mathrm{s}$

ii $a = 0.6\,\mathrm{m\,s^{-2}}$, $v = -21\,\mathrm{m\,s^{-1}}$, $t = 3.5\mathrm{s}$

b Find the values of t and v when:

i $u = 13\,\mathrm{m\,s^{-1}}$, $a = -1.2\,\mathrm{m\,s^{-2}}$, $s = 60$ m

ii $u = 20\,\mathrm{m\,s^{-1}}$, $a = -3\,\mathrm{m\,s^{-2}}$, $s = 40$ m

c Find the values of a and t when:

i $u = 12\,\mathrm{m\,s^{-1}}$, $v = 6\,\mathrm{m\,s^{-1}}$, $s = 120\mathrm{m}$

ii $u = -3\,\mathrm{m\,s^{-1}}$, $v = 16\,\mathrm{m\,s^{-1}}$, $s = 120\mathrm{m}$

d Find the values of a and s when:

i $u = 5\,\mathrm{m\,s^{-1}}$, $v = -6\,\mathrm{m\,s^{-1}}$, $t = 6\mathrm{s}$

ii $u = 8\,\mathrm{m\,s^{-1}}$, $v = -5\,\mathrm{m\,s^{-1}}$, $t = 10$ s

2 A car accelerates uniformly from rest to $12.45\,\mathrm{m\,s^{-1}}$ in 6.5 seconds. Find:

a the acceleration

b the distance travelled during this time.

3 A cyclist is travelling at the speed of $4.2\,\mathrm{m\,s^{-1}}$. She accelerates uniformly at $0.3\,\mathrm{m\,s^{-2}}$ for 7 seconds. Find:

a her final speed

b the distance she travels in the 7 seconds.

4 A particle reduces its speed from $16.3\,\mathrm{m\,s^{-1}}$ to $7.5\,\mathrm{m\,s^{-1}}$ while travelling 120 m.

a Find the constant acceleration of the particle.

b Find the distance the particle would travel in another 5 seconds.

5 A particle starts from rest and accelerates uniformly at $2.5\,\mathrm{m\,s^{-2}}$. How long will it take to travel 250 m?

6 A particle reduces its speed from $20\,\mathrm{m\,s^{-1}}$ to $8.2\,\mathrm{m\,s^{-1}}$ while travelling 100 m. Assuming it continues to move with the same constant acceleration, how long will it take to travel another 20 m?

7 A particle moves with constant deceleration of $3.6\,\mathrm{m\,s^{-2}}$. It travels $350\,\mathrm{m}$ while its speed halves. Find the time it takes to do this.

8 A car reduces its speed from $18\,\mathrm{m\,s^{-1}}$ to $9\,\mathrm{m\,s^{-1}}$ while travelling 200 m. Assuming the car continues to move with the same uniform acceleration, how much further will it travel before it stops?

9 **a** A particle moves in a straight line with constant acceleration $a = -3.4\,\mathrm{m\,s^{-2}}$. At $t = 0$ its velocity is $u = 6\,\mathrm{m\,s^{-1}}$.

Find its maximum displacement from the starting point.

b Explain why this is not the maximum distance from the starting point.

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Section 3: Vertical motion under gravity

When an object is thrown in the air, the force of gravity acts on it: if the object is moving upwards the force of gravity slows it down; if it is moving downwards the force of gravity speeds it up.

The force of gravity produces acceleration. This acceleration is the same regardless of the mass of the object, and is called **gravitational acceleration** or **acceleration of freefall**. On or near the surface of Earth, this value is denoted by g and is approximately $9.8\,\mathrm{m\,s^{-2}}$. In fact, this value varies slightly with the geographical position and height above sea level, but in this course you can assume that it is constant.

Explore

What relationship did Newton discover between the gravitational force between two objects and the distance between them?

The motion of the object may also be affected by air resistance. However, if the object is modelled as a particle, you can assume that the air resistance is sufficiently small as not to affect the resulting motion significantly. With these modelling assumptions (that any other force can be ignored), an object moving vertically under gravity moves in a straight line with constant acceleration. You can therefore use the equations from Section 2 to calculate its velocity and displacement.

Focus on ...

See Focus on ... Modelling 4 if you want to explore the effects of air resistance and variable g.

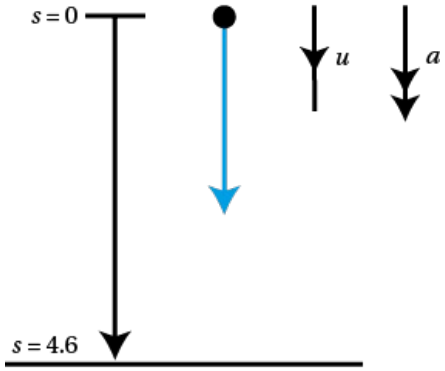
WORKED EXAMPLE 20.7

A small ball is thrown straight downwards from a window $4.6\,\mathrm{m}$ above the ground with an initial velocity of $1.2\,\mathrm{m\,s^{-1}}$. Air resistance can be ignored.

a How long does it take for it to reach the ground? Give your answer to 2 s.f.

b How would your answer change if air resistance were included?

a



Draw a diagram to show the directions of displacement, velocity and acceleration.

$s = 4.6$

$u = 1.2$

$a = 9.8$

$t = ?$

Write down what you are given and what you are trying to find. Since the displacement is measured downwards, the ground is at $s = +4.6$ m.

Select the equation involving s, u, a and t:
$s = ut + \frac{1}{2}at^2$

$$s = ut + \frac{1}{2}at^2$$

$$4.6 = 1.2t + 4.9t^2$$

$$4.9t^2 + 1.2t - 4.6 = 0$$

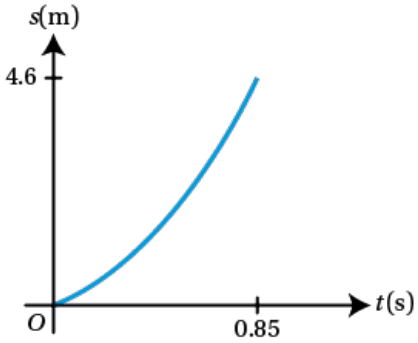
$$t = 0.854 \text{ or } -1.1$$

$\therefore t = 0.854$ s

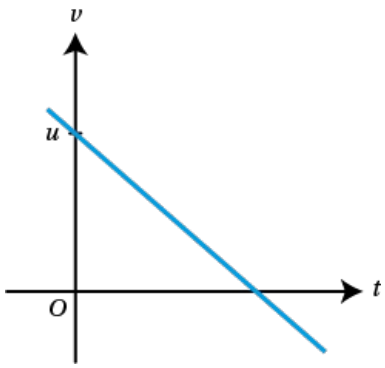
We want the positive root.

b The time taken would be greater as air resistance would decrease the acceleration.

In Worked example 20.7 we took the positive direction to be downward, so the acceleration, velocity and displacement were all positive. The displacement–time graph is a positive parabola with equation $s = 4.9t^2 + 1.2t$. Only the part of the parabola from $t = 0$ to $t = 0.85$ is relevant in this situation. You can see that the velocity increases as the ball approaches the ground.



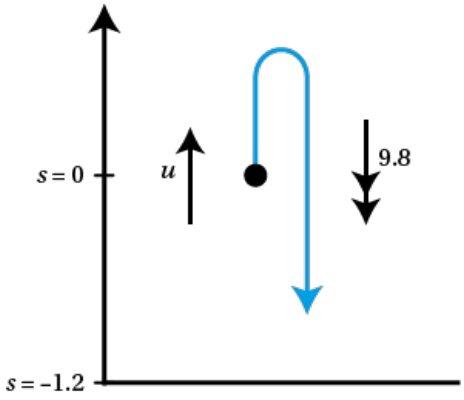
If an object is thrown vertically upwards, it makes more sense to measure displacement in the upward direction. In this case the initial velocity is positive but the acceleration, which always points downwards, is negative: $a = -9.8 \text{ m s}^{-2}$. The velocity-time graph has equation $v = u - 9.8t$. As can be seen from the graph, if the object is allowed to move freely under gravity, the velocity will eventually become negative, corresponding to the object changing direction and starting to fall.



WORKED EXAMPLE 20.8

A stone is thrown upwards with a velocity of 5.6 m s^{-1} from a platform 1.2 m above ground. Find the velocity of the stone when it hits the ground.

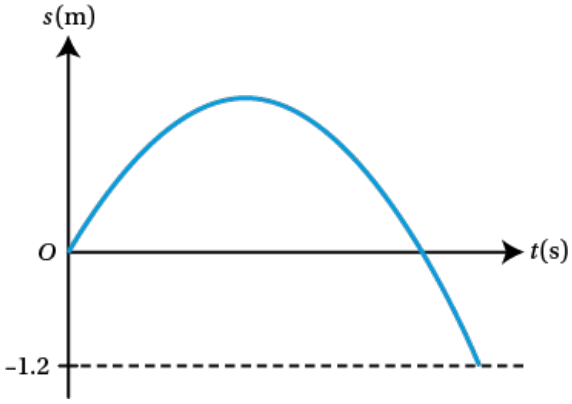
Draw a diagram to show the directions. Since the initial direction of motion is upwards, take this to be the positive direction.



$s = -1.2$	Write down what you are given and what you are trying to find.
$u = 5.6$	
$a = -9.8$	The displacement is measured upwards, so the ground is at $s = -1.2\text{m}$.
$v = ?$	
	Acceleration is downwards so a is negative.
$v^2 = u^2 + 2as$	Select the equation involving s, u, v and a: $v^2 = u^2 + 2as$
$v^2 = 5.6^2 + 2(-9.8)(-1.2)$	
$v^2 = 54.88$	
$v = \pm 7.41$	
The final velocity of the stone is $-7.41\,\text{m s}^{-1}$.	Before hitting the ground the stone is moving downwards, so its velocity is negative.

The displacement–time graph for the motion of the stone is a negative parabola with equation $s = -4.9t^2 + 5.6t$. Its vertex corresponds to the highest point reached by the stone. Note that the value of s at this point is not the maximum height of the stone above ground, because the displacement is measured from the platform.

The part of the graph below the horizontal axis represents the motion of the stone below the platform until it hits the ground.



WORKED EXAMPLE 20.9

For the stone from the previous example, find:

a the greatest height above the ground

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b how long it takes to reach the highest point

c for how long it is falling.

a $u = 5.6$

$v = 0$

$a = -9.8$

$s = ?$

Write down what you are given and what you are trying to find.

At the greatest height, $v = 0$.

$v^2 = u^2 + 2as$

$0 = 5.6^2 + 2(-9.8)s$

$19.6s = 31.36$

$s = 1.6$

You can use the same equation as in Worked example 20.8.

$h = s + 1.2$

$= 2.8\text{m}$

The maximum height of the stone is 2.8 m.

s is the displacement from the starting point, which is itself $1.2\ \text{m}$ above the ground.

b $u = 5.6$

$v = 0$

$a = -9.8$

$t = ?$

You now want to find t rather than s.

$v = u + at$

$0 = 5.6 - 9.8t$

$t = 0.571s$

Select the equation involving u, v, a and t: $v = u + at$

c $u = 0$

$s = 2.8$

$a = 9.8$

$t = ?$

The stone falls from its highest points until it hits the ground.

When it is at the highest point, the stone is at rest $(u = 0)$ and $2.8\ \text{m}$ above ground (from part a).

The direction of motion is downwards, so take that as the positive direction. This means that both s and a are positive.

$s = ut + \frac{1}{2}at^2$

$2.8 = 0 + \frac{1}{2}(9.8)t^2$

$t^2 = 0.571$

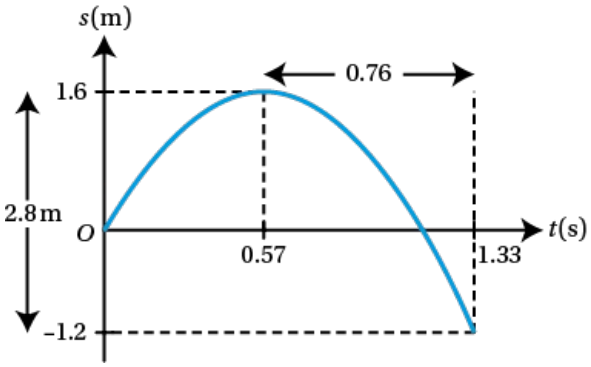
$t = 0.756$

The stone spends 0.756 seconds falling.

Select the equation involving u, s, a and t.

You can see all this information on the displacement-time graph.

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WORK IT OUT 20.1

A stone is thrown vertically upwards from the top of a cliff with speed $10\,\mathrm{m\,s^{-1}}$. It hits the sea below with speed $33\,\mathrm{m\,s^{-1}}$. Find the time taken, giving your answer to 2 s.f.

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

Solution 1

$$
\begin{aligned}
u &= 10 \\
v &= 33 \\
a &= 9.8 \\
v &= u + at \\
33 &= 10 + 9.8t \\
23 &= 9.8t \\
t &= \frac{23}{9.8} = 2.4\mathrm{s}
\end{aligned}
$$

Solution 2

$$
\begin{aligned}
u &= 10 \\
v &= 33 \\
a &= 9.8 \\
v &= u + at \\
-33 &= 10 - 9.8t \\
-43 &= -9.8t \\
t &= \frac{43}{9.8} = 4.4\mathrm{s}
\end{aligned}
$$

Solution 3

Highest point when $v = 0$:

$u = 10$

v $= 0$

$a = -9.8$

$v = u + at$

$0 = 10 - 9.8t$

$-10 = -9.8t$

$t = \dfrac{10}{9.8} = 1.0\,\text{s}$

Same time to come down:

$\therefore t = 1.0 \times 2$

$= 2.0\,\text{s}$

WORKED EXAMPLE 20.10

An object is thrown upwards from ground level with initial velocity u.

a Find an expression in terms of u and g for the time it takes to

i reach the highest point

ii return to the ground.

b Find the speed of the object when it hits the ground.

a i At the highest point, $v = 0$:

$v = u + at$

$o = u + (-g)t$

$\Rightarrow t = \dfrac{u}{\text{g}}$

At the highest point, $v = 0$. You need an equation involving u, v, a and t.

If the displacement is measured upwards, u is positive and a is negative: $a = -\text{g}$.

ii When the object returns to the ground: s $= 0$:

$s = ut + \dfrac{1}{2}at^2$

$0 = ut - \dfrac{1}{2}\text{g}t^2$

$\Leftrightarrow 0 = t\left(u - \dfrac{1}{2}\text{g}t\right)$

$\Leftrightarrow t = 0 \text{ or } t = \dfrac{2u}{\text{g}}$

$\therefore t = \dfrac{2u}{\text{g}}$

When the object returns to the ground, $s = 0$. You need an equation involving u, a, s and t.

The expression on the right can be factorised.

$t = 0$ is the starting point, so you need the other value of t.

b $v = u + at$

$= u + (-\text{g})\left(\dfrac{2u}{\text{g}}\right)$

$= u - 2u$

$= -u$

$\therefore$ The speed is u.

You now know the time when the object hits the ground, so you can use it to find the velocity.

You should notice two things here, summarised in Key point 20.5.

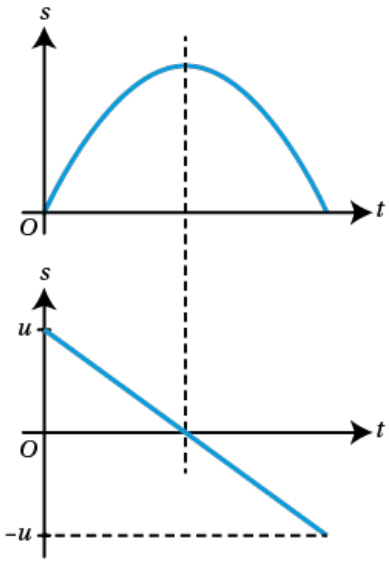
Key point 20.5

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For an object thrown upwards from ground level:

- the time taken to return to the ground is twice the time to the highest point; the time to go up is the same as the time to come down
- the speed of the object when it hits the ground equals its initial speed.

This can also be seen from the displacement–time graph and the velocity–time graph, which are both symmetrical.



Tip

'The speed of the particle when it hits the ground' means its speed at the moment of impact. It is not zero!

EXERCISE 20C

1 A ball is thrown vertically downwards with the given initial speed. Assume the air resistance can be ignored, and that the ball does not reach the ground. Find the speed and the distance travelled by the ball at the specified time.

a i initial speed $= 19\ \text{m s}^{-1}$, $t = 0.3$ seconds

ii initial speed $= 23\ \text{m s}^{-1}$, $t = 0.8$ seconds

b i initial speed $= 0\ \text{m s}^{-1}$, $t = 3.1$ seconds

ii initial speed $= 0\ \text{m s}^{-1}$, $t = 6$ seconds

2 A ball is thrown vertically upwards with the given initial speed. Assume that air resistance can be ignored. Find the magnitude and direction of the velocity, and the height above the projection point, at the specified time.

a i initial speed $= 9\ \text{m s}^{-1}$, $t = 2.5$ seconds

ii initial speed $= 12\ \text{m s}^{-1}$, $t = 0.8$ seconds

b i initial speed $= 35\ \text{m s}^{-1}$, $t = 3.1$ seconds

ii initial speed $= 20\ \text{m s}^{-1}$, $t = 1.1$ seconds

3 A stone is dropped from rest from a height of 25 m. Find its velocity when it hits the ground.

4 A ball is thrown vertically upwards from ground level with a speed of $18.5\ \text{m s}^{-1}$. Assuming that air resistance can be ignored, find:

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a how long the ball takes to reach the height of 15 m

b the speed of the ball at this time.

c Explain how your answers to parts **a** and **b** would change if air resistance was included?

5 An object was projected vertically upwards with velocity u m s^{-1}. When it has reached the height of 5.6 m its velocity is 3.5 m s^{-1}. Find the value of u.

Worksheet

For more examples of vertical motion see Support sheet 20.

6 A ball is projected vertically upwards from the top of a 15 m tall cliff with a velocity of 28 m s^{-1}.

a Find the maximum height of the ball above ground.

b How long does the ball take to reach the ground and how fast is it going when it hits it?

7 A bunch of flowers is projected vertically upwards towards a window 8.3 m above the point from which it is thrown.

a Given that the initial speed is 6.5 m s^{-1}, and assuming that air resistance can be ignored, will it reach the window?

b Find the minimum projection speed required for the flowers to reach the window.

c How would your answer to part b change if the air resistance were included?

8 A ball is thrown vertically upwards from a window 12 m above ground level. The initial velocity of the ball is 16 m s^{-1}.

a After what time will the ball reach the highest point?

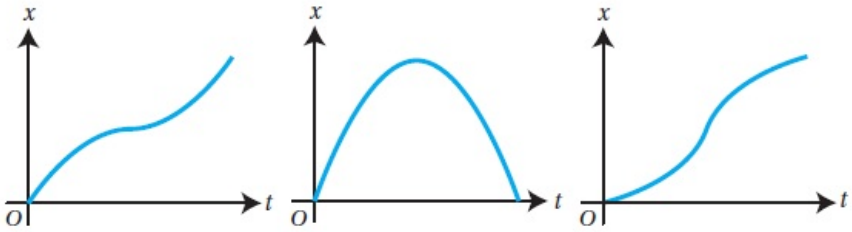
b How long does the ball take to fall to the ground?

c Sketch the velocity–time graph for the ball's motion.

9 A particle is projected upwards from ground level with initial speed u. Air resistance can be ignored.

a Find an expression, in terms of u, for the time it takes the particle to return to ground level.

b Which of the following graphs shows the distance travelled by the particle as a function of time?



10 A small stone is projected vertically upwards from the top of a cliff, with speed 8 m s^{-1}. It hits the ground with speed 18 m s^{-1}. How high is the cliff?

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Section 4: Multi-stage problems

You may have information about two separate stages of the motion, which will often lead to simultaneous equations.

WORKED EXAMPLE 20.11

Three cameras are positioned on a straight road. The distance between the first and the second camera is $50\ \text{m}$ and the distance from the second to the third camera is 30 m.

A car passes the first camera with speed $u\ \text{m s}^{-1}$ and immediately starts braking. It passes the second camera 3.1 seconds later and the third camera 2.3 seconds after that. Assuming the deceleration remains constant, find the value of u.

From the first to the second camera:

$s = 50\text{m}$

$t = 3.1s$

$u = u$

Write down the information you have for the first stage of motion.

$$s = ut + \frac{1}{2}at^2$$

$$50 = 3.1u + \frac{1}{2}a\left(3.1^2\right)$$

$$50 = 3.1u + 4.805a$$

This is not enough information to find u. However, you also have some information about the second stage of the motion.

You can write down an equation involving s, t and u (for example, the second one from the table), and hope you can use the extra information to find a.

From the first to the third camera:

$s = 80$

$t = 5.4$

$u = u$

Now write down the information for both stages together. t and s are still measured from the first camera.

$$s = ut + \frac{1}{2}at^2$$

$$80 = 5.4u + \frac{1}{2}a\left(5.4^2\right)$$

$$80 = 5.4u + 14.58a$$

You still can't find u, but you can write another equation.

$$\therefore \begin{cases} 3.1u + 4.805a = 50 \\ 5.4u + 14.58a = 80 \end{cases}$$

$$\Rightarrow u = 17.9\ \text{m s}^{-1}$$

You now have two simultaneous equations. You can solve them on your calculator, or substitute for a from one equation into the other.

In Worked example 20.11 you were told that the acceleration remains constant throughout the 5.4 seconds. But there are many situations where the acceleration changes part-way through the motion. For example, changing gears while driving a car might change its acceleration. Because the acceleration is not constant, you need to consider the two stages of motion separately.

WORKED EXAMPLE 20.12

A car starts from rest and accelerates at a constant rate of $2.7\ \text{m s}^{-2}$ for 6 seconds. It then changes to a higher gear and accelerates for 10 seconds, reaching a speed of $28.2\ \text{m s}^{-1}$. Find the acceleration for the second stage of motion.

First stage:

$u = 0$

$a = 2.7$

$t = 6$

$$\begin{aligned} v &= u + at \\ &= 0 + 2.7 \times 6 \\ &= 16.2\,\text{m s}^{-1} \end{aligned}$$

For the second stage, you have the time and the final speed. To work out the acceleration, you need the initial speed. But this is the same as the final speed for the first stage.

Second stage:

$u = 16.2$

$t = 10$

$v = 28.2$

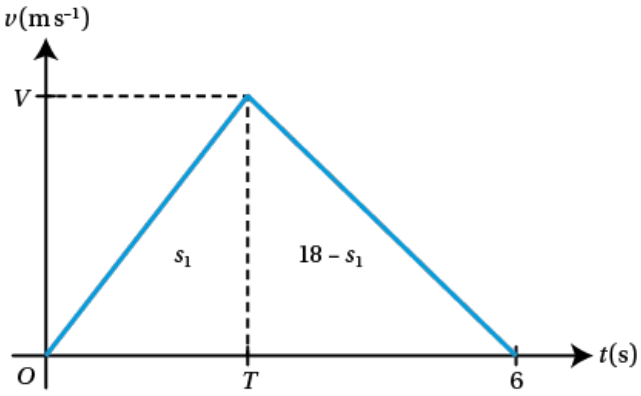
$$\begin{aligned} v &= u + at \\ 28.2 &= 16.2 + 10a \\ a &= 1.2\text{s}^{-2} \end{aligned}$$

Using the value of v from the first part as u in the second, you can find a.

Sometimes it is not possible to find all the information for the first stage before using it in the second stage.

WORKED EXAMPLE 20.13

A dog accelerates uniformly from rest, reaching a speed of $v\ \text{m s}^{-1}$. It then decelerates back to rest. If the dog ran the total distance of $18\ \text{m}$ in 6 seconds, find the value of V.



You only know the total distance and total time, but you still need to split the motion into two separate stages as the acceleration changes.

First stage:

$$s_1 = \frac{VT}{2} \qquad (1)$$

Let the first stage of the motion be from $t = 0$ to $t = T$ and the distance covered be s_1. You can write an equation involving T, V and s_1 by using the area of the triangle on the left.

Second stage:

$$18 - s_1 = \frac{1}{2}(V)(6 - T)$$

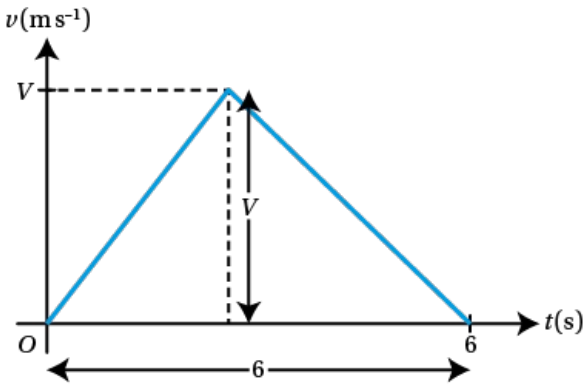
$$18 - s_1 = 3V - \frac{VT}{2} \qquad (2)$$

The second stage is from $t = t$ to $t = 6$, so the time taken for this stage is $6 - T$. The distance covered is $18 - s1$. You can use the area of the triangle on the right to write another equation.

Now substitute s_1 from equation (1) into equation (2).

$$18 - \frac{VT}{2} = 3V - \frac{VT}{2}$$
$$18 = 3V$$
$$V = 6$$

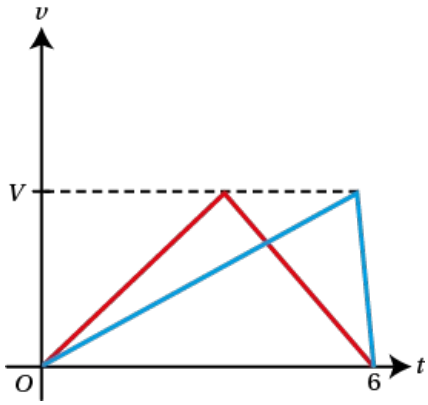
The calculations in Worked example 20.13 are quite involved, and at the beginning it looked as if there wasn't enough information to answer the question. For this example, you can arrive at the answer more quickly if you use the velocity–time graph.



Although you don't know t or s1, you can use the fact that the total distance is the area under the graph:

$$18 = \frac{1}{2} \times 6 \times V$$
$$V = 6$$

Notice that you never found T, so you can't tell when the dog changes from accelerating to decelerating. In fact, there isn't sufficient information in the question to determine this. You can see from the following graph that the dog could accelerate and decelerate for equal amounts of time (red graph), or it could accelerate at a lower rate for longer and then stop suddenly (blue graph). In both cases it covers the same distance (as both triangles have the same area).



Many two-stage problems involve vertical motion. You need to be very careful about the direction of velocity and acceleration.

Tip

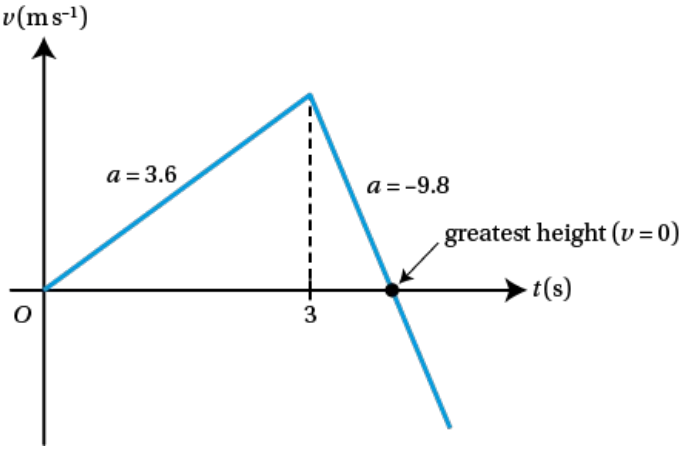
If the equations look too complicated, you should always try sketching the graph to see if it gives you a simpler method.

WORKED EXAMPLE 20.14

A toy rocket with an engine starts from rest at ground level and moves vertically upwards with constant acceleration of $3.6\,\mathrm{m\,s^{-2}}$. After three seconds the engine is turned off and the rocket moves freely under gravity. Find:

a the greatest height reached by the rocket

b the total time the rocket spends in the air.

a 

There are two stages: first the rocket has upward acceleration of 3.6 m s–2 and then downward acceleration of 9.8 m s–2.

First stage:

$u = 0$

$a = 3.6$

$t = 3$

$v = u + at$

$= 0 + 3.6 \times 3$

$= 10.8\,\mathrm{m\,s^{-1}}$

The greatest height is when $v = 0$ and is reached during the second stage, so you need to find the initial velocity for that stage, which is the final velocity for the first stage.

$s = ut + \frac{1}{2}at^2$

$= 0 + \frac{1}{2}(3.6)\left(3^2\right)$

$= 16.2\,\mathrm{m}$

You also need the height reached during the first stage, because the displacement for the second stage is measured from there.

Second stage:

$u = 10.8$

$v = 0$

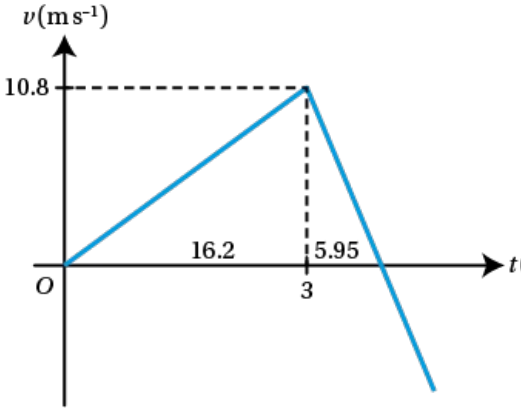
$a = -9.8$

$v^2 = u^2 + 2as$

$0 = 10.8^2 + 2(-9.8)s$

$s = 5.95m$

The initial velocity is upwards but the acceleration is now downwards so it is negative.



The height is $16.2 + 5.95 = 22.2m$

Calculate the total height.

b Second stage:

$u = 10.8$

$a = -9.8$

$s = -16.2$

The total time spent in the air equals 3 seconds from the first stage, plus the time it takes to reach the ground in the second stage. The ground is $16.2\,\mathrm{m}$ below the starting point for the second stage.

$$s = ut + \frac{1}{2}at^2$$

$$-16.2 = 10.8t - 4.9t^2$$
$$4.9t^2 - 10.8t - 16.2 = 0$$
$$t = -1.02 \text{ or } 3.23$$
$$\therefore t = 3.23 \text{ s}$$

Total time:

$3.23 + 3 = 6.23$ s

You need to add the time from the first stage.

EXERCISE 20D

1 A car accelerates from rest for 8.3 seconds, reaching a speed of $12.8\ \mathrm{m\,s^{-1}}$. It then travels for another 12 seconds with deceleration $0.8\ \mathrm{m}$ s−2. Find the total distance travelled by the car.

2 A fox is running in a straight line. It passes tree A with a speed of $8.3\ \mathrm{m\,s^{-1}}$ and starts accelerating uniformly. It passes tree B, $120\ \mathrm{m}$ away, 13 seconds later. It immediately starts decelerating, coming to rest next to tree C, $250\ \mathrm{m}$ from tree B.

a Find the speed of the fox when it passes tree B.

b Find the deceleration of the fox.

3 A runner starts with speed u and accelerates uniformly. She covers the first $10\ \mathrm{m}$ in $2.1\ \mathrm{s}$ and the next $10\ \mathrm{m}$ in another 1.1 s. Find the value of u.

4 A cyclist starts at the bottom of a hill moving at a speed of $13.5\ \mathrm{m\,s^{-1}}$. She moves with a constant deceleration of $0.9\ \mathrm{m\,s^{-2}}$, reaching the top of the hill 9.2 seconds later. She then accelerates down the hill at $1.6\,\mathrm{m\,s^{-2}}$ for 86 m. Find the speed of the cyclist when she reaches the bottom of the hill.

5 A ball is dropped from a height of $2.6\ \mathrm{m}$ above the surface of a water well and falls freely under gravity. After it enters the water, the ball's acceleration decreases to $1.2\,\mathrm{m\,s^{-2}}$. It reaches the bottom of the well 0.9 seconds later. Assuming the acceleration through the water is constant over a short period of time, find the depth of the water in the well.

6 A rocket is projected vertically upwards with a speed of $260\ \mathrm{km\,h^{-1}}$. After 6 seconds the engines are switched on and the rocket starts accelerating at $2.8\,\mathrm{m\,s^{-2}}$. Find the total time from the launch it takes for the rocket to reach a height of 400 m.

7 A car starts from rest at time $t = 0$. It accelerates uniformly until its speed reaches $v\ \mathrm{m\,s^{-1}}$. It travels at constant speed for 12 seconds and then decelerates uniformly, coming to rest when $t = 26$. The total distance travelled by the car is 840 m. Find the value of V.

8 A model rocket starts from rest. It has an engine that produces an upward acceleration of $5.2\ \mathrm{m\,s^{-2}}$. When the rocket has reached the height of $25\ \mathrm{m}$ the engine is switched off. Find the maximum height of the rocket and its speed when it returns to the ground.

9 A ball is dropped (with zero initial velocity) from a window $30\ \mathrm{m}$ above the ground. Half a second later, another ball is projected vertically upwards from the ground, vertically below the window. The balls collide when they are $15\ \mathrm{m}$ above the ground. Find the initial velocity of the second ball.

10 A motorbike is overtaking a lorry on a straight horizontal road. The length of the lorry is $15.2\ \mathrm{m}$ and the motorbike is modelled as a particle. Initially the motorbike and the lorry are moving at a constant velocity of $18.6\ \mathrm{m\,s^{-1}}$. The lorry continues to move with constant velocity. The motorbike starts $35\ \mathrm{m}$ behind the lorry, accelerates at a constant rate until it reaches speed $v\ \mathrm{m\,s^{-1}}$, then decelerates at a constant rate back to $18.6\ \mathrm{m\,s^{-1}}$. It ends up $40\ \mathrm{m}$ in front of the lorry at the point they again have the same velocity. The overtaking takes 28 seconds.

a How much further does the motorbike travel than the lorry during the 28 seconds?

b On the same axes, sketch the velocity–time graphs for the lorry and the motorbike. Hence find the value of V.

Checklist of learning and understanding

- When an object is moving with constant acceleration you can use the following equations:
  - $v = u + at$
  - $s = ut + \frac{1}{2}at^2$
  - $s = vt - \frac{1}{2}at^2$
  - $s = \frac{1}{2}(u + v)t$

- $v^2 = u^2 + 2as$

where a is the acceleration, u is the initial velocity, v is the velocity at time t and s is the displacement from the starting position.

- You can derive these equations using integration or from a straight line velocity–time graph.
- In these equations all the quantities (usually with the exception of t) can be either positive or negative.
  - The object is instantaneously at rest, and may be changing direction, when $v = 0$.
  - The object returns to the starting point when s $= 0$.
- A special case of motion with constant acceleration is vertical motion under gravity. The acceleration is directed downwards and has magnitude $\mathrm{g} = 9.8\ \mathrm{m\ s^{-2}}$.
  - The object reaches maximum height when $v = 0$.
  - For projection from ground level, time going up equals time going down, and the object hits the ground with the same speed with which it was projected.
- The constant acceleration model requires two assumptions:
  - air resistance can be ignored (true if the object is modelled as a particle)
  - g is constant (true for small heights).

Mixed practice 20

1 A particle is moving with a speed of $12\ \text{m s}^{-1}$ when it starts to accelerate uniformly at $1.6\ \text{m s}^{-2}$.

a Find how long it takes for the particle's speed to increase to $26\ \text{m s}^{-1}$.

b How far does the particle travel in that time?

2 A stone is projected vertically upwards from the ground with a speed of $11\ \text{m s}^{-1}$.

a Find the speed and the direction of motion of the stone after 2 seconds.

b Find the height of the stone above ground at this time.

3 A cyclist passes point P with a speed of $6.2\ \text{m s}^{-1}$ and starts to decelerate uniformly at $2.1\ \text{m s}^{-2}$. How fast is she moving after she has travelled 8 m?

4 An object is projected vertically upwards with speed $7\ \text{m s}^{-1}$. Calculate:

a the speed of the object when it is $2.1\ \text{m}$ above the point of projection

b the greatest height above the point of projection reached by the object

c the time after projection when the object is travelling downwards with speed $5.7\ \text{m s}^{-1}$.

OCR, GCE Mathematics, Paper 4728, June 2009 [Question part reference style adapted]

5 A car travels on a straight horizontal road. It passes point A with a speed of $32\ \text{km h}^{-1}$ and starts to decelerate uniformly until it reaches speed $v\ \text{km/h}$. It then accelerates uniformly. When it reaches point B its speed is $32\ \text{km h}^{-1}$ again.

a Draw the velocity–time graph representing the car's journey.

b Given that the distance AB is $550\ \text{m}$ and the journey takes 1.2 minutes, find the value of V.

c Find the average speed of the car during its journey from A to B.

6 **a** Use the formulae $v = u + at$ and $s = ut + \frac{1}{2}at^2$ to derive the formula $v^2 = u^2 + 2as$.

b A ball is projected vertically downwards from the top of the building, with a speed of $8.5\ \text{m s}^{-1}$. It reaches the ground with a speed of $110\ \text{m s}^{-1}$. Find the height of the building.

7 Two particles are projected simultaneously with a speed of $15.4\ \text{m s}^{-1}$. The first particle is projected vertically upwards from ground level. The second particle is projected vertically downwards from a height of 20 m. The two particles move on the same straight line. Find:

a the height above ground where the particles collide

b the speed of each particle at the moment they collide.

8 A particle moves with constant acceleration a. When $t = 0$ it passes point O with velocity u. Let s be the displacement from O at time t.

Use integration to show that $s = ut + \frac{1}{2}at^2$.

9 A particle P is projected vertically upwards, from horizontal ground, with speed $8.4\ \text{m s}^{-1}$.

a Show that the greatest height above the ground reached by P is 3.6 m.

A particle Q is projected vertically upwards, from a point $2\ \text{m}$ above the ground, with speed $u\ \text{s}^{-1}$. The greatest height above ground reached by Q is also 3.6 m.

b Find the value of u.

It is given that P and Q are projected simultaneously.

c Show that, at the instant when P and Q are at the same height, the particles have the same speed and are moving in opposite directions.

OCR, GCE Mathematics, Paper 4728, June 2007 [Question part reference style adapted]

10 Two cars start from rest, from the same start line, and accelerate uniformly along a racetrack running perpendicular to the start line. After 5 seconds the first car is $30\,\mathrm{m}$ in front of the second car. How far in front is it after another 5 seconds?

11 A ball is projected vertically upwards from ground level with speed u_1. At the moment when this first ball is at its maximum height, a second ball is projected vertically upwards from ground level with speed u_2. The two balls fall back on the ground at the same time without colliding in the air. Find the ratio $u_1 : u_2$.

12 A particle travels in a straight line and decelerates uniformly at $2\,\mathrm{m\,s^{-2}}$. When $t = 0$ its velocity is $u\ \mathrm{m\,s^{-1}}$ and when $t = 100$ its velocity is $-v\,\mathrm{m\,s^{-1}}$ (where $u > v > 0$). The average speed of the particle over the 100 seconds is $62.5\,\mathrm{m\,s^{-1}}$. Find the values of u and V.

Worksheet

For a selection of more challenging problems using constant acceleration formulae see Extension sheet 20.

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21 Force and motion

In this chapter you will learn:

- what causes motion and the concept of a force (Newton's first law)
- how force is related to acceleration (Newton's second law)
- what happens when several forces act on an object
- about different types of forces, including gravity
- how to determine whether a particle is in equilibrium.

Before you start...

Chapter 20	You should know how to use the constant acceleration formulae.	1 A particle accelerates from $3\ \text{ms}^{-1}$ to $8\ \text{ms}^{-1}$ in 12 seconds with constant acceleration. a Find its acceleration. b Find the distance the particle travels in this time.
Chapter 12	You should know how to work with vectors in component form.	2 a Add the vectors $\begin{pmatrix}1\\-3\end{pmatrix}$, $\begin{pmatrix}2\\2\end{pmatrix}$ and $\begin{pmatrix}-5\\7\end{pmatrix}$. b Find vector $\mathbf{v}$ so that $3.5\mathbf{v} = -14\mathbf{i} + 7\mathbf{j}$.
Chapter 12	You should know how to find the magnitude and direction of a vector from its components.	3 Find the magnitude of each vector and the angle it makes with the direction of vector $\mathbf{i}$: a $1.2\mathbf{i} + 2.5\mathbf{j}$ b $\begin{pmatrix}4\\-1\end{pmatrix}$

What causes motion?

In Chapters 19 and 20 you derived formulae to describe how the displacement, velocity and acceleration of a particle vary with time. So far nothing has been said about the causes of motion: Why should a particle start to move, or change its velocity? You probably already know that motion is caused by forces and you are familiar with some types of forces – such as gravitational, electromagnetic and frictional forces. In this chapter you will investigate the relationship between force and acceleration for forces acting in one and two dimensions. You will also see how to work out the combined effect of several forces; this requires the application of vectors from Chapter 12.

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Section 1: Newton's laws of motion

Imagine a box lying on a table. If you want the box to move, you need to push or pull it – you need to act on it with a **force**. If you do not apply a force, the box will remain at rest. Once the box is moving, a force is required to change its velocity. For example, a friction force might cause it to slow down, or you may continue to push it to make it accelerate. If there is no force at all, the box will continue to move at a constant speed. This may be difficult to imagine – objects around us are constantly subject to forces like gravity and friction – but it may help to think about, for example, a stone sliding across ice. It takes a very long time to slow down because the forces acting on it are very small.

Any change in velocity of an object is caused by a force. This is one of the most important principles of mechanics.

Key point 21.1

Newton's first law: An object continues to move with constant velocity, or remains at rest, unless acted upon by an external force.

Notice that the law refers to the **velocity**, rather than just the speed of the object. So a force is required to change the **direction** of motion, as well as the speed.

Once you apply the force, the box will start to accelerate. If you want to produce greater acceleration, you need to push or pull harder. But you also know from experience that heavier objects are more difficult to move. So the force required to produce a given acceleration depends on the mass of the object; it is, in fact, directly proportional to it.

Key point 21.2

Newton's second law: The force, $\mathbf{F}$ newtons, required to make an object of mass $m\ \mathrm{kg}$ move with acceleration $\mathbf{a}\,\mathrm{m\,s^{-2}}$ is given by the equation $\mathbf{F} = m\mathbf{a}$.

Force and acceleration are both vectors. The direction of the acceleration is the same as the direction of the force. If the direction of the force is the same as the direction of motion, the object will continue to move in the same direction but its speed will change. However, if the force acts in a direction different from the direction of motion, it will also change the direction of the velocity vector.

The magnitude of a force is measured in **newtons** (N). The equation $F = ma$ tells you how the newton is related to the fundamental units of the S.I. system: $1\ \mathrm{N} = 1\ \mathrm{kg\,m\,s^{-2}}$.

Tip

F and a denote the **magnitudes** of the vectors $\mathbf{F}$ and $\mathbf{a}$.

Did you know?



The force you feel when holding an apple is about 1 newton. Sir Isaac Newton (1642–1727) was the English physicist and mathematician whose work laid the foundations for classical mechanics.

WORKED EXAMPLE 21.1

A truck of a mass 3.2 tonnes is moving in a straight line under the action of a constant driving

force. Find the magnitude of this force when the truck is:

a accelerating at a constant rate of $1.6\,\mathrm{m\,s^{-2}}$

b moving at a constant speed of $58\,\mathrm{km\,h^{-1}}$.

a $F = ma$

Use Newton's second law in scalar form.

$m = 3.2\,\text{tonnes} = 3200\,\mathrm{kg}$

$F = 3200 \times 1.6 = 5120\,\mathrm{N}$

To apply the equation, the mass must be in kilograms and the acceleration in $\mathrm{m\,s^{-2}}$.

b Constant speed $\Rightarrow F = 0\,\mathrm{N}$

According to Newton's first law, if the velocity is constant, there is no force acing on the object.

Notice that this also fits in with Newton's second law: if $v = \text{constant}$ then $a = 0$, so $F = m \times 0 = 0$.

WORKED EXAMPLE 21.2

A box of mass $3.5\,\mathrm{kg}$ is being acted on by a single force $\mathbf{F} = \begin{pmatrix} 14 \\ -21 \end{pmatrix}$ newtons. Find:

a the vector acceleration of the box

b the magnitude of the acceleration.

a $\underline{F} = m\,\underline{a}$

Use Newton's second law in vector form.

$$\begin{pmatrix} 14 \\ -21 \end{pmatrix} = 3.5\,\underline{a}$$

$$\Rightarrow \underline{a} = \frac{1}{3.5}\begin{pmatrix} 14 \\ -21 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -6 \end{pmatrix}\,\mathrm{ms^{-2}}$$

b $|\underline{a}| = \sqrt{4^2 + 6^2} = 7.21\,\mathrm{ms^{-2}}\ (3\text{ s.f.})$

Use the components to find the magnitude.

If you know the force acting on an object you can find the equations of motion: use $F = ma$ to find the acceleration, then use integration (or the constant acceleration formulae) to find equations for velocity and displacement.

Fast forward

A In Student Book 2 you will extend constant acceleration formulae to work with two-dimensional vectors.

WORKED EXAMPLE 21.3

A block of mass $1.2\,\mathrm{kg}$ slides across the floor with the initial speed of $3.6\mathrm{m\,s^{-1}}$. It slows down due to a constant friction force of magnitude $9\,\mathrm{N}$. How far does the block travel before coming to rest?

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You first need to find the deceleration. You can then use one of the constant acceleration equations to find the distance. Take the direction of the initial motion to be positive.

$F = ma$
$-9 = 1.2a$

Use Newton's second law, with $F = -9$ as the force is acting against the positive direction.

$\Rightarrow a = -7.5\,\text{ms}^{-2}$
$u = 3.6$
$v = 0$
$a = -7.5$
$s = ?$

You now know the acceleration (which is negative!), the initial and final velocities, and want to find the displacement; look for an equation involving u, v, a and s.

$v^2 = u^2 + 2as$
$0 = 3.6^2 + 2(-7.5)s$
$\Rightarrow 15s = 12.96$
$s = 0.864\,(3\,\text{s.f.})$

The block travels $0.864\,\text{m}$ before coming to rest.

EXERCISE 21A

1 Find the magnitude of the force, in newtons, acting on the object in each case.

a A crate of mass $53\,\text{kg}$ moves with constant acceleration of $2.6\,\text{m s}^{-2}$.

b A stone of mass $1.5\,\text{kg}$ is pushed across ice and decelerates at a constant rate of $0.3\,\text{m s}^{-2}$.

c A truck of mass 6 tonnes accelerates uniformly at $1.2\,\text{m s}^{-2}$.

d A toy car of mass $230\,\text{g}$ moves with constant acceleration of $3.6\,\text{m s}^{-2}$.

e A box of mass $32\,\text{kg}$ is dragged across the floor in a straight line, at a constant speed of $5.2\,\text{m s}^{-1}$.

f A ball of mass $120\,\text{g}$ falls with a constant acceleration of $9.8\,\text{m s}^{-2}$.

g A book of mass $340\,\text{g}$ rests on a horizontal table.

2 Find the acceleration of the object in the following examples:

a A constant force of magnitude $86\,\text{N}$ acts on a box of mass $36\,\text{kg}$.

b A toy truck of mass $400\,\text{g}$ is pushed with the force of $7.3\,\text{N}$.

c A car of mass 1.5 tonnes moves under the action of a constant force of magnitude $600\,\text{N}$.

d A ball of mass $120\,\text{g}$ is slowed down by a force of magnitude $2.6\,\text{N}$.

3 In each question a particle of mass m moves with constant acceleration $\mathbf{a}$ under the action of a constant force $\mathbf{F}$.

a i $m = 3\text{kg},\ \mathbf{a} = \begin{pmatrix} 1.6 \\ -2.5 \end{pmatrix}\ \text{m s}^{-2}$, find $\mathbf{F}$

ii $m = 5\text{kg},\ a = \begin{pmatrix} -0.7 \\ 1.3 \end{pmatrix}\ \text{m s}^{-2}$, find $\mathbf{F}$.

b i $m = 0.6\,\text{kg}, \mathbf{F} = \begin{pmatrix} 3.6 \\ 1.2 \end{pmatrix}\ \text{N}$, find $\mathbf{a}$.

ii $m = 6.3\ \text{kg}, \mathbf{F} = \begin{pmatrix} -12.6 \\ 9 \end{pmatrix}\ \text{N}$, find $\mathbf{a}$.

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c **i** $\mathbf{a} = \begin{pmatrix} 0.5 \\ -1.5 \end{pmatrix}\ \mathrm{m\,s^{-2}}$, $\mathbf{F} = \begin{pmatrix} 0.7 \\ -2.1 \end{pmatrix}\ \mathrm{N}$, find m.

ii $\mathbf{a} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}\ \mathrm{m\,s^{-2}}$, $\mathbf{F} = \begin{pmatrix} 2.5 \\ 1 \end{pmatrix}\ \mathrm{N}$, find m.

4 Discuss the following questions in class:

a Imagine a car driving at a constant speed around a bend. Is there a force acting on the car? What is its direction?

b Newton's first law states that an object will continue to move with constant velocity if there is no force acting on it. Is this ever the case?

c In many questions in this and the next chapter you will state that forces like friction or air resistance can be ignored. How realistic are these assumptions?

5 A car of mass $900\ \mathrm{kg}$ accelerates from rest to $15\ \mathrm{km\,h^{-1}}$ in 3.5 seconds. Assuming the driving force is constant, find its magnitude.

6 A stone of mass $120\ \mathrm{g}$ is pushed across ice with a speed of $3.2\ \mathrm{m\,s^{-1}}$. It comes to rest 8 seconds later. Find the magnitude of the friction force acting on the stone.

7 A crate of mass $28\ \mathrm{kg}$ is pulled across a horizontal floor. The pulling force acting on the crate is $260\ \mathrm{N}$. Assuming that any friction forces can be ignored, how long does it take for the crate to accelerate from rest to $2.5\ \mathrm{m\,s^{-1}}$?

8 Find, in vector form, the force required to move an object of mass $1.8\ \mathrm{kg}$ with acceleration $\left(0.6\mathbf{i} + 1.1\mathbf{j}\right)\mathrm{m\,s^{-2}}$.

9 A particle of mass $6.5\ \mathrm{kg}$ accelerates under the action of force $\mathbf{F} = \begin{pmatrix} -8.5 \\ 6.5 \end{pmatrix}\ \mathrm{N}$. Find:

a the acceleration in vector form

b the magnitude of the acceleration.

10 A van of mass 2.3 tonnes, travelling in a straight line, decelerates under the action of a constant braking force. Its speed decreases from $50\ \mathrm{km\,h^{-1}}$ to $30\ \mathrm{km\,h^{-1}}$ while it covers the distance of $650\ \mathrm{m}$. Find the magnitude of the braking force.

11 A girl pulls a toy truck with a constant horizontal force of $23\ \mathrm{N}$. The truck starts from rest and accelerates uniformly, travelling $16\ \mathrm{m}$ in 3 seconds. Find the mass of the truck.

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Section 2: Combining forces

In many situations there is more than one force acting on an object. If you are pushing a box across a carpet you are providing a force to accelerate it, but there is also a frictional force slowing it down. A light suspended from a ceiling is being pulled down by the force of gravity but pulled up by the tension in the wire.

When several forces are acting on an object, their combined effect is to produce a certain acceleration. You can find a single force that would produce the same acceleration; this force is called the **resultant force** or **net force**.

Fast forward

In Student Book 2 you will learn about geometrical methods of adding forces.

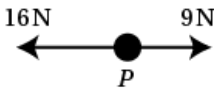
Key point 21.3

The resultant or net force is a single force that produces the same acceleration as several forces acting together. It is found by adding vectors representing the original forces.

When all forces act along the same straight line it is straightforward to add the vectors: you can treat forces as scalars, except that you use $+$ and $-$ signs to indicate the direction. If you know the direction of motion of the particle, take that as positive; otherwise take the positive direction to be to the right or up.

WORKED EXAMPLE 21.4

Two forces act on a particle P, as shown in the diagram. Find the direction and magnitude of the resultant force.



$|\underline{F}| = 16 - 9$
$= 7\,\text{N}$
The direction is to the left.

Both forces act along the same line but in opposite directions. So the resultant is in the direction of the larger force.

When forces are acting in two dimensions, the easiest way to add them is using components. You can then use the components to find the magnitude and direction of the resultant force.

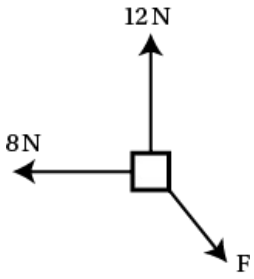
Rewind

You learnt about the magnitude and direction of a vector in Chapter 12, Section 1.

WORKED EXAMPLE 21.5

Three forces act in a vertical plane, with the unit vectors $\mathbf{i}$ and $\mathbf{j}$ directed to the right and up, respectively. The force $\mathbf{F}$ equals $(5\mathbf{i} - 8\mathbf{j})$ N. Find the magnitude of the resultant force, and the angle it makes with the horizontal direction.

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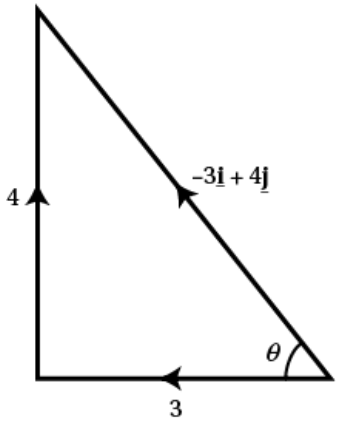
$\underline{F} = (-8\,\underline{i}) + (12\,\underline{j}) + (5\,\underline{i} - 8\mathbf{j})$

Write all three forces in vector form. Note that the horizontal force is to the left, so it is $-8\mathbf{i}$.

$= (-3\,\underline{i} + 4\,\underline{j})\,\mathrm{N}$

Add the $\mathbf{i}$ and $\mathbf{j}$ components separately.

You can now use the components to find the magnitude and direction. Draw a diagram to help.



The magnitude is

$|\underline{F}| = \sqrt{3^2 + 4^2} = 5\,\mathrm{N}$

For the direction:

$\tan\theta = \dfrac{4}{3}$

$\theta = 53.1^\circ$

So the angle from the horizontal is $180 - 53.1 = 126.9^\circ$

The angle is measured from the *positive* horizontal direction.

Once you have found the resultant force, you can use Newton's second law to find the acceleration.

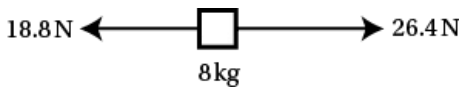
WORKED EXAMPLE 21.6

Two children are pulling a box of mass $8\,\mathrm{kg}$ in opposite directions, as shown in the diagram.

a Find the acceleration of the box.

A third child joins in, pulling with the force $\mathbf{F}_3$.

b The acceleration of the box is now $0.7\,\mathrm{m\,s^{-2}}$ to the right. Find the magnitude and direction of $\mathbf{F}_3$.



a $F = ma$

$26.4 - 18.8 = 8a$

$\Rightarrow a = 0.95\,\text{ms}^{-2}$

Use $F = ma$ where F is the magnitude of the resultant force. Take the positive direction to be to the right (since that is the direction of the larger force, so you expect the box to move in that direction).

b $F = ma$

$26.4 - 18.8 + F_3 = 8 \times 0.7$

Use $F = ma$ again, but now with all three forces. You don't know the direction of the new force, so should you put $+F_3$ or $-F_3$ in the equation?

Let's assume that it's to the right (so use $+F_3$).

$\Rightarrow F_3 = -2$

The magnitude of $-F_3$ is 2 N and its direction is to the left.

The negative sign means that the force is actually to the left.

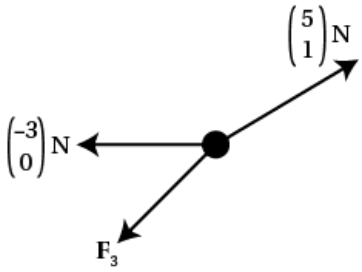
If the resultant force equals zero, the object will remain at rest or continue to move with constant speed (this is Newton's first law). In case of forces in two dimensions, this means that both components equal zero.

Fast forward

When the resultant force is zero, you say that the object is in equilibrium. You will see more examples of this in Section 5.

WORKED EXAMPLE 21.7

A particle is subject to three forces, as shown in the diagram. Given that the particle moves with constant speed, find $\mathbf{F}_3$ in vector form.



$\begin{pmatrix}-3\\0\end{pmatrix} + \begin{pmatrix}5\\1\end{pmatrix} + \underline{F}_3 = \begin{pmatrix}0\\0\end{pmatrix}$

$\Rightarrow \underline{\mathrm{F}}_3 = \begin{pmatrix}3\\0\end{pmatrix} + \begin{pmatrix}-5\\-1\end{pmatrix}$

$= \begin{pmatrix}-2\\-1\end{pmatrix}$ N

Since the particle does not accelerate, the resultant force must be zero.

EXERCISE 21B

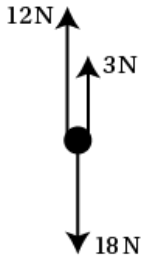
1 Find the magnitude and direction of the resultant force in each case.

a i 5N ⟵ ● ⟶ 3N

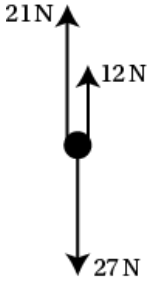
ii 7N ⟵ ● ⟶ 12N

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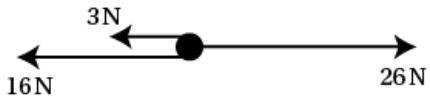
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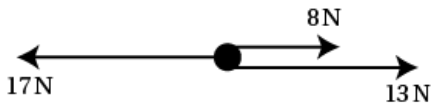
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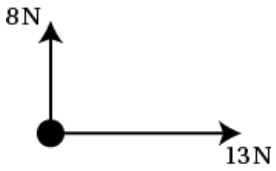


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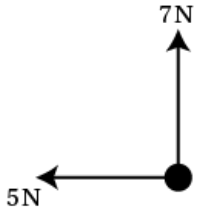


2 Find the resultant force in the form $p\mathbf{i} + q\mathbf{j}$.

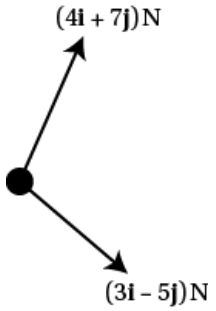
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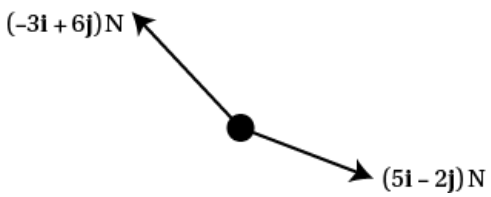
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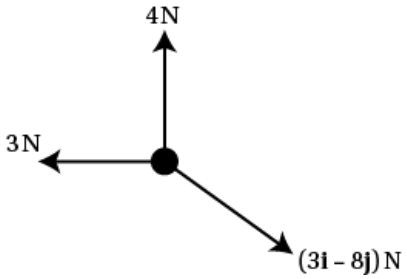


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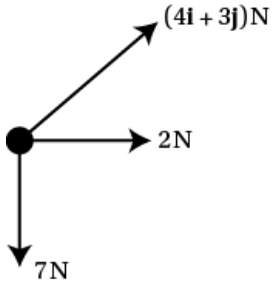


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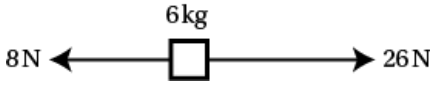
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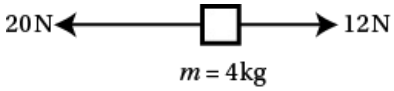
3 Find the magnitude and direction of each resultant force in question 2.

4 For each object shown in the diagram, find the magnitude and direction of acceleration.

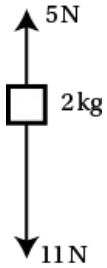
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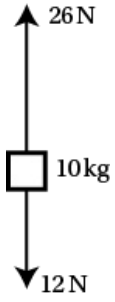
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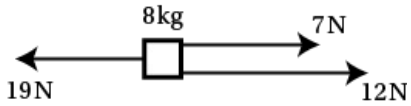
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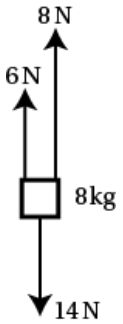
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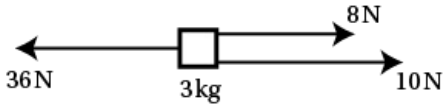
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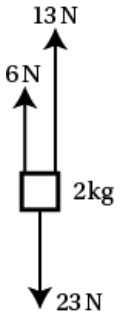
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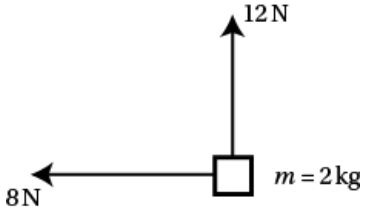


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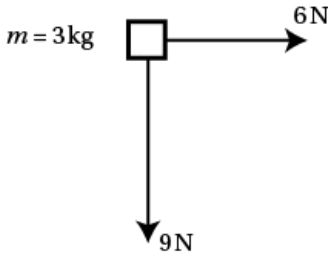


5 Find the acceleration in the form $p\mathbf{i} + q\mathbf{j}$.

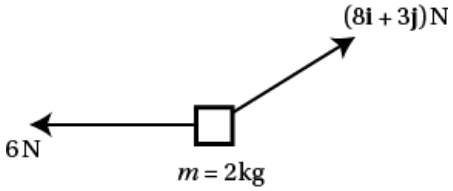
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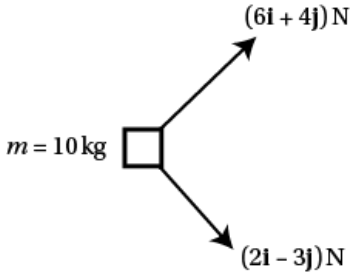
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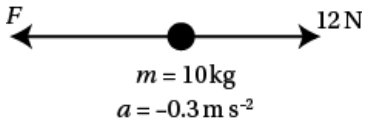


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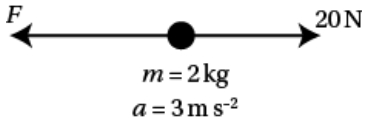


6 In each diagram, the mass of the object and the acceleration are given, with the positive direction to the right or up. Find the magnitude of the force marked F.

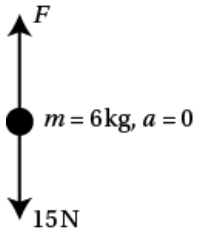
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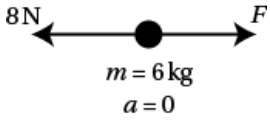
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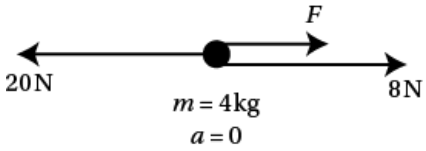
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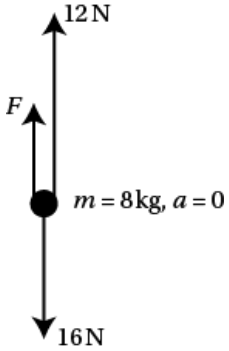
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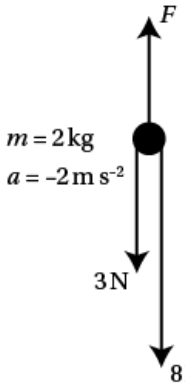
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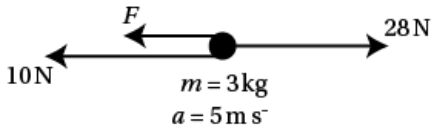
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ii



Worksheet

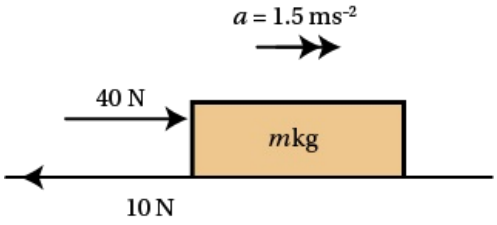
For more examples like this see Support sheet 21.

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7 Two people attempt to push-start a car on a horizontal road. One person pushes with a force of $100\ \mathrm{N}$; the other with a force of $80\ \mathrm{N}$. The car starts to accelerate constantly at $0.15\mathrm{m\ s^{-2}}$. Assuming these are the only horizontal forces acting, find the mass of the car.

8 A sledge of mass $m\ \mathrm{kg}$ is pushed horizontally through the snow by a force of $40\ \mathrm{N}$. There is resistance to its motion of magnitude $10\ \mathrm{N}$ as shown in the diagram.

If the sledge is accelerating at $1.5\mathrm{m\ s^{-2}}$, find its mass.



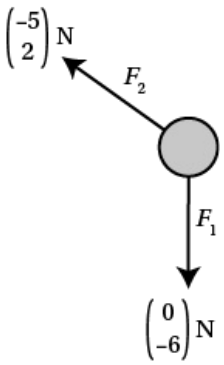
9 Two forces F_1 and F_2 act on a particle as shown.

A third force, F_3, is added so that the resultant force on the particle is $2\ \mathrm{N}$ to the right.

Find:

a the magnitude of F_3

b the direction F_3 makes with the direction of motion.



Section 3: Types of forces

There are several common examples of forces you should be familiar with. You need to be able to identify all forces acting on an object in a given situation and draw a force diagram, before using techniques from the previous section to do calculations.

Many examples you will meet involve moving vehicles. A **driving force** accelerates the vehicle. A **braking force**, acting in the direction opposite to the velocity, will slow the vehicle down.

When an object is sliding across a surface, there is normally some **friction** (or **frictional force**) resisting the motion. The frictional force always acts in the direction opposite to the velocity of the object. There are other types of forces that resist the motion: for example, **air resistance**. The magnitude of each of these resistance forces depends on many different factors, which are beyond the scope of the AS course. For now, you will usually be told the magnitude of any resistance forces.

Sometimes the frictional force is so small that it can be ignored. This can be the case, for example, when you consider an object sliding on ice. We say that the contact between the object and the ice is **smooth** to indicate that friction can be ignored.

Fast forward

A You can learn more about friction in Student Book 2.

WORKED EXAMPLE 21.8

a A car moves under the action of a driving force of $1740\,\text{N}$. The total resistance to motion equals $600\,\text{N}$. Given that the acceleration of the car is $1.2\,\text{m s}^{-2}$, find its mass.

b The car starts to brake and decelerates at $2.5\,\text{m s}^{-2}$. Assuming that the total resistance force remains the same, find the magnitude of the braking force.

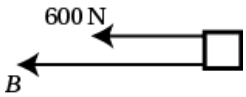
a 

Draw a diagram showing all the relevant forces. You don't need to include the car's weight because the question is only about forces and motion in the horizontal direction.

$1740 - 600 = m \times 1.2$

$\Rightarrow m = 950\,\text{kg}$

Use $F = ma$ with the positive direction to the right.

b 

The braking force is in the opposite direction to the driving force.

$-B - 600 = 950 \times (-2.5)$

$\Rightarrow B = 1775\,\text{N}$

The acceleration is negative.

WORKED EXAMPLE 21.9

A small toy of mass $230\,\text{g}$ moves along the floor with an initial speed of speed of $8\,\text{m s}^{-1}$.

a The contact between the toy and the floor is modelled as smooth. Predict the time it would take the toy to travel $6\,\text{m}$.

b The toy actually takes 0.82 seconds to travel $6\,\text{m}$. Find the magnitude of the frictional force, assuming it is constant.

a No force, so constant speed.

$t = \frac{s}{v} = \frac{6}{8} = 0.75\,s$

There are no horizontal forces acting on the toy, so its speed remains constant. This means that you can use $\text{speed} = \frac{\text{distance}}{\text{time}}$.

b $u = 8$
$s = 6$
$t = 0.82$
$a = ?$

You first need to find the acceleration and then use $F = ma$.

$s = ut + \frac{1}{2}at^2$

$6 = (8 \times 0.82) + \frac{1}{2}a\left(0.82^2\right)$

$6 = 6.56 + 0.3362\,a$

Find the constant acceleration formula involving u, s, t and a.

c $\Rightarrow a = -1.67\,\text{m s}^{-2}$

The acceleration is negative because the friction is slowing the toy down.

$F = ma$
$= 0.23 \times 1.67$
$= 0.383\,\text{N}$

You can now find the force. Since you only want its magnitude, you can ignore the minus sign (which tells you that the direction is opposite to the direction of motion).

WORKED EXAMPLE 21.10

A box of mass $24\,\text{kg}$ moves on a rough horizontal floor under the action of a constant horizontal force $(16\mathbf{i} + 11\mathbf{j})\,\text{N}$. Find, in vector form, the frictional force acting on the box when its acceleration is $\left(0.7\mathbf{i} - 1.1\mathbf{j}\right)\text{ms}^{-2}$.

The net force acting on the box is $(16\mathbf{i} + 11\mathbf{j}) + \mathbf{F}$, where $\mathbf{F}$ is the frictional force.

Identify all the forces acting on the box.

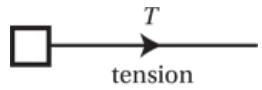
Newton's second law:

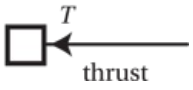
$24(0.7\mathbf{i} - 1.1\mathbf{j}) = (16\mathbf{i} + 11\mathbf{j}) + \mathbf{F}$

$\mathbf{F} = (16.8\mathbf{i} - 26.4\mathbf{j}) - (16\mathbf{i} + 11\mathbf{j})$
$= (0.8\mathbf{i} - 37.4\mathbf{j})\text{N}$

Apply Newton's second law in vector form: $m\mathbf{a} = \text{net force}$.

If you are pulling a box using a rope, you are not acting on the box directly. You are pulling on the rope and the rope pulls the box; the force exerted by the rope on the box is called **tension**, and is directed away from the box (towards you). If you use a stick, a rod or a tow bar instead of a rope, then you could push the box as well as pull. The pushing force provided by the tow bar is called **thrust**, and it is directed towards the box.





WORKED EXAMPLE 21.11

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A boy is using a light horizontal stick to pull a toy box of mass $3.2\,\text{kg}$ across rough carpet. The tension in the stick is $18\,\text{N}$ and the friction force is $7\,\text{N}$.

a Find the acceleration of the box, and the time it takes for it to accelerate from rest to $2.1\,\text{m s}^{-1}$.

b Assuming that the friction force remains the same, what tension is required for the box to maintain the constant speed of $2.1\,\text{m s}^{-1}$?

The boy now makes the box slow down by applying a different constant force though the stick, and the box comes to rest after travelling 0.8 metres. The friction force is still the same.

c Find the magnitude of the force in the stick, and state whether it is tension or thrust.

a 

Draw a diagram showing all the forces acting on the box.

$18 - 7 = 3.2a$
$\Rightarrow a = 3.44\,\text{ms}^{-2}$
$u = 0$
$v = 2.1$
$a = 3.44$
$t = ?$

$F = ma$. The box is moving to the right, so take that as the positive direction.

$v = u + at$
$2.1 = 0 + 3.44t$
$t = 0.611\,\text{s}$

Use a constant acceleration formula involving u, v, a and t.

b $T - 7 = 0$
$\Rightarrow T = 7\,\text{N}$

If the box is moving at constant speed, the acceleration is now zero.

c $u = 2.1$
$v = 0$
$s = 0.8$
$a = ?$

$v^2 = u^2 + 2as$
$\Rightarrow a = -2.76\,\text{ms}^{-2}$

To use $F = ma$ you need to find the acceleration. Use a constant acceleration formula.

$T - 7 = 3.2 \times (-2.76)$

Use $F = ma$. The box is still moving to the right, but now the acceleration is negative. You don't know the direction of the force in the stick; try taking it as positive.

$\Rightarrow T = -1.82$

The force in the stick is a thrust of magnitude $1.82\,\text{N}$.

The negative sign means that the force is in fact to the left. Since this is directed towards the box, it is a thrust rather than tension.

You may wonder whether you should include the mass of the rope or the rod in your calculations. In practice, its mass is almost always a lot smaller than the mass of the object you are trying to move, so you can ignore it. You say that you are modelling the rope (or the rod, tow bar, and so on) as **light**. When you are using a rope or a string, you also need to assume that it does not stretch; otherwise the two ends could move with a different velocity. You say that the string or rope is **inextensible**. You will look at the importance of this assumption further in Chapter 22, Section 4.

EXERCISE 21C

In each question draw a force diagram first.

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1 A child pushes a box of mass 8 kg horizontally with a constant force of 28 N. The frictional force between the box and the floor is 12 N. Find the acceleration of the box.

2 A car moves in a straight line on a horizontal road. The driving force has magnitude 1200 N and the total resistance to the motion is 500 N. Given that the acceleration of the car is 2.6 m s^{-2}, find its mass.

3 A truck of mass 6.2 tonnes moves in a straight line with a constant acceleration of 1.2 m s^{-2}. The driving force is 8200 N.

a Find the total resistance to the motion.

Now assume that the resistance can be ignored.

b How much less time would it take for the truck to accelerate from rest to 40 km h^{-1}?

4 A box of mass 8 kg moves on rough horizontal floor under the action of a constant horizontal force $(p\mathbf{i} + q\mathbf{j})\text{ N}$. When the acceleration of the box is $(2.5\mathbf{i} + 3.1\mathbf{j})\text{ m s}^{-2}$, the frictional force between the box and floor is $(6.8\mathbf{i} - 5.2\mathbf{j})\text{ N}$. Find the values of p and q.

5 A particle rests on a rough horizontal floor. Two perpendicular horizontal forces, shown in the diagram, act on the particle. Given that the particle is in equilibrium, find the magnitude of the frictional force and the angle it makes with the 16 N force.



6 A box of mass 45 kg moves on a rough horizontal floor under the action of a constant horizontal force of magnitude 62 N. Find the magnitude of the acceleration of the box when the frictional force has magnitude 18 N and makes a 170° angle with the 62 N force.

7 Two girls are pulling a crate using two horizontal ropes. The mass of the crate is 56 kg. The tension in one of the ropes is 120 N and the friction force is 80 N. Given that the crate moves with constant acceleration of 0.8 m s^{-2} and that the girls are pulling in exactly the same direction find the tension in the other rope.

8 Two men are pushing a car, in a straight line each using an equal force of magnitude $F\text{ N}$. The resistance to motion has magnitude 420 N. The mass of the car is 850 kg and it is moving at a constant speed of 6 km h^{-1}. Find the value of F.

9 A car of mass 950 kg is moving with a speed of 15.3 m s^{-1} when the driver applies the brakes. The total resistance force, excluding the braking force, is 320 N. The car travels 120 m in a straight line before coming to rest. Find the magnitude of the braking force.

10 A box of mass 45 kg is pulled across a horizontal floor using a light inextensible rope. The rope is horizontal and the tension in the rope is 180 N. The box starts from rest.

a The contact between the box and the floor is modelled as smooth. According to this model, how long will it take for the box to travel 25 m?

b The box in fact travels 25 m in 4.2 seconds. Find the magnitude of the friction force between the box and the floor.

11 A car travels along a straight horizontal road. The total resistance to the car's motion is constant and has magnitude 320 N.

a The driving force is constant at 1200 N. The car passes point A with a speed of 12 m s^{-1} and

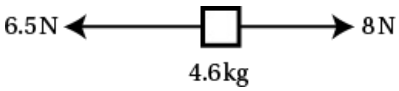
accelerates to $26\ \mathrm{m\,s^{-1}}$ in 7 seconds. Show that the mass of the car is $440\ \mathrm{kg}$.

b The driving force is reduced so that the car travels at a constant speed of $26\ \mathrm{m\,s^{-1}}$. State the magnitude of the driving force.

c The car travels at the constant speed for 12 seconds. Then the driver turns off the engine and the car stops at point B. Find the distance AB.

12 An object of mass $4.6\ \mathrm{kg}$ rests on a horizontal surface. It is acted on by two parallel pulling forces of magnitudes $8\ \mathrm{N}$ and $6.5\ \mathrm{N}$, as shown in the diagram, and it starts to accelerate uniformly.

After 4 seconds a third pulling force is added, acting parallel to the other two. When the object has moved a further $2\ \mathrm{m}$ in the same direction as before, its speed is $1.24\ \mathrm{m\,s^{-1}}$. Find the magnitude and direction of the third force.



13 In this question vectors $\mathbf{i}$ and $\mathbf{j}$ lie in the horizontal plane and point east and north, respectively.

Two people are pulling a box of weight $1400\ \mathrm{N}$ using two ropes. The ropes are modelled as light and inextensible, and the forces in the ropes are $(p\mathbf{i} + q\mathbf{j})\ \mathrm{N}$ and $(p\mathbf{i} - q\mathbf{j})\ \mathrm{N}$, as shown in the diagram. The friction force of magnitude $180\ \mathrm{N}$ is directed west. The box starts from rest.

a Explain why the box moves in a straight line in the east direction.

b Given that the acceleration of the box is $\left(0.003p\right)\mathrm{m\,s^{-2}}$, find the value of p.

c The tension in each rope has magnitude $200\mathrm{N}$. Find the value of q.

Worksheet

Extension sheet 21 looks at estimating magnitudes of different types of forces.

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Section 4: Gravity and weight

If you pick up a ball and let it go, it falls downwards. This is because the Earth exerts gravitational force on it. The force with which the Earth attracts any object is called the **weight** of the object. You know from Chapter 20 that all objects under the influence of gravity alone move with the same acceleration, $\mathrm{g} \approx 9.8\,\mathrm{ms}^{-2}$.

You should be very careful to use the correct terminology here. In everyday language, you tend to use the word 'weight' to mean 'mass'; so you would say, for example, that the bag of apples weighs $1.2\,\mathrm{kg}$. In mechanics, the $1.2\,\mathrm{kg}$ is the mass of the bag; its weight is a **force** with approximate magnitude $1.2 \times 9.8 = 11.76\,\mathrm{N}$.

Key point 21.4

The magnitude of the weight force on an object of mass m is $W = m\mathrm{g}$ where $\mathrm{g} \approx 9.8\,\mathrm{ms}^{-2}$.

Its direction is towards the centre of the Earth (which is normally described as 'downwards').

The mass is a property of the object itself, independent of where it is. The magnitude of the weight force depends both on the object and on the properties of the Earth. On a different planet the weight would be different: for example, the gravitational acceleration near the surface of Mars is about $3.7\,\mathrm{m\,s}^{-2}$ so the weight of the same bag of apples on Mars would be $1.2 \times 3.7 = 4.44\,\mathrm{N}$.

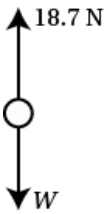
You should know that gravitational acceleration is not exactly the same everywhere on Earth. It depends on the latitude (this is because the Earth is not perfectly spherical, and also because of its rotation); it also depends on the altitude: it is lower on a mountain than at sea level. The variation is between around $9.76\,\mathrm{m\,s}^{-2}$ and $9.83\,\mathrm{m\,s}^{-2}$, which both round to 9.8. The average is normally quoted as $9.81\,\mathrm{m\,s}^{-2}$. The approximate figure of $9.8\,\mathrm{m\,s}^{-2}$ is appropriate up to the height of about 10 km, so you can use it in all the questions involving balls being thrown in the air or lifts taking you to the top floor of a building.

Focus on ...

Focus on ... Modelling 4 explores the effect of changing the value of g.

WORKED EXAMPLE 21.12

In his house on Planet X, Zixo has a crystal ball suspended from the ceiling by a light inextensible string. The mass of the ball is $1.6\,\mathrm{kg}$ and the tension in the string is $18.7\,\mathrm{N}$. Find the magnitude of the gravitational acceleration on Planet X.



Always draw a diagram showing all the forces. In this case there are two forces acting on the ball: its weight and the tension in the string.

$W - 18.7 = 0$

$\therefore W = 18.7\,\mathrm{N}$

Since the ball is not moving, the net force is zero.

$W = m\mathrm{g} \Rightarrow 1.6\,\mathrm{g} = 18.7$

$\therefore \mathrm{g} = 11.7\,\mathrm{ms}^{-2}$

Whenever there is a possibility of an object moving in the vertical direction, you should include weight on your force diagram. You need to be very careful about the direction of acceleration.

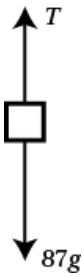
WORKED EXAMPLE 21.13

A crate of mass $87\,\text{kg}$ is being lowered using a light inextensible rope.

a Find the acceleration of the crate when the tension in the rope is $750\,\text{N}$.

b Find the tension in the rope if the crate is being lowered at constant speed.

c Find the tension in the rope required to decelerate the crate from $1.2\,\text{m s}^{-1}$ to rest in 3.5 seconds.



a $ma = mg - T$

$87a = 87 \times 9.8 - 750 = 102.6$

$\Rightarrow a = 1.18\,\text{ms}^{-2}$

To find acceleration, use $F = ma$.

The crate is being lowered, so take the positive direction to be downwards.

b $0 = 87 \times 9.8 - T$

$\Rightarrow T = 853\,\text{N}$

Constant speed means that $a = 0$.

c $v = u + at$

$0 = 1.2 + 3.5a$

$\Rightarrow a = -0.343\,\text{ms}^{-2}$

You first need to find the acceleration, given that $u = 1.2$, $v = 0$ and $t = 3.5$. We need to pick one of the constant acceleration formulae.

You expect a to be negative because the crate is decelerating.

$ma = mg - T$

$87(-0.343) = 87 \times 9.8 - T$

$\Rightarrow T = 29.8 + 853 = 882\,\text{N}$

You took the positive direction to be downwards, so you should use negative a here.

WORKED EXAMPLE 21.14

A ball falling through the air is subject to a constant resistance force of magnitude $0.3\,\text{N}$. The ball starts from rest and takes 1.6 seconds to fall 12 metres. Find the mass of the ball.

Newton's second law for the ball:

$ma = 9.8m - 0.3$

The forces acting on the ball are its weight and the air resistance.

To find the acceleration:

$u = 0,\ t = 1.6,\ s = 12,\ a = ?$

You can find the acceleration from the information about the time and displacement.

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Using $s = ut + \frac{1}{2}at^2$:

$12 = 0.5a(1.6^2)$

$a = 9.375\,\text{ms}^{-2}$

So:

Now use this in Newton's second law equation.

$9.375m = 9.8m - 0.3$

$0.425m = 0.3$

$m = 0.706\,\text{kg}(3\,\text{s.f.})$

The mass of the ball is about 706 grams.

EXERCISE 21D

1 The diagram shows an object of mass $m\,\text{kg}$ suspended by a light inextensible string. The magnitude of the tension in the string is T N. Find the direction and magnitude of the acceleration.

a **i** $m = 3\,\text{kg}, T = 32\,\text{N}$

ii $m = 8\,\text{kg}, T = 92\,\text{N}$

b **i** $m = 0.6\,\text{kg}, T = 4.5\,\text{N}$

ii $m = 3.2\,\text{kg}, T = 27\,\text{N}$



2 An object of mass m is suspended by a vertical rope. The acceleration of the object is $a\text{m s}^{-2}$ in the stated direction. Find the tension in the rope.

a **i** $m = 12\,\text{kg}$, $a = 0.6\,\text{ms}^{-2}$ downwards

ii $m = 3.2\,\text{kg}$, $a = 1.2\,\text{ms}^{-2}$ downwards

b **i** $m = 8\,\text{kg}$, $a = 1.4\,\text{ms}^{-2}$ upwards

ii $m = 3\,\text{kg}$, $a = 2\,\text{ms}^{-2}$ upwards

3 An object is suspended by a string, with tension $T\,\text{N}$. The acceleration of the object is $a\text{m s}^{-2}$ in the stated direction. Find the mass of the object.

a **i** $T = 26\,\text{N}$, $a = 1.2\,\text{ms}^{-2}$ downwards

ii $T = 18.6\,\text{N}$, $a = 0.7\,\text{ms}^{-2}$ downwards

b **i** $T = 26\,\text{N}$, $a = 1.2\,\text{ms}^{-2}$ upwards

ii $T = 18.6\text{N}$, $a = 0.7\,\text{ms}^{-2}$ upwards

4 A crate of mass $98\,\text{kg}$ is being lifted using a rope which can be modelled as light and inextensible. The tension in the rope is $367\,\text{N}$. Find the acceleration of the crate.

5 A ball is at rest, suspended from a ceiling by a light inextensible cable. The tension in the cable is $12\,\text{N}$.

a Find the mass of the ball.

The tension in the cable is increased to $15\,\text{N}$.

b Find the magnitude and direction of the acceleration of the ball.

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6 A crate is lowered from a window of a space ship on Mars, using a rope. The tension in the rope is $328\ \mathrm{N}$ and the crate is descending at a constant speed. Given that the gravitational acceleration on Mars is $3.7\ \mathrm{m\,s^{-2}}$, find the mass of the crate.

7 A box of mass $1.2\ \mathrm{kg}$ falls vertically downwards with a constant acceleration of $8.9\ \mathrm{m\,s^{-2}}$. Find the magnitude of the air resistance acting on the box.

8 A ball falls vertically downwards, starting from rest, from a height of $24\,\mathrm{m}$. It takes 2.3 seconds to reach the ground. Given that the ball is subject to a constant air resistance of magnitude 0.25 newtons, find the mass of the ball.

9 A stone is dropped vertically from the top of a $25\ \mathrm{m}$ tall cliff.

a A simple model assumes that air resistance can be ignored. According to this model, how long does it take for the stone to reach the ground?

b The stone actually takes 2.42 seconds to reach the ground. Given that the mass of the stone is $1.2\ \mathrm{kg}$, find the magnitude of the air resistance.

10 A crane is lifting a $350\ \mathrm{kg}$ load, which is initially at rest on the ground, using a light inextensible cable. It takes 6 seconds to raise it to the height of $84\,\mathrm{m}$. Find the tension in the cable, assuming it is constant.

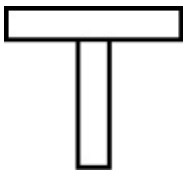
11 A crate of mass $124\ \mathrm{kg}$ is being lowered using a light inextensible rope. The crate is decelerating at the rate of $1.8\ \mathrm{m\,s^{-2}}$. Find the tension in the rope.

12 A fisherman is lifting a crate caught on the end of his fishing line. The mass of the crate is $3.5\ \mathrm{kg}$ and it is being lifted vertically through the water.

a While the crate is moving through the water, the water exerts a net force of magnitude $5\ \mathrm{N}$, acting downwards. Given that he is raising the crate at a constant speed, find the tension in the fishing line.

b The crate breaks the surface of the water and the tension in the string remains unchanged. Air resistance can be ignored. Find the acceleration of the crate at that moment.

13 A horizontal platform of mass $120\ \mathrm{kg}$ is supported by a vertical steel rod, as shown in the diagram. The platform is being lowered and decelerating from $3.2\ \mathrm{m\,s^{-1}}$ to rest in 4.5 seconds. Find the thrust in the rod.



Section 5: Forces in equilibrium

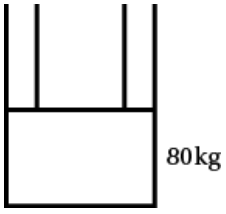
You saw in Section 1 that when several forces act on an object, you can find its acceleration by using $\mathbf{F} = m\mathbf{a}$ with $\mathbf{F}$ being the resultant force. A special case of this is when the resultant force is zero: in this case, there is **no** acceleration. According to Newton's first law, the object remains at rest or continues to move with constant velocity. We say that the object is **in equilibrium**.

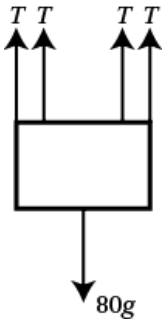
Key point 21.5

If an object is in equilibrium then the resultant force is zero.

WORKED EXAMPLE 21.15

A large box of mass $80\ \text{kg}$ hangs in equilibrium supported by four cables, as shown in the diagram. The tension in each cable has magnitude T. Find the value of T.





Draw a force diagram. The forces are the four tensions and the weight of the box.

$80\,\text{g} - 4T = 0$

$\Rightarrow T = \dfrac{80\,\text{g}}{4} = 196\,\text{N}$

Since the box is in equilibrium, the resultant force is zero. Take positive direction to be downwards.

If forces are acting in two dimensions, both horizontal and vertical components need to be zero.

Explore

A falling object can be in equilibrium. Find out about terminal velocity. Why do astronauts sometimes feel 'weightless'?

WORKED EXAMPLE 21.16

A particle is in equilibrium under the action of three forces: $\mathbf{F}_1 = \begin{pmatrix} 40 \\ 2x \end{pmatrix}$, $\mathbf{F}_2 = \begin{pmatrix} y \\ -32 \end{pmatrix}$ and $\mathbf{F}_3 = \begin{pmatrix} x \\ y \end{pmatrix}$.

Find the values of x and y.

$\underline{\mathrm{F}}_1 + \underline{\mathrm{F}}_2 + \underline{\mathrm{F}}_3 = 0$

Since the particle is in equilibrium, the resultant force is zero.

$\Rightarrow \begin{cases} 40 + y + x = 0 \\ 2x - 32 + y = 0 \end{cases}$

You can write separate equations for the horizontal and vertical components to get two simultaneous equations.

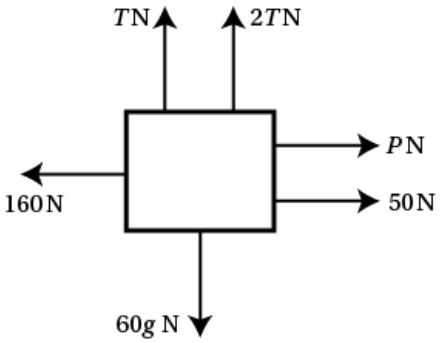
$\Rightarrow \begin{cases} x + y = -40 \\ 2x + y = 32 \end{cases}$

$\Rightarrow x = 72, y = -112$

Sometimes all forces are either horizontal or vertical. In that case you don't need to use vector notation; you can simply write separate equations for horizontal and vertical forces.

WORKED EXAMPLE 21.17

A box of mass $60\,\mathrm{kg}$ hangs in equilibrium supported by five light inextensible cables, as shown in the diagram. The tensions in the two vertical cables are $T\,\mathrm{N}$ and $2T\,\mathrm{N}$ and the tensions in the horizontal cables are $160\,\mathrm{N}$, $50\,\mathrm{N}$ and $P\,\mathrm{N}$. Find the values of T and P.



Vertically:

$3T = 60\,\mathrm{g}$

$\Rightarrow T = 196$

You can write separate equilibrium equations for horizontal and vertical directions.

Horizontally:

$P + 50 = 160$

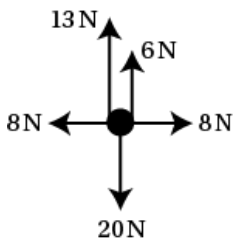
$\Rightarrow P = 110$

The equilibrium equation is $3T - 60\,\mathrm{g} = 0$. However, sometimes it is easier to think of it as 'forces up = forces down'.

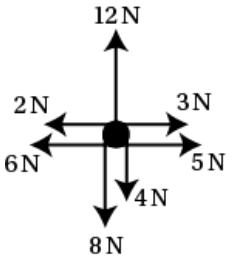
EXERCISE 21E

1 Determine which of these particles are in equilibrium:

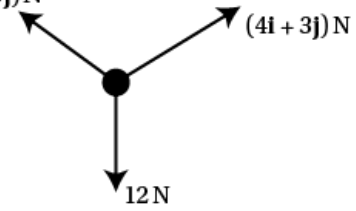
a



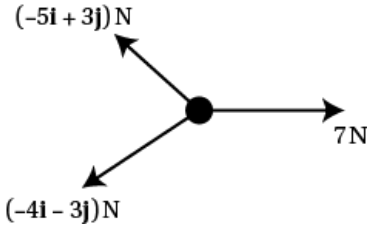
b



c $(-4\mathbf{i}+9\mathbf{j})$ N

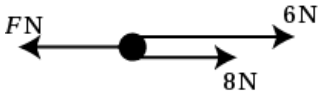


d $(-5\mathbf{i}+3\mathbf{j})$ N

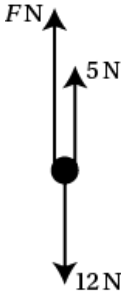


2 Each diagram shows a particle in equilibrium. Find the magnitudes of the forces marked with letters. Assume that the unknown forces are vertical or horizontal, as depicted.

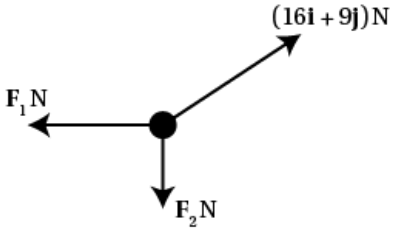
a



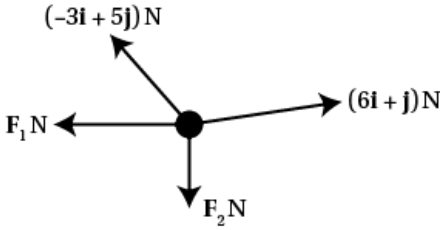
b



c



d



3 A particle is in equilibrium under the action of the three given forces. Find the values of x and y.

a **i** $\mathbf{F}_1 = 12\mathbf{i}+8\mathbf{j}$, $\mathbf{F}_2 = x\mathbf{i}–15\mathbf{j}$, $\mathbf{F}_3 = 6\mathbf{i}–y\mathbf{j}$

ii $\mathbf{F}_1 = 25\mathbf{i}+18\mathbf{j}$, $\mathbf{F}_2 = x\mathbf{i}+12\mathbf{j}$, $\mathbf{F}_3 = –31\mathbf{i}+y\mathbf{j}$

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b **i** $\mathbf{F}_1 = \begin{pmatrix} 3 \\ 8 \end{pmatrix}, \mathbf{F}_2 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}, \mathbf{F}_3 = \begin{pmatrix} x \\ y \end{pmatrix}$

ii $\mathbf{F}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \mathbf{F}_2 = \begin{pmatrix} 11 \\ 3 \end{pmatrix}, \mathbf{F}_3 = \begin{pmatrix} x \\ y \end{pmatrix}$

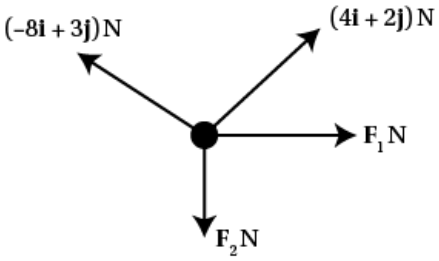
c **i** $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}, \mathbf{F}_2 = -4\mathbf{j}, \mathbf{F}_3 = 8\mathbf{i}$

ii $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}, \mathbf{F}_2 = 15\mathbf{i}, \mathbf{F}_3 = -9\mathbf{j}$

d **i** $\mathbf{F}_1 = \begin{pmatrix} x \\ -7 \end{pmatrix}, \mathbf{F}_2 = \begin{pmatrix} 3y \\ 2x \end{pmatrix}, \mathbf{F}_3 = \begin{pmatrix} -11 \\ y \end{pmatrix}$

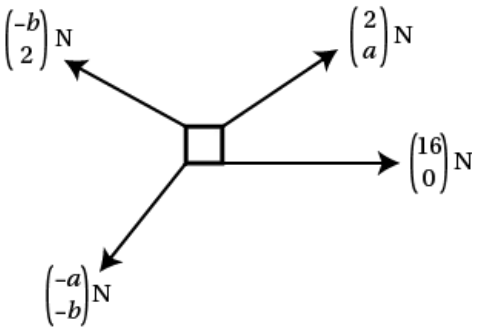
ii $\mathbf{F}_1 = \begin{pmatrix} y \\ 5 \end{pmatrix}, \mathbf{F}_2 = \begin{pmatrix} 3x \\ 4y \end{pmatrix}, \mathbf{F}_3 = \begin{pmatrix} -2 \\ 25x \end{pmatrix}$

4 A particle is in equilibrium under the action of the forces shown in the diagram. Find the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$, which are horizontal and vertical respectively.

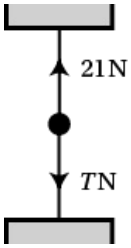


5 A particle is acted on by three forces, $(3\mathbf{i} - 6\mathbf{j})\text{N}, (-5\mathbf{i} + 2\mathbf{j})\text{N}$ and $(x\mathbf{i} + y\mathbf{j})\text{N}$. Given that the particle is in equilibrium, find the values of x and y.

6 The diagram shows a particle in equilibrium under the action of four forces. Find the values of a and b.



7 A ball of mass 1.4 kg is attached to the floor and the ceiling by two light inextensible strings, as shown in the diagram.

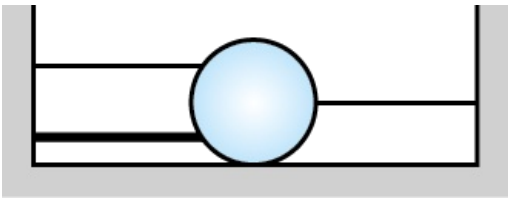


Given that the ball is in equilibrium, find the value of T.

8 A crate rests on a rough horizontal floor. Two people are pulling the crate in opposite directions using two horizontal ropes. The tensions in the ropes are of 16 N and 28 N. Given that the crate is in equilibrium, find the magnitude of the friction force between the crate and the floor.

9 A particle rests on a smooth horizontal floor between two walls. It is attached to the wall on the right by a light inextensible string and the tension in this string is 63 N. It is attached to the wall on the left by a light inextensible string and a light rod. The tension in the string is 82 N. Both strings

and the rod are horizontal.



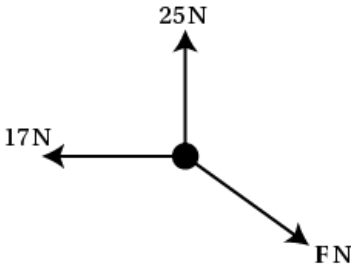
Find the force in the rod and state whether it is a tension or a thrust.

10 A ball of mass $m\,\text{kg}$ is attached to the ceiling by two identical light, inextensible strings. The strings make equal angles with the horizontal. The force on the ceiling from the left string is $\begin{pmatrix} a \\ 7b \end{pmatrix}$ N. The force on the ceiling from the right string is $\begin{pmatrix} a-12 \\ 9b+7 \end{pmatrix}$ N.

Determine the values of a, b and m.

11 Two people are trying to move a heavy box lying at rest on a carpet, using two light inextensible ropes. The tensions in the ropes are $(32\mathbf{i}+12\mathbf{j})$N and $(25\mathbf{i}-18\mathbf{j})$N, where $\mathbf{i}$ and $\mathbf{j}$ point east and north respectively. The box remains at rest. Find the magnitude of the frictional force.

12 A particle is in equilibrium under the action of three forces shown in the diagram.



a Find the horizontal and vertical components of $\mathbf{F}$.

b Find the magnitude of $\mathbf{F}$ and the angle it makes with the horizontal.

Checklist of learning and understanding

- A force can start or stop the motion of an object, change the magnitude or the direction of its velocity. Force is a vector and its magnitude is measured in newtons $\left(1\,\text{N}=1\,\text{kg}\,\text{ms}^{-2}\right)$.
- Newton's first law states that an object remains at rest or continues to move with a constant velocity unless a force acts on it.
- Newton's second law states that the force required to produce a given acceleration is proportional to the acceleration and the mass of the object: $\mathbf{F}=m\mathbf{a}$. The acceleration is in the same direction as the force.
- If several forces are acting on an object, their combined effect is represented by the resultant force. This force is found by adding the vectors corresponding to all the original forces.
- The examples of forces you have met in this chapter are: driving force, braking force, resistance forces (including friction and air resistance), tension, thrust and weight. Whenever you draw a force diagram, you should consider which of these forces need to be included.
- The weight of an object is $W=mg$, where m is the object's mass and g is the gravitational acceleration (on the surface of the Earth $g\approx 9.8\,\text{ms}^{-2}$). The mass of an object is fixed but its weight depends on its location in the universe.
- An object is in equilibrium if the resultant force is zero.
- When working with force vectors, you can consider horizontal and vertical components

separately.

Mixed practice 21

1 A car moves on a straight horizontal road, under the action of a constant driving force of magnitude $1360\,\text{N}$. It accelerates from rest to the speed of $12.6\,\text{m s}^{-1}$ in 8 seconds.

a Assuming that any resistance forces can be ignored, find the mass of the car.

b How would your answer change if a resistance force were included?

2 A box of mass $13\,\text{kg}$ slides across a rough horizontal floor with an initial speed of $2.6\,\text{m s}^{-1}$ and moves in a straight line. It comes to rest after it has travelled $3.7\,\text{m}$. Find the magnitude of the frictional force between the box and the floor.

3 A van of mass $1600\,\text{kg}$ travels on a straight horizontal road under the action of a constant driving force of magnitude $2170\,\text{N}$. The total resistance force on the van is $865\,\text{N}$.

a Find the acceleration of the van.

b Given that the van starts from rest, find the time taken for it to travel $260\,\text{m}$.

Worksheet

For a reminder and more practice of questions like this that combine Newton's second law with the constant acceleration formulae, see Support sheet 21.

4 **a** A box has weight $268\,\text{N}$. Find its mass.

b The box is being pulled vertically upwards using a light inextensible rope and accelerates uniformly with $a = 0.9\,\text{m s}^{-2}$. Find the tension in the rope.

c The box is now lowered at a constant speed. Find the tension in the rope.

d Explain how you have used the modelling assumption that the rope is:

i light

ii inextensible.

5 A particle of mass $2.5\,\text{kg}$ is in equilibrium under the action of three forces, $\mathbf{F}_1 = (6.3\mathbf{i} + 1.7\mathbf{j})\text{N}$, $\mathbf{F}_2 = (-3.7\mathbf{i} + 2.1\mathbf{j})\text{N}$ and $\mathbf{F}_3 = (p\mathbf{i} + q\mathbf{j})\text{N}$.

a Find the values of p and q.

The force $\mathbf{F}_3$ is now removed.

b Find, in vector form, the acceleration of the particle.

6 In the sport of curling, a heavy stone is projected in a straight line across a horizontal ice surface. A player projects a stone of mass $21\,\text{kg}$, and it comes to rest $14\,\text{s}$ after the instant of projection, having travelled $32\,\text{m}$.

a Calculate the deceleration of the stone.

b Find the magnitude of the frictional force acting on the stone.

7 A car of mass $750\,\text{kg}$ accelerates uniformly from $30\,\text{km h}^{-1}$ to $40\,\text{km h}^{-1}$ while travelling $200\,\text{m}$ in a straight line. The resistance to the motion of the car has magnitude $380\,\text{N}$.

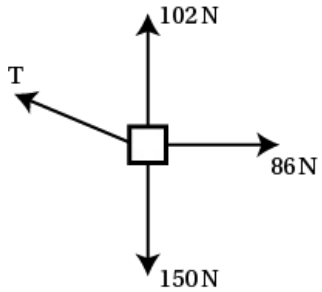
a Find the magnitude of driving force.

The car now starts to brake with a braking force of $620\,\text{N}$. The resistance force remains unchanged.

b How long does it take for the car to stop?

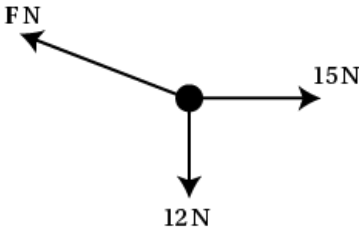
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8 A box rests in equilibrium on a smooth horizontal floor. Four children pull the box using light inextensible ropes, all in the horizontal plane. The tensions in the ropes are shown in the diagram.



Find the magnitude of the force marked $\mathbf{T}$ and the angle it makes with the $102\ \mathrm{N}$ force.

9 A particle is in equilibrium under the action of the three forces shown in the diagram.



Find the magnitude of $\mathbf{F}$ and the angle it makes with the direction of the $15\ \mathrm{N}$ force.

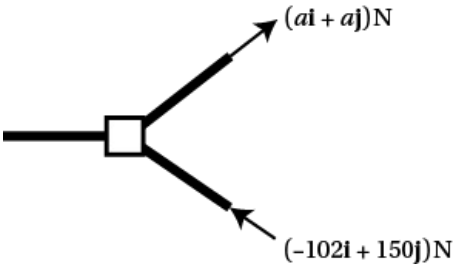
10 Two horizontal forces $\mathbf{X}$ and $\mathbf{Y}$ act at a point O and are at right angles to each other. $\mathbf{X}$ has magnitude $12\ \mathrm{N}$ and acts along a bearing of 090°. $\mathbf{Y}$ has magnitude $15\ \mathrm{N}$ and acts along a bearing of 000°.

a Calculate the magnitude and bearing of the resultant of $\mathbf{X}$ and $\mathbf{Y}$.

b A third force $\mathbf{E}$ is now applied at O. The three forces $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{E}$ are in equilibrium. State the magnitude of $\mathbf{E}$ and give the bearing along which it acts.

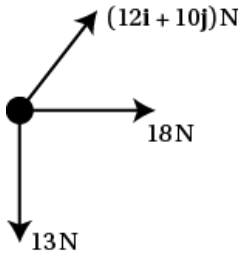
© OCR, GCE Mathematics, Paper 4728, January 2008 [Question part reference style adapted]

11 A box of mass $24\ \mathrm{kg}$ is held in equilibrium in mid-air. It is supported by three light rigid rods, as shown in the diagram. All forces act in the vertical plane.



Find the magnitude of the force in the horizontal rod and determine whether it is a tension or a thrust.

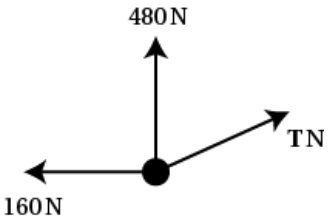
12 **a** A particle of mass $4\ \mathrm{kg}$ moves on a smooth horizontal surface under the action of the three forces shown in the diagram. All three forces act in the horizontal plane.



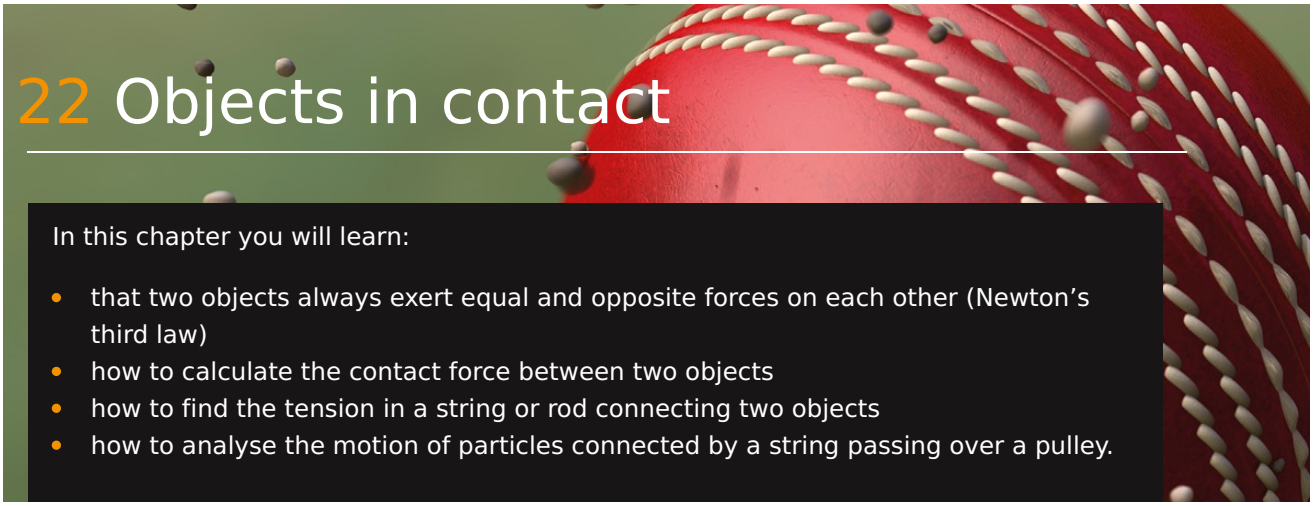
Find, in vector form, the acceleration of the particle.

b A fourth force is added so that now the particle moves with constant velocity. Find the magnitude of the new force and the angle it makes with the $18\ \mathrm{N}$ force.

13 A particle of mass $53\ \mathrm{kg}$ is attached to three light inextensible strings as shown in the diagram. The particle hangs in equilibrium in the vertical plane.



Find the magnitude and direction of $\mathbf{T}$.

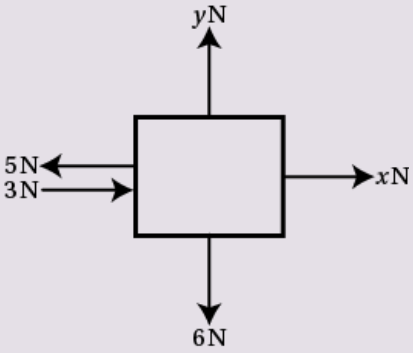


22 Objects in contact

In this chapter you will learn:

- that two objects always exert equal and opposite forces on each other (Newton's third law)
- how to calculate the contact force between two objects
- how to find the tension in a string or rod connecting two objects
- how to analyse the motion of particles connected by a string passing over a pulley.

Before you start...

Chapter 21	You should know how to find the resultant force and use it in Newton's second law.	1 A particle of mass $2.4\,\text{kg}$ is acted upon by two horizontal forces, $25\,\text{N}$ to the left and $32\,\text{N}$ to the right. Find the acceleration of the particle.
Chapter 21	You should know how to calculate and use the weight of an object.	2 Find the weight of a box with mass $34\,\text{kg}$.
Chapter 21	You should know that if a particle is in equilibrium then the resultant force is zero.	3 The particle in the diagram is in equilibrium. Find the values of x and y. 
Chapter 20	You should know how to use the constant acceleration formulae.	4 A particle accelerates uniformly from $2\,\text{ms}^{-1}$ to $8\,\text{ms}^{-1}$ while travelling $75\,\text{m}$ in a straight line. a Find the acceleration. b How long does the journey take?

What is different about objects in contact?

In many situations, two objects are in contact or connected in some way. For example, several boxes stacked on top of each other, or a car pulling a trailer. There are contact forces acting between the objects that need to be taken into account. In this chapter you will meet the normal reaction force and

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use the tension/thrust force in a connecting string, cable or rod. You will also look at modelling assumptions relating to the type of contact or connection between the objects.

Another important force that exists when two surfaces are in contact is friction. You will learn more about it if you study the full A level course.

Section 1: Newton's third law

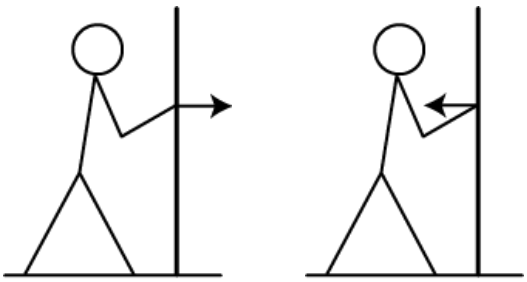
If you push against the wall, you can feel the wall 'pushing you back'. If you were standing on ice you would probably slide backwards, although you are pushing towards the wall, not away from it. What is the force that you feel? The answer is given by Newton's third law.

Key point 22.1

Newton's third law: If object A exerts a force on object B, then object B exerts a force on object A, with the same magnitude but opposite direction.

This means that whatever force you are exerting on the wall (which is directed towards the wall), the wall exerts a force on you. Since this force is away from the wall, this force stops you moving through the wall.

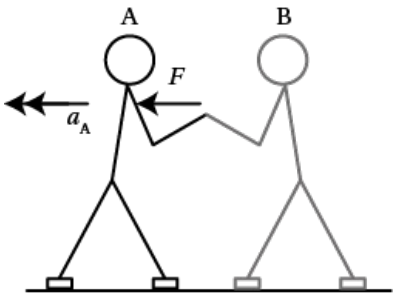
Newton's third law is commonly stated as: 'Each action has an equal and opposite reaction.' An important point to remember is that the two forces do **not** act on the same object so they do not cancel each other. It is a good idea to draw two separate force diagrams, one for each object.



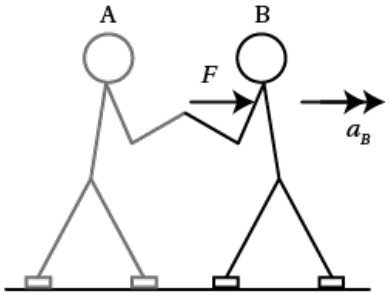
force on wall force on man

WORKED EXAMPLE 22.1

Two skaters are standing on ice. They push against each other and start to move away from each other. Skater A, whose mass is $75\ \text{kg}$, moves with acceleration of $3.7\ \text{m s}^{-2}$. Skater B's mass is $63\ \text{kg}$. Assuming that any frictional forces can be ignored, find the acceleration of skater B.



force on A



force on B

Draw two separate force diagrams. The force on each skater is away from the other one (i.e. backwards).

Skater A:

$F = m_A a_A$

$= 75 \times 3.7 = 277.5N$

According to Newton's third law, the two forces have equal magnitude. You can find this using Newton's second law for skater A.

Skater B:

$277.5 = 63a_B$

$\therefore a_B = 4.40 \text{ m s}^{-2}$

Now you can use Newton's second law for skater B.

EXERCISE 22A

1 Two bumper cars collide. Their masses are 265 kg and 280 kg. While the cars are in contact, the acceleration of the first car is 5.8 m s^{-2}. Assuming any resistance forces can be ignored, find the acceleration of the second car.

2 Two skaters stand on ice facing each other. They push off each other and start to accelerate backwards. The mass of the first skater is 78 kg and his acceleration is 3.5 m s^{-2}. The acceleration of the second skater is 4.2 m s^{-2}. Find her mass.

3 Two robots with long extendible arms push against each other with a constant force of 215 N. They start next to each other, and slide away from each other in a straight line. The first robot has mass 120 kg and the friction force between its feet and the floor is 96 N. The second robot has mass 90 kg and the friction force between its feet and the floor is 65 N. How far are the robots from each other after 2 seconds?

4 Two skaters, of masses 52 kg and 68 kg, stand on ice facing each other and holding hands with their arms outstretched, 1.2 m apart. They pull towards each other so that the acceleration of the first skater is 0.4 m s^{-2}.

a Find the acceleration of the second skater.

The skaters keep holding hands and pulling with the same force.

b How long does it take for them to come together?

5 An apple of mass 120 g falls from the third floor window, 10 m above the ground.

a Find the magnitude of the force with which the Earth attracts the apple.

b How long does the apple take to fall the 10 metres?

c State the magnitude of the force with which the apple attracts the Earth.

d The mass of the Earth is $5.97 \times 10^{24} \text{ kg}$. If no other forces acted on the Earth, how much would it move in the time it takes the apple to fall 10 m?

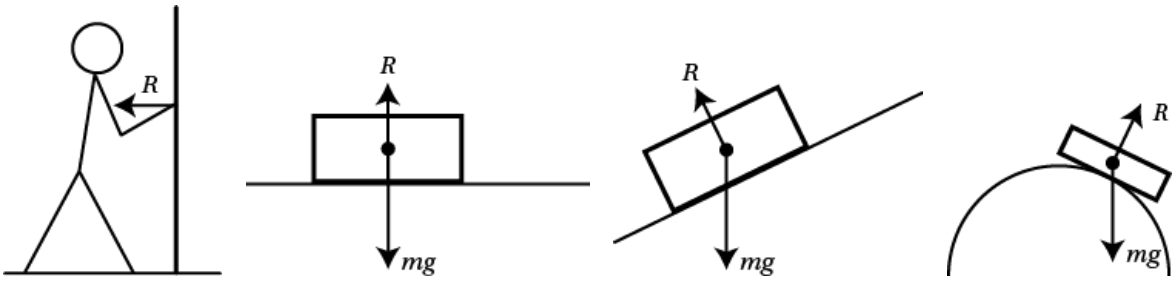
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Section 2: Normal reaction force

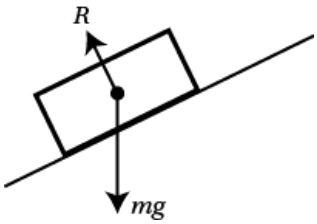
Look at this book resting on your desk. You know that there is weight acting on it, so why is it not accelerating downwards? There must be another, upward force to make the net force zero. This force, exerted by the table on the book, is called the **normal reaction force**.

Key point 22.2

Whenever an object is in contact with a surface, the surface exerts a normal reaction force on it. This force acts in the direction perpendicular to the surface and away from it.



You need to include the normal reaction force, as well as the object's weight, on your force diagram.



Did you know?

If an object is in contact with a curved surface, you can find the direction of the normal reaction force by calculating the gradient of the normal, as you learnt to do in Chapter 14, Section 1.

WORKED EXAMPLE 22.2

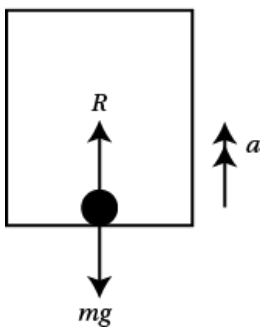
A person of mass $76\ \text{kg}$ is standing in a lift. Find the magnitude of the normal reaction force exerted by the floor of the lift on the person when the lift is

a moving upwards with acceleration $2.6\ \text{m s}^{-2}$

b moving downwards with acceleration $2.6\ \text{m s}^{-2}$

c moving downwards with deceleration $2.6\ \text{m s}^{-2}$.

a



The forces acting on the person are the weight (down) and the normal reaction force (up).

The acceleration of the person is the same as the acceleration of the lift.

Since the lift is moving upwards, take the positive direction to be up.

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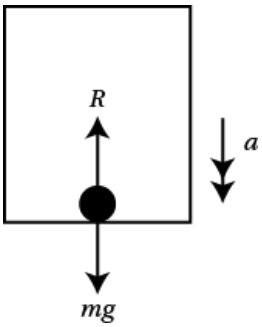
$F = ma$

$R - mg = ma$

$R - 76 \times 9.8 = 76 \times 2.6$

$\Rightarrow R = 942\ \text{N}$

b



Now it makes sense to take the positive direction to be down.

$mg - R = ma$

$76 \times 9.8 - R = 76 \times 2.6$

$\Rightarrow R = 547\text{N}$

c $76 \times 9.8 - R = 76 \times (-2.6)$

$\Rightarrow R = 942\ \text{N}$

The positive direction is still down, but the acceleration is negative.

Notice that the normal reaction force is larger when the lift is accelerating upwards or decelerating downwards, than when it is accelerating downwards. Think about what you would feel if you were standing in the lift. If it is accelerating upwards, it feels as if the floor is pushing against your feet. But if it's accelerating downwards you feel as if the floor is moving away from you. This is because the normal reaction force in the two cases is different.

Be careful not to confuse normal reaction force with Newton's third law; a common misconception is that normal reaction is the 'reaction' to the object's weight. However, the weight and the normal reaction are both acting on the object itself, so they are not a Newton's third law pair. In the example of a book on the table, the reaction to the book's weight is the force with which the book acts on the Earth. The reaction to the normal reaction force is the downward force with which the book pushes the table.

WORKED EXAMPLE 22.3

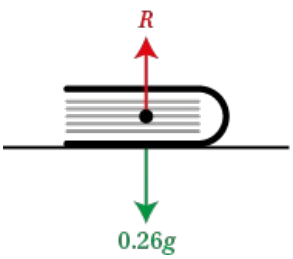
A book of mass $260\ \text{g}$ rests on a horizontal table of mass $65\ \text{kg}$. The table has four legs, and the thrust in each leg is the same. Assume that the table legs are light.

a Draw two separate diagrams showing the forces acting on the book and the table top.

b Find the thrust in each leg.

c State the magnitude and direction of the force exerted on the ground by the table top.

a

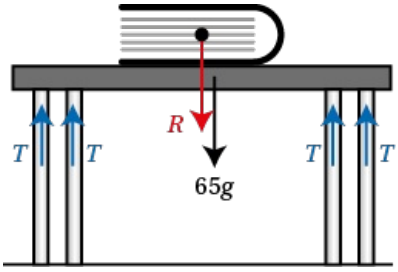


Forces on book

In the diagrams, the Newton's third law pairs of forces are shown in the same colour.

The forces on the book are its weight and the normal reaction from the table top.

The forces on the table top are the table top's weight, normal reaction from the book and the thrusts in the legs.



Forces on table top

b Forces on the book:

$0.26g = R$

Forces on the table top:

$65g + R = 4T$

$\Rightarrow 4T = 65g + 0.26g$

$\Rightarrow T = 160\ \text{N}$

Both the book and the table top are in equilibrium, so the net force on each must be zero. This means that forces up = forces down.

c The force on the ground from the table top equals $4T = 640\ \text{N}$, directed downwards.

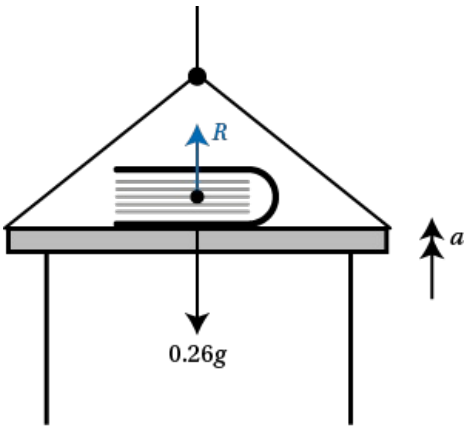
The force exerted on the ground by the table top has the same magnitude as the normal reaction on the table top from the ground.

In Worked example 22.3, the normal reaction force exerted by the table on the ground is equal to the total weight of the book and the table: the ground 'feels' the combined weight of the table and the book. However, this is only the case because all the objects are at rest.

If the table were being lifted using a cable attached to it, the normal reaction forces would change depending on the acceleration. You can write Newton's second law equations for both the book and the table, keeping in mind that they have the same acceleration.

WORKED EXAMPLE 22.4

A book of mass $260\ \text{g}$ rests on a horizontal table of mass $65\ \text{kg}$. The table is being lifted by a cable attached to it and accelerates upwards. The tension in the cable is $710\ \text{N}$. Find the normal reaction force between the book and the table.



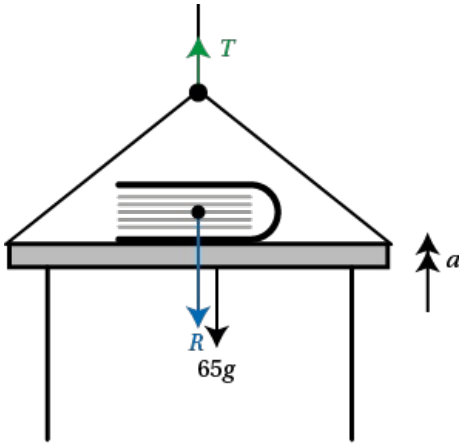
forces on book

Draw separate force diagrams for the book and the table.

Forces on the book are its weight and the normal reaction.

Forces on the table are its weight, normal reaction and the tension in the cable.

The book is not attached to the cable, so the tension does not act on it directly.



forces on table

Table: $65a = 710 - 65g - R$

Book: $0.26a = R - 0.26g$

$65.26a = 710 - 65g - 0.26g$

$65.26a = 70.452$

$a = 1.08ms^{-2}$

$R = 0.26a + 0.26g$

$R = 2.83N$

Newton's second law for each object: the positive direction is upwards.

You have two simultaneous equations. You may be able to solve them on the calculator. Otherwise it's easiest to find a by adding the two equations and then substitute it back to find R.

Notice that the equation you got for acceleration in Worked example 22.4 can be written as $65.26a = 710 - 65.26g$. This is the same equation you would get if you considered the table and the book as a single object with mass $65.26\ \text{kg}$ (which is the combined mass of the table and the book) being pulled up using the tension of $710\ \text{N}$. The normal reaction forces do not appear in this equation.

Key point 22.3

When two objects are in contact and moving with the same acceleration, you can consider them as a single particle. Newton's second law, $\mathbf{F} = ma$, applies with m being the combined mass of the two objects and $\mathbf{F}$ being the resultant of all the external forces.

Tip

'External force' excludes any forces acting between the two objects. So, for example, the normal reaction forces should not be included in the resultant force.

WORKED EXAMPLE 22.5

A person of mass $72\ \text{kg}$ stands in a lift of mass $540\ \text{kg}$. The lift is supported by a cable that can be modelled as light and inextensible.

a Draw two diagrams showing all the forces acting on the person and the lift.

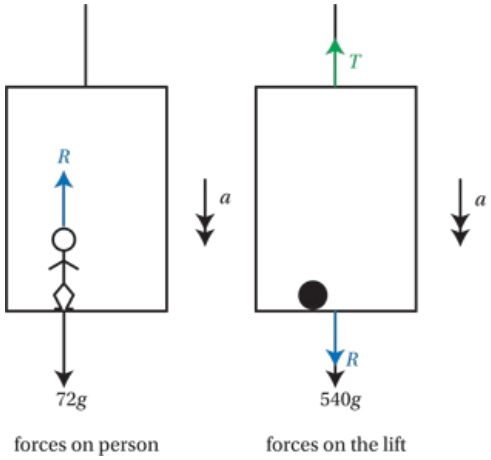
The lift is moving downwards and decelerating at $2.1\ \text{m s}^{-2}$.

b Find the tension in the cable.

c Find the magnitude of the normal reaction force exerted on the person by the floor of the lift.

The forces on the person are weight and normal reaction.

a



The forces on the lift are its weight, tension (T) and normal reaction.

According to Newton's third law, the two normal reactions have equal magnitudes but opposite directions.

b Newton's second law for the two objects together:

$612 \times (-2.1) = 612\text{g} - T$

$T = 7280\text{N}$

To find the tension, which is an external force on the system, you can treat the lift and the person as a single object of mass $540 + 72 = 612\ \text{kg}$.

The two normal reaction forces are not included.

Take the positive direction to be downwards, so the acceleration is negative.

c For the person:

$72 \times (-2.1) = 72\text{g} - R$

$R = 857N$

To find the normal reaction force consider only the forces acting on the person. However, the acceleration is the same as for the whole system.

If you want to find the contact force, you need to consider each object separately.

EXERCISE 22B

1 For each situation draw a separate force diagram for each object.

a i A book of mass $300\ \text{g}$ rests on a table of mass $18\ \text{kg}$. The table is on the floor.

ii A book of mass $120\ \text{g}$ rests on a shelf of mass $25\ \text{kg}$. The shelf is on the floor.

b i A box of mass $2\ \text{kg}$ rests on a table of mass $12\ \text{kg}$. A vertical cable is attached to the table and the tension in the cable is $15\ \text{N}$. The table is not in contact with the floor.

ii A box of mass $5\ \text{kg}$ rests on top of a crate of mass $20\ \text{kg}$. The crate is suspended by a vertical cable. The tension in the cable is T.

c i A person of mass $65\ \text{kg}$ stands in a lift of mass $200\ \text{kg}$. The lift is suspended by a cable and the tension in the cable is T.

ii A person of mass $80\ \text{kg}$ stands in a lift of mass $200\ \text{kg}$. The lift is suspended by a cable and the tension in the cable is $2800\ \text{N}$.

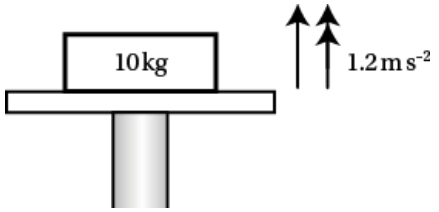
d i A box of mass $12\ \text{kg}$ is suspended by a string from the ceiling of a lift of mass $400\ \text{kg}$. The lift is suspended by a vertical cable.

ii A box of mass $15\ \text{kg}$ is suspended by a string from the ceiling of a lift of mass $350\ \text{kg}$. The lift is suspended by a vertical cable.

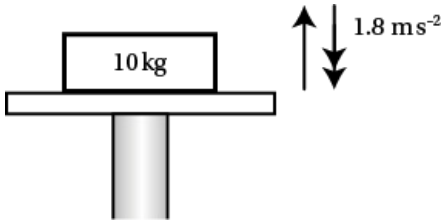
2 Each diagram shows an object of mass $10\ \text{kg}$ resting on a platform. The platform is moving in the direction shown by a single arrow, with acceleration shown by a double arrow. Find the magnitude of the normal reaction force exerted on the object by the platform.

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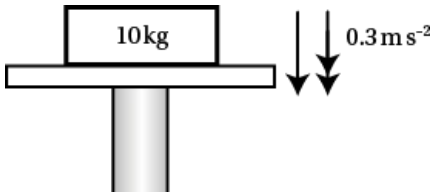
a i



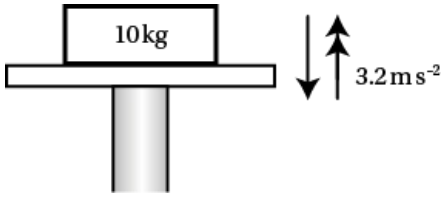
ii



b i



ii



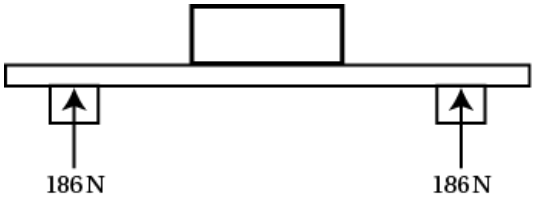
3 A book of mass $0.6\ \text{kg}$ rests on a horizontal table. A child pushes down on the book with a force of $12\ \text{N}$. Find the magnitude of the normal reaction force between the book and the table.

4 A box of mass $15\ \text{kg}$ rests on horizontal ground. The box is attached to a vertical cable that can be modelled as light and inextensible. The magnitude of the normal reaction force between the box and the ground is $68\ \text{N}$. Find the tension in the cable.

5 A crate of mass $120\ \text{kg}$ lies on the horizontal floor of a lift. The lift accelerates upwards at $0.4\ \text{m s}^{-2}$. Find the magnitude of the normal reaction force between the crate and the floor of the lift.

6 A basket of mass $750\ \text{g}$ rams is attached to a light inextensible rope and is being lowered at a constant speed. A box of mass $120\ \text{g}$ rams rests at the bottom of the basket. Find the magnitude of the normal reaction force between the box and the basket.

7 A horizontal plank of mass $27\ \text{kg}$ rests on two light vertical supports. A box rests on top of the plank. The thrust in each support is $186\ \text{N}$.



Find the mass of the box.

8 A child of mass $35\ \text{kg}$ stands in a lift of mass $500\ \text{kg}$. The lift is suspended by a light inextensible cable and accelerates upwards at $0.6\ \text{m s}^{-2}$. Find:

a the tension in the cable

b the magnitude of the normal reaction force between the child's feet and the floor of the lift.

9 A person of mass $85\ \text{kg}$ stands in a lift of mass $360\ \text{kg}$. The lift is suspended by a light inextensible cable. Find the magnitude of the normal reaction force between the person's feet and the floor of

the lift when the lift is:

a moving downwards and decelerating at $4.2\ \mathrm{m\ s^{-2}}$

b moving upwards at a constant speed.

10 A lift of mass $520\ \mathrm{kg}$ is supported by a steel rod attached to its bottom. The rod can withstand the maximum thrust $15400\ \mathrm{N}$. The lift can accelerate at $2.5\ \mathrm{m\ s^{-2}}$ and decelerate at $7.8\ \mathrm{m\ s^{-2}}$. Find the maximum allowed load in the lift.

11 A woman of mass $63\ \mathrm{kg}$ stands in a lift of mass $486\ \mathrm{kg}$. The lift is supported by a cable and moves with acceleration $2.2\ \mathrm{m\ s^{-2}}$. The magnitude of the normal reaction force between the woman's feet and the floor of the lift is $756\ \mathrm{N}$.

a Is the lift going up or down?

b Find the tension in the cable.

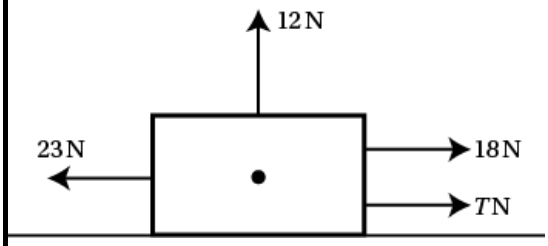
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Section 3: Further equilibrium problems

In Chapter 21 you met the idea of equilibrium, where all the forces on an object balance. When the object is in contact with a surface you need to include a normal reaction force in the calculations.

WORKED EXAMPLE 22.6

A box of mass $16\ \text{kg}$ rests on a smooth horizontal table. Four light inextensible strings are attached to the box. The tensions in the string are $12\ \text{N}, 23\ \text{N}, 18\ \text{N}$, and $T\ N$, as shown in the diagram.



Given that the box is in equilibrium, find:

a the value of T

b the magnitude of the normal reaction force between the box and the table.

a Horizontal components:

$18 + T - 23 = 0$

$\Rightarrow T = 5$

All the horizontal forces add up to zero. The components to the right are taken as positive and those to the left as negative.

b Vertical components:

$12 + R - 16\text{g} = 0$

$\Rightarrow R = 16\text{g} - 12 = 145\ \text{N}$

Forces in the vertical direction are tension, weight and normal reaction.

All the vertical forces add up to zero. The components upwards are taken as positive.

The normal reaction force only acts as long as the object is in contact with the surface. When there is another force pulling the object away from the surface the normal reaction force will decrease. If it reaches zero then the object is no longer in contact with the surface.

WORKED EXAMPLE 22.7

A book of mass $320\ \text{g}$ rests on a horizontal table. A girl pushes vertically down on the book with a force of $8.2\ \text{N}$. The book is attached to a light inextensible string, and a boy pulls the string vertically upwards so that the tension in the string is $T\ \text{N}$.

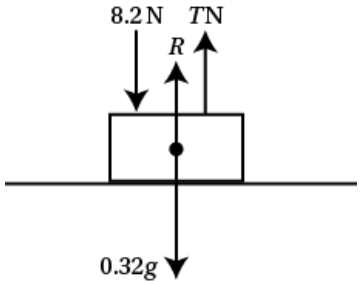
a Express the normal reaction force in terms of T.

b Find the smallest value of T required to lift the book off the table.

c Find the acceleration of the book when $T = 12$.

The forces on the book are: its weight, the normal reaction force, the tension and the pushing force of $8.2\ \text{N}$.

a



$mg + 8.2 - R - T = 0$

$\therefore R = 3.136 + 8.2 - T$

$= 11.336 - T$

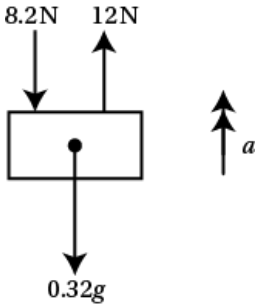
When the book is equilibrium, the resultant force is zero. Write the equation taking the positive direction to be down.

b $11.336 - T = 0$

$\Leftrightarrow T = 11.3$

The normal reaction force cannot be negative. The book will leave the table when $R = 0$.

c



$F = ma$ with the positive direction upwards:

$0.32a = 12 - 0.32g - 8.2$

$0.32a = 0.664$

$a = 2.08 ms^{-2}$

Since $T > 11.3$ the book is no longer in contact with the table so $R = 0$. The equilibrium is broken and the book has upward acceleration.

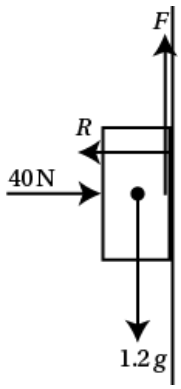
Remember that the normal reaction force does not need to act in a vertical direction. It is always perpendicular to the contact surface, and acting away from it.

WORKED EXAMPLE 22.8

A box of mass $1.2\ \text{kg}$ is pushed against a rough vertical wall with a force of $40\ \text{N}$ and rests in equilibrium. Find:

a the normal reaction force between the box and the wall

b the magnitude and direction of the friction force between the box and the wall.



The forces on the box are its weight, the pushing force, the normal reaction force (away from the wall) and the friction force (up along the wall, stopping the box from slipping downwards).

a Horizontally: $R = 40\ \text{N}$

b Vertically: $F = 1.2g$

$= 11.8\ \text{N}$ upwards

Write equilibrium equations for horizontal and vertical directions separately.

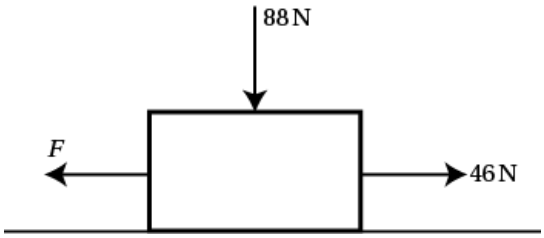
Fast forward

A In Student Book 2 you will find out how the magnitude of the friction force is related to the normal reaction.

EXERCISE 22C

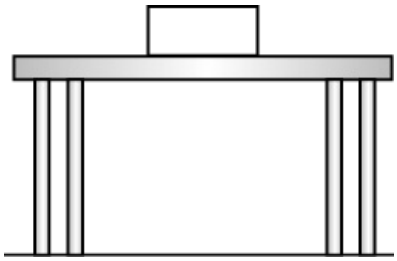
1 A crate of mass $150\ \text{kg}$ rests on a horizontal floor. A vertical cable is attached to the crate and the tension in the cable is $820\ \text{N}$. Find the normal reaction force between the crate and the floor.

2 A box of mass $68\ \text{kg}$ rests in equilibrium on a horizontal table under the action of three forces shown in the diagram.



Find the value of F and the magnitude of the normal reaction force.

3 A box rests in equilibrium on a horizontal table. The mass of the box is $16\ \text{kg}$ and the mass of the table top is $86\ \text{kg}$. The table is supported by four light legs, as shown in the diagram.

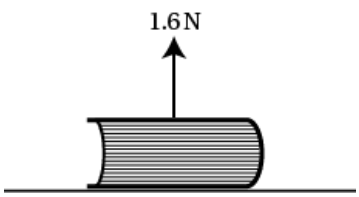


Find the thrust in each leg, assuming they are all the same.

4 A book of mass $320\ \text{g}$ rests in equilibrium on a horizontal table. Find the magnitude of the normal reaction force between the book and the table in each situation.

a A light inextensible string is attached to the book, as shown in the diagram. The string is vertical and the tension in the string is $1.6\ \text{N}$.

b The string is now removed and a girl pushes vertically down on the book with a force of $8.2\ \text{N}$.



5 A ball of mass $6.5\ \text{kg}$ is suspended by a vertical string and is in contact with a horizontal table. The string can be modelled as light and inextensible.

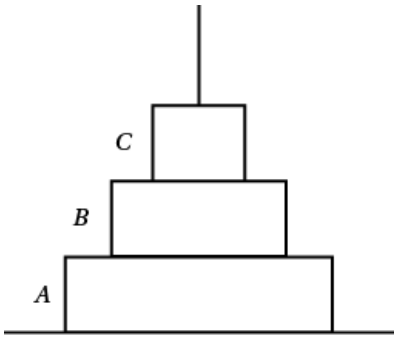
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a Find the magnitude of the normal reaction force between the ball and the table when the tension in the string is $38\ \mathrm{N}$.

b Find the minimum tension force required to lift the ball off the table.

6 Blocks A, B and C, of masses $13\ \mathrm{kg}, 21\ \mathrm{kg}$ and $18\ \mathrm{kg}$, are stacked on top of each other, as shown in the diagram. A light inextensible string is attached to block C and the system is in equilibrium.

The magnitude of the normal reaction force between blocks A and B is $260\ \mathrm{N}$. Find the tension in the string and the magnitude of the normal reaction force between blocks B and C.

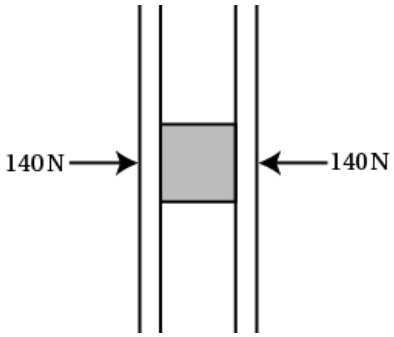


7 A small box of mass $1.2\ \mathrm{kg}$ is held in equilibrium between two rough planks. The planks are vertical and the friction forces between the box and the two planks are equal. Each plank is held in position by a horizontal force of magnitude $140\ \mathrm{N}$.

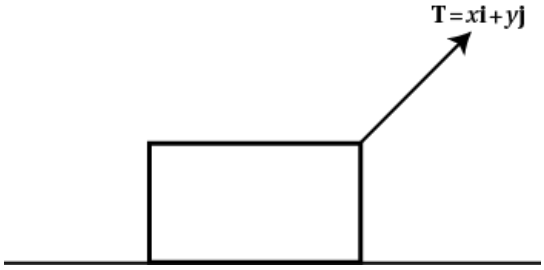
Find:

a the magnitude of friction force between each plank and the box

b the normal reaction force between each plank and the box.



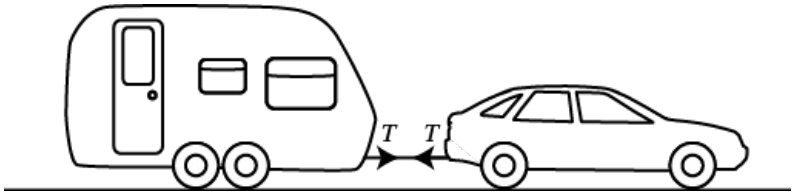
8 A box of mass $34\ \mathrm{kg}$ rests in equilibrium on a rough horizontal floor. A light inextensible string is attached to the box as shown in the diagram.



The friction force has magnitude $220\ \mathrm{N}$ and the magnitude of the normal reaction force is $180\ \mathrm{N}$. Find the magnitude of the tension in the string.

Section 4: Connected particles

In Chapter 21, Section 2, you looked at examples of objects being pulled using a rope or a stick, and you saw that one of the forces acting on the object is tension. When two objects are connected by a taut rope or string then the tension acts on both of them. The magnitude of the tension is the same at both of its ends. However, the directions at the two ends are different, because the tension acts away from the object that is attached to its end.



If the rope is inextensible, the connected objects will move with the same speed and the same acceleration. Any external forces, such as driving or resistance forces, can be different for each object. Remember that, to find the acceleration, you can consider them as a single object and not include the connecting tension forces.

Two main modelling assumptions have been made here, and it is important that you know what specific effect each has.

Tip

The term string is used for anything that transmits tension but not compression forces, such as a rope, wire, cable, chain.

The term rod is used for an object that transmits tension and compression forces, such as a tow bar or plank. A rod cannot stretch or compress itself (it is rigid).

Key point 22.4

Modelling assumptions about a string connecting two particles:

Assumption	What it means	How it is used
Inextensible	The string cannot stretch, e.g. not an elastic band or spring.	The magnitude of the acceleration and velocity of each connected particle is the same (if the string is taut).
Light	The string has a mass that is negligible in the context of the system.	1 When treating the whole system as a single particle the mass of the string is ignored. 2 In vertical systems the tension of the string is the same throughout. This is because lower parts do not 'pull' on upper parts.

WORKED EXAMPLE 22.9

A car of mass $780\ \text{kg}$ is pulling a trailer of mass $560\ \text{kg}$ using a light, inextensible cable. The driving force on the car is constant and has magnitude $1800\ \text{N}$. The total resistance forces on the car and the trailer are $800\ \text{N}$ and $600\ \text{N}$, respectively.

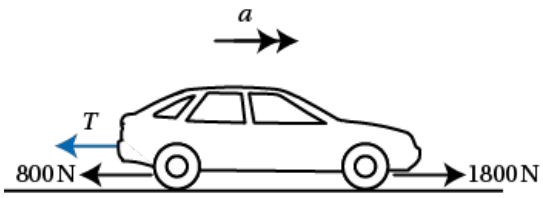
a Find the acceleration of the car.

b Find the tension in the cable.

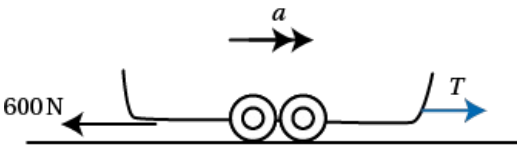
c Explain how you used the assumption that the cable is:

i light

ii inextensible.



forces on car



forces on trailer

Draw separate force diagrams for the car and the trailer.

The tension is the same magnitude at both ends of the cable, and both the car and the trailer have the same acceleration.

The driving force acts on the car only.

a For car and trailer together $(m = 1340kg)$:

$1340a = 1800 - 800 - 600$

$a = 0.299ms^{-2}$

Write $F = ma$ for the combined object, taking into account only the external forces.

b For the trailer:

$560 \times 0.299 = T - 600$

$T = 767N$

To find the tension you need to consider each object separately. You can choose which equation to use. Here the trailer equation is used, as there are fewer forces to include.

c **i** The mass of the cable wasn't included in the total mass of the two objects.

ii The car and the trailer have the same acceleration.

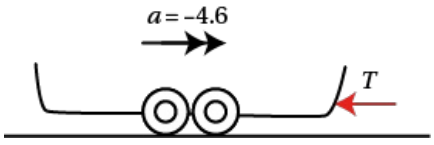
'Light' means that you can ignore the mass of the cable.

If the cable could stretch, the two ends could have different accelerations.

What would happen in Worked example 22.9 if the car started to brake? The braking force is acting on the car only, so there is nothing to slow down the trailer. The cable would go slack and the trailer would get closer to the car. This is why for towing we use rigid objects such as a tow bar, which can exert thrust as well as tension.

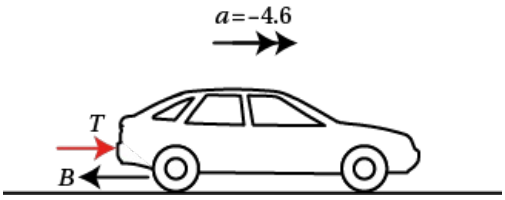
WORKED EXAMPLE 22.10

A trailer is attached to a car by a light tow bar. The mass of the trailer is $350\ \text{kg}$ and the mass of the car is $680\ \text{kg}$. The car starts to brake and decelerates at $4.6\ \text{m s}^{-2}$. Assuming all other resistance forces can be ignored, find the thrust in the tow bar.



Draw a separate force diagram for each object. The thrust force is directed towards the object. The braking force acts on the car only.

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car: $T - B = 680 \times (-4.6)$

trailer: $-T = 350 \times (-4.6)$

$T = 1610N$

Since you want to find the connecting thrust force between the two objects, write separate $F = ma$ equations for each. The acceleration is negative.

You only need the second equation to find T.

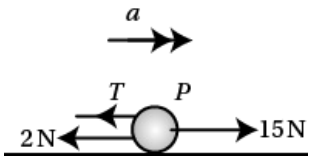
WORK IT OUT 22.1

Particles P and Q, of masses $0.6\,\text{kg}$ and $0.4\,\text{kg}$, are connected by a light inextensible string. The particles are initially at rest on horizontal ground and the string is taut. A force of magnitude $15\,\text{N}$ starts acting on P, in the direction away from Q. The friction force between P and the ground is $2\,\text{N}$ and the contact between Q and the ground is smooth. Find the acceleration of the two particles.

Decide whether each solution is correct or not. Can you identify the errors made in the incorrect solutions?

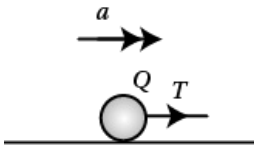
Solution 1

Considering P and Q as a single particle: $m = 1\,\text{kg}$



$1a = 15 - T - 2$

Forces on Q:



$0.4a = T$

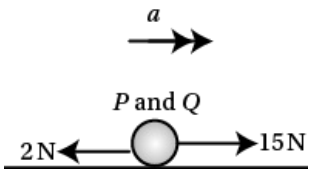
So: $1a = 15 - 0.4a - 2$

$1.4a = 13$

$a = 9.3\text{m s}^{-2}$

Solution 2

Considering P and Q as a single particle: $m = 1\,\text{kg}$

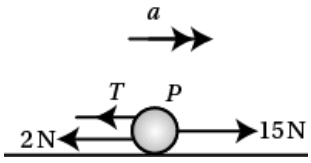


$1a = 15 - 2$

$a = 13\ \mathrm{m\,s^{-2}}$

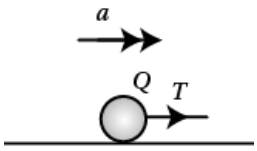
Solution 3

For P:



$0.6a = 15 - T - 2$

For Q:



$0.4a = T$

So: $0.6a = 15 - 0.4a - 2$

$a = 13 \mathrm{m} s^{-2}$

EXERCISE 22D

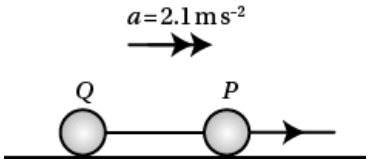
1 A car of mass $750\ \mathrm{kg}$ is towing a trailer of mass $350\ \mathrm{kg}$. The driving force on the car has magnitude $15\ \mathrm{kN}$. The resistance forces on the car and the trailer are $600\ \mathrm{N}$ and $400\ \mathrm{N}$, respectively. Find:

a the acceleration of the car and the trailer

b the tension in the tow bar.

2 Particle P of mass $4.5\ \mathrm{kg}$ is being pulled by a light inextensible string. Another light inextensible string is attached to the other side of P and particle Q, of mass $6\ \mathrm{kg}$, is attached to the other end of this string. The particles move with acceleration $2.1\ \mathrm{m\,s^{-2}}$ in a straight line on a smooth horizontal table.

Find the tension in each string.

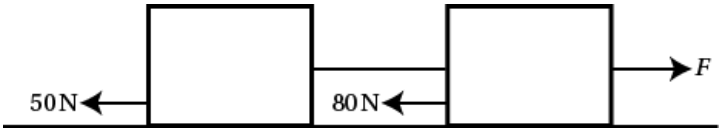


3 Two identical boxes, each of mass $10\ \mathrm{kg}$, are connected by a light inextensible cable. One box is pushed away from the other one with a force of $75\ \mathrm{N}$. The boxes move in a straight line at constant

velocity on a rough horizontal table. Find the magnitude of the friction force between each box and the table.

4 A car of mass $1200\ \text{kg}$ is towing a trailer of mass $400\ \text{kg}$ using a light tow bar. The resistance forces acting on the car and the trailer are $500\ \text{N}$ and $300\ \text{N}$, respectively. The car starts to brake and decelerates at $1.2\ \text{m s}^{-2}$. Find the magnitude of the braking force.

5 A box of mass $24\ \text{kg}$ is pulled across a rough horizontal floor with a force of $F\ \text{N}$. The friction force between this box and the floor is $80\ \text{N}$. Another box, of mass $15\ \text{kg}$, is attached to the first box by a light inextensible string. The friction force between the second box and the floor is $50\ \text{N}$.



a The string connecting the two boxes will break if the tension exceeds $120\ \text{N}$. Find the largest possible value of F.

b The string breaks when the boxes are moving with a speed of $2.4\ \text{m s}^{-1}$. Assuming the two boxes do not collide, how long does it take for the second box to stop?

6 A crate of mass $35\ \text{kg}$ is suspended by a light inextensible cable. Another crate, of mass $50\ \text{kg}$, is attached to the bottom of the first crate by another light inextensible cable. Find the tensions in the two cables when the crates are:

a being raised with acceleration $0.8\ \text{m s}^{-1}$

b being lowered at constant speed.

7 A train is made up of a locomotive of mass $4500\ \text{kg}$ and two carriages of mass $2500\ \text{kg}$ each. The train is accelerating at $0.9\ \text{m s}^{-2}$. The resistance force acting on the locomotive is $1200\ \text{N}$ and the resistance force acting on each carriage is $500\ \text{N}$. Find:

a the driving force on the locomotive

b the tension in each coupling.

8 A train consists of a locomotive of mass $4000\ \text{kg}$ and two carriages of mass $2000\ \text{kg}$ each. The train is decelerating at $2\ \text{m s}^{-2}$. The resistance forces are: $9000\ \text{N}$ on the locomotive, $6000\ \text{N}$ on the first carriage and $3000\ \text{N}$ on the second carriage.

a Determine whether the locomotive is driving or braking.

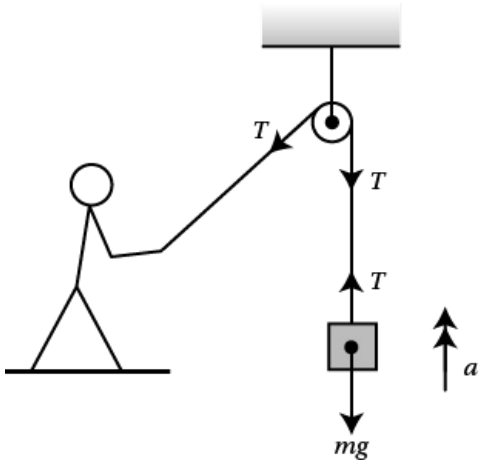
b Find the magnitude of the force in each coupling, stating whether it is a tension or a thrust.

IRANINOVIN

Section 5: Pulleys

A rope or string connecting two objects doesn't need to be straight. For example, you could be using a rope passing over a pulley to lift a crate.

As before, if the string is inextensible then the magnitude of acceleration of both objects will be the same. If the pulley has some mass, or if there is friction between the pulley and the axis it rotates around, then the magnitude of the tension will not be the same at both ends of the string. However, if you make some further modelling assumptions then the tension is the same throughout the string.



Key point 22.5

Modelling assumptions for a pulley:

Assumption	What it means	How it is used
Smooth	There is no friction between the pulley and its axis.	The tension in the string is the same on both sides of the pulley.
Light	The mass of the pulley can be ignored.	

The presence of a smooth, light pulley doesn't alter the method used for connected particles in the previous section.

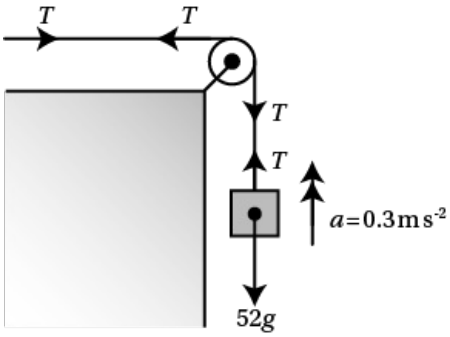
Tip

Note that there must be some friction between the pulley and the string – otherwise the string would be slipping over the pulley.

WORKED EXAMPLE 22.11

A crate of mass $52\,\text{kg}$ is attached to a light rope hanging over the edge of a wall and passing over a smooth pulley. A man pulls the other end of the rope, keeping it horizontal. The crate is moving upwards with acceleration $0.3\,\text{m}\,\text{s}^{-2}$. Find the tension in the rope.

Draw a diagram showing all the forces.



crate: $T - 52g = 52 \times 0.3$

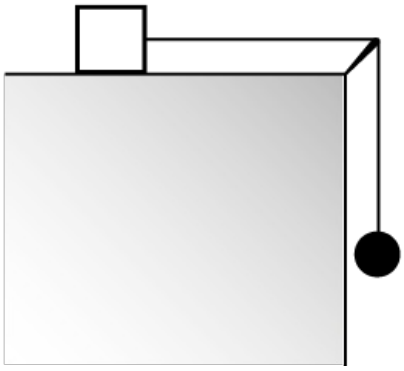
$\Rightarrow T = 525$ N

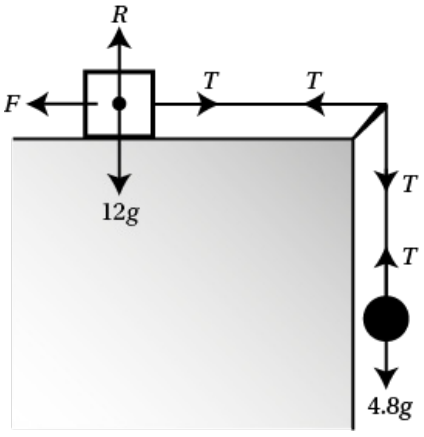
Write the force equation for the crate, taking the positive direction to be up.

A string could also be passed over a fixed peg instead of a pulley. If the peg is smooth (so that the string slides freely over it) the tension on either side of the peg will have the same magnitude.

WORKED EXAMPLE 22.12

A box of mass 12 kg rests on a rough horizontal table. It is attached to one end of a light inextensible string that passes over a smooth peg. The other end of the string is attached to a small object of mass 4.8 kg, as shown in the diagram. Given that the system is in equilibrium, find the magnitude of the frictional force between the box and the table.





Draw a diagram showing forces acting on each object. Since the peg is smooth, the tension force is the same on either side.

Ball: $4.8g = T \Rightarrow T = 47.04N$

Box, horizontally: $F = T$

The magnitude of the frictional force is 47.04 N.

Because the system is in equilibrium, the net force on each object is zero.

You can also say that $R = 12g = 117.6$N, but this is not required in the question.

You should remember that the tension in a rope or string exists only as long as it is taut. Once a string goes slack the tension force becomes zero.

WORKED EXAMPLE 22.13

Particles P and Q are connected by a light inextensible string passing over a smooth pulley. The mass of P is $2.3\ \text{kg}$ and the mass of Q is $3.1\ \text{kg}$. Initially Q is held at rest $80\ \text{cm}$ above the floor and P is $0.9\ \text{m}$ below the pulley.

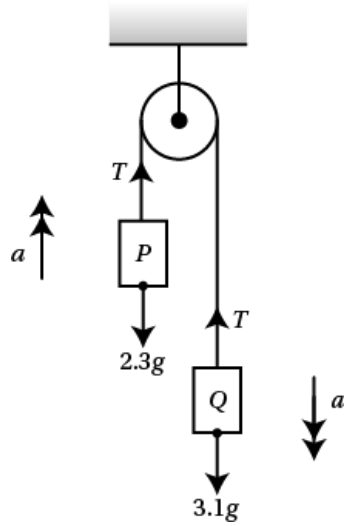
The particles are released from rest and the system moves freely under gravity.

a Find the speed of Q when it reaches the ground.

Once Q reaches the ground, P continues to move upwards.

b Find the time it takes P to reach the pulley from the moment it is first released.

a



To find the speed of Q you need to know its acceleration.

Draw a force diagram for each particle.

$P : T - 2.3\text{g} = 2.3a$

$Q : 3.1\text{g} - T = 3.1a$

Write an equation for each particle, taking the positive direction to be its direction of motion.

$$0.8\text{g} = 5.4a$$

$$\Rightarrow a = 1.45 ms^{-2}$$

You only want to find a, so eliminate T by adding the two equations.

$u = 0$

$a = 1.45$

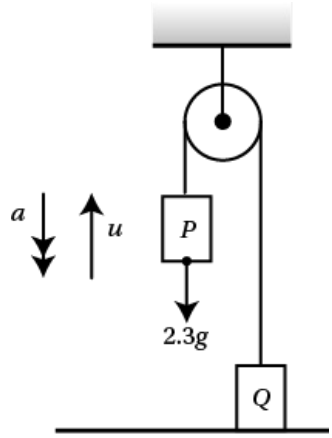
$s = 0.8$

$v = ?$

You need the final speed given the initial speed, acceleration and distance.

$$v^2 = u^2 + 2as$$
$$= 0 + 2(1.45)(0.8) = 2.32$$
$$v = \sqrt{2.32} = 1.52\ \text{m s}^{-1}$$

b



Once Q is on the ground there is no more tension in the string, so the only force acting on P is its weight. This means that its acceleration is g downwards. However, P has an upward speed (equal to the speed Q had when it hit the ground), so it will continue to move upwards for a while. It will reach the pulley before stopping and falling down again.

$u = 1.52$

$s = 0.1$

$a = -9.8$

$t = ?$

$s = ut + \frac{1}{2}at^2$

$0.1 = 1.52t - 4.9t^2$

P has moved $0.8\ \text{m}$ from its initial position, so it is now $0.1\ \text{m}$ from the pulley. Its speed is $1.52\ \text{m s}{-1}$ upwards.

$4.9t^2 - 1.52t + 0.1 = 0$

$t = 0.0947 \text{ or } 0.216$

This is a quadratic equation for t.

P reaches the pulley 0.0947 seconds after Q hits the ground.

P is moving freely under gravity so it will pass the pulley twice, once on the way up and once on the way down. You want the first of the two times.

Time taken for Q to reach the ground:

You need to add the time it took for Q to reach the ground. To do this, you need to use the information from part a.

$u = 0$

$a = 1.45$

$v = 1.52$

$t = ?$

$v = u + at$

$1.52 = 0 + 1.45\text{t}$

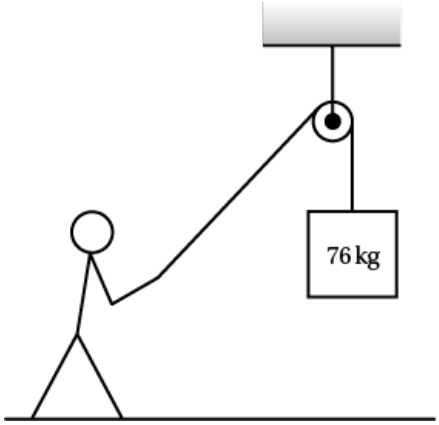
$\Rightarrow t = 1.05$

Total time taken for P to reach the pulley:

$t = 1.05 + 0.0947 = 1.14s$

EXERCISE 22E

1 A man is holding a crate of mass $76\ \text{kg}$ using a light inextensible rope passing over a smooth pulley.

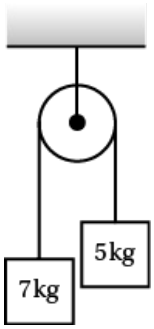


Given that the crate is in equilibrium, find the magnitude of the force exerted by the man on the rope.

Worksheet

For more examples and further practice with pulleys see Support sheet 22.

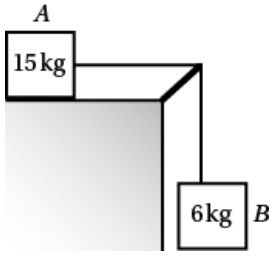
2 Two particles of masses 5 kg and 7 kg are connected by a light inextensible string that passes over a fixed smooth pulley. The system is released from rest with both ends of the string vertical and taut. Find the acceleration of each particle and the tension in the string.



3 Box A, of mass 15 kg, rests on a rough horizontal table. It is connected to one end of a light inextensible string which passes over a fixed smooth peg. Box B, of mass 6 kg, is attached to the other end of the string.

a Given that the system is in equilibrium, find the magnitude of the frictional force between box A and the table.

b Given instead that the contact between box A and the table is smooth, find the acceleration of the system and the tension in the string.



4 A box of mass 7 kg lies on a smooth horizontal table. It is attached to one end of a light inextensible string, which passes over a smooth pulley placed at the edge of the table. A ball hangs on the other end of the string. The system is released from rest with the string taut and the ball starts to move downwards. The tension in the string is 18.2 N. Find the mass of the ball.

5 Two particles have masses m and km, with $k > 1$. The particles are connected by a light inextensible string passing over a smooth pulley. The system is released from rest and the particles

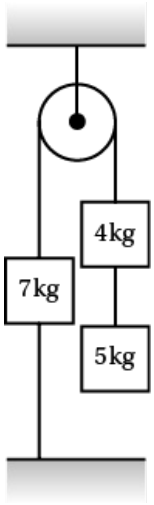
move with acceleration $\frac{2}{3}\mathrm{g}$. Find the value of k.

6 A particle of mass $4\ \mathrm{kg}$ is attached by light inextensible strings to two other particles of masses $7\ \mathrm{kg}$ and $5\ \mathrm{kg}$. The string connecting the $4\ \mathrm{kg}$ particle to the $7\ \mathrm{kg}$ particle passes over a smooth pulley, as shown in the diagram.

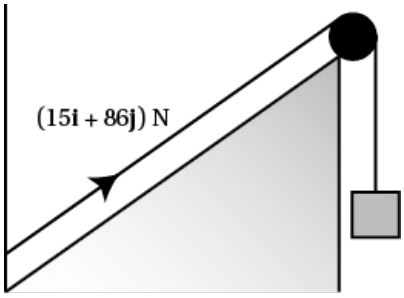
The particle of mass $7\ \mathrm{kg}$ is attached to the floor by another light inextensible string.

a Given that the system is in equilibrium, find the tension in the string connecting the $7\ \mathrm{kg}$ particle to the floor.

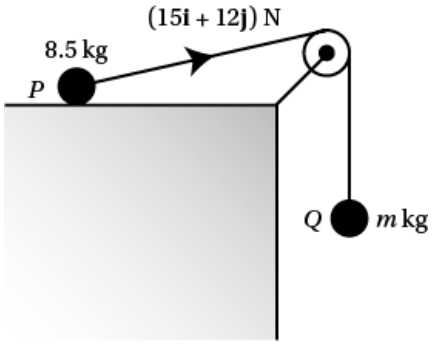
b This string is now removed. Find the acceleration of the system.



7 A crate is held in equilibrium by a light inextensible rope. The rope passes over a smooth peg. The other end of the rope is attached to a wall, as shown in the diagram. The force exerted by the rope on the wall is $(15\mathrm{i} + 86\mathrm{j})\ \mathrm{N}$. Find the mass of the crate.



8 Particle P, of mass $8.5\ \mathrm{kg}$, is attached to one end of a light inextensible string and rests on a rough horizontal table. The string passes over a smooth light pulley. Particle Q, of mass $m\ \mathrm{kg}$, is attached to the other end of the string and hangs with the string vertical. When the system is in equilibrium, the force exerted by the string on particle P is $\mathrm{T} = (15\mathrm{i} + 12\mathrm{j})\ \mathrm{N}$.



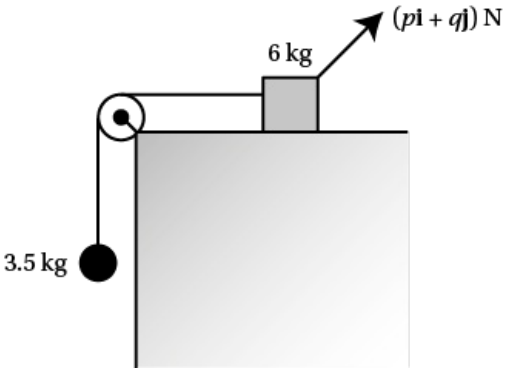
a Find the magnitude of T. Hence find the value of m.

b Find the magnitude of:

i the normal reaction force

ii the frictional force between P and the table.

9 A small box of mass $6\ \text{kg}$ rests on a smooth horizontal table. It is attached to one end of a light inextensible string which passes over a smooth pulley at the edge of the table. The other end of the string is attached to a particle of mass $3.5\ \text{kg}$, which hangs with the string vertical. The system is held in equilibrium by a force $\mathbf{F} = (p\mathbf{i} + q\mathbf{j})\ \text{N}$, as shown in the diagram.



The normal reaction exerted on the box by the table is $28\ \text{N}$. Find the magnitude of F.

10 Particles P and Q, of masses $0.8\ \text{kg}$ and $m_Q\ \text{kg}$, are connected by a light inextensible string passing over a smooth pulley. The particles are held at rest, both $2\ \text{m}$ above the floor, with the string taut. The particles are more than $2\ \text{m}$ below the pulley. When the system is released from rest, it takes the heavier particle 1.6 seconds to reach the floor. Find the two possible values of m_Q.

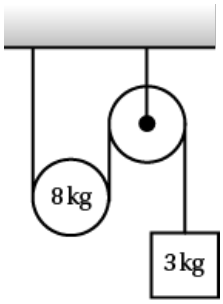
11 The diagram shows a tape passing over a fixed smooth pulley. One end of the tape is fixed to the ceiling and the other is attached to a box of mass $3\ \text{kg}$. A smooth cylinder of mass $8\ \text{kg}$ is placed in a loop formed by the tape.

The system is released from rest and the cylinder starts to accelerate downwards.

a The cylinder moves downwards a distance x. How far upwards does the box move in that time?

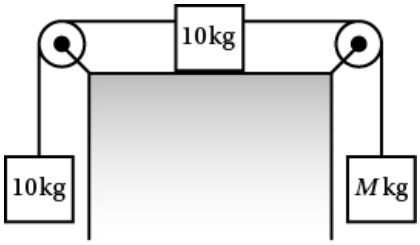
b The acceleration of the cylinder is a. Write down an expression, in terms of a, for the acceleration of the box.

c Find the tension in the tape.



12 A box of mass $10\ \text{kg}$ rests on a rough horizontal table. It is attached to two other boxes, of masses $10\ \text{kg}$ and $M\ \text{kg}$, by two light inextensible strings. Each string passes over a smooth pulley, as shown in the diagram.

The system is in equilibrium. Given that the maximum possible magnitude of the friction force between the box and the table is $49\ \text{N}$, find the range of possible values of M.



13 Discuss which of the modelling assumptions for connected particles hold in the following situations:

a Two cars colliding and one pushing the other a further $5\ \mathrm{m}$.

b A steel chain being used to lift a chair.

c A steel chain being used to lift a car.

d A pendulum hanging on an old clock.

e A water-skier being pulled by a boat.

f A glass-blower shaping a bottle between two pliers.

g A caravan being towed by a car.

h A steel cable hanging between supports on a bridge.

Checklist of learning and understanding

- Newton's third law states that if object A exerts a force on object B, then object B exerts a force on object A, with the same magnitude but opposite direction.
- Whenever an object is in contact with a surface, the surface exerts a normal reaction force. This force is perpendicular to the surface and directed away from it.
  - Newton's third law implies that when two objects are in contact, each object exerts a normal reaction force on the other one.
- If two objects are connected by a taut string then the tension is the same throughout the string.
  - The tension at the point where the string is attached to an object is directed away from the object.
  - If the string is inextensible, the two objects have the same acceleration.
  - If the string passes over a light smooth pulley, or a fixed smooth peg, the tension is the same on either side of the pulley/peg.
- If the string is replaced by a light rod, then the force can be a thrust as well as a tension. The thrust force is directed towards the object.
- Two connected objects move with the same acceleration and same speed.
  - To find the acceleration you can treat them as a single particle, but to find the normal reaction or tension force you need to consider each object separately.

IRANINOVIN

Mixed practice 22

1. Two skaters, of masses $58\ \text{kg}$ and $M\ \text{kg}$, stand facing each other on ice. They push away from each other and move with initial accelerations of $3.6\ \text{m s}^{-2}$ and $4.1\ \text{m s}^{-2}$. Find the value of M.

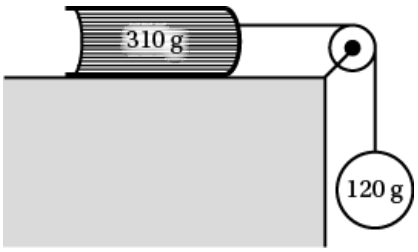
2. A car of mass $850\ \text{kg}$ is pulling a trailer of mass $320\ \text{kg}$. The driving force on the car has magnitude $1800\ \text{N}$. The resistance forces acting on the car and the trailer are $450\ \text{N}$ and $220\ \text{N}$, respectively. Find:

   a the acceleration of the car

   b the tension in the tow bar.

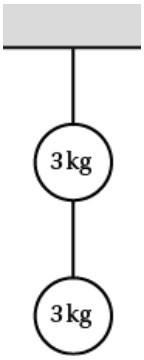
3. A crate of mass $80\ \text{kg}$ lies on a horizontal platform. The platform is being raised and decelerates at $2.6\ \text{m s}^{-2}$. Find the magnitude of the normal reaction force between the crate and the platform.

4. A book of mass $310\ \text{g}$ lies on a rough horizontal table. A light inextensible string is attached to the book. The string passes over a smooth pulley fixed at the edge of the table. A ball of mass $120\ \text{g}$ is attached to the other end of the string.



   The system is in equilibrium with the string taut. Find the magnitude of the friction force between the book and the table.

5. Two balls are connected and suspended from the ceiling by two light inextensible strings, as shown in the diagram.



   Given that both balls have mass $3\ \text{kg}$, find the tension in each string.

6. A man of mass $70\ \text{kg}$ stands on the floor of a lift which is moving with an upward acceleration of $0.3\ \text{m s}^{-2}$. Calculate the magnitude of the force exerted by the floor on the man.

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7. A car is pulling a trailer using a light rigid tow bar. The mass of the car is $1200\ \text{kg}$ and the mass of the trailer is $350\ \text{kg}$. Assume that any resistances to motion can be ignored.

   a The car is moving with a speed of $9.2\ \text{m s}^{-1}$ when it starts to accelerate at $1.8\ \text{m s}^{-2}$. Find the driving force acting on the car.

   b The car continues to accelerate uniformly for 4 seconds. It then starts to brake, with uniform deceleration, until it comes to rest. During the braking phase, it travels $26\ \text{m}$. Find the magnitude of the thrust in the tow bar during the braking phase.

8 A person of mass $75\ \text{kg}$ stands in a lift of mass $450\ \text{kg}$. The lift is suspended by a light inextensible cable and moving downwards.

a The lift is decelerating at $5.2\ \text{m s}^{-2}$. Find the normal reaction force between the person's feet and the floor of the lift.

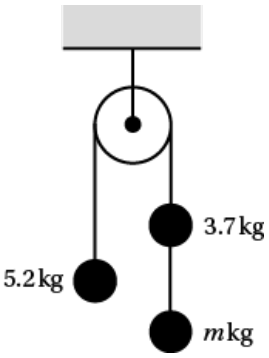
b Given instead that the normal reaction force between the person's feet and the floor of the lift is $577.5\ \text{N}$:

i find the magnitude and direction of the acceleration of the lift

ii calculate the tension in the cable.

9 Two skaters stand on ice $5\ \text{m}$ from each other, holding onto the ends of a light inextensible rope. They pull at the rope with a constant force and come together in 1.5 seconds. Any friction can be ignored. Given that the mass of the first skater is $62\ \text{kg}$, and that he moves with acceleration $1.8\ \text{m s}^{-2}$, find the mass of the second skater.

10 In the diagram the three strings can be modelled as light and inextensible and the pulley can be modelled as smooth. The masses of the balls are $5.2\ \text{kg}, 3.7\ \text{kg}$ and $m\ \text{kg}$. The system hangs in equilibrium in the vertical plane.

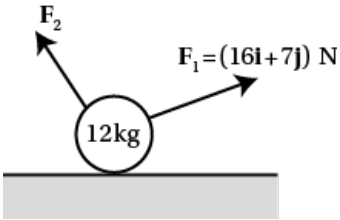


a Find the value of m.

b Find the tension in each string.

11 A particle of mass $12\ \text{kg}$ rests in equilibrium on a rough horizontal table, under the action of two forces, $F_1 = (16\mathbf{i} + 7\mathbf{j})\ \text{N}$ and F_2, as shown in the diagram.

The magnitude of the normal reaction force between the particle and the table is $72\ \text{N}$ and the magnitude of the friction force is $9\ \text{N}$. Find the two possible values for the magnitude of F_2.

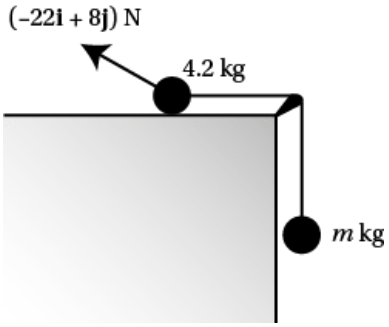


12 A particle of mass $4.2\ \text{kg}$ rests on a rough horizontal table. The magnitude of the frictional force between the particle and the table is $4\ \text{N}$. The particle is attached to one end of a light inextensible string which passes over a smooth peg at the edge of the table. Another particle, of mass $m\ \text{kg}$, is attached to the other end of the string. The system is held in equilibrium by a force $(-22\mathbf{i} + 8\mathbf{j})\ \text{N}$, as shown in the diagram.

a Find the magnitude of the normal reaction force exerted on the table by the particle.

b Find two possible values for the magnitude of the tension in the string.

c Hence find two possible values of m.



13 A trailer of mass $500\ \text{kg}$ is attached to a car of mass $1250\ \text{kg}$ by a light rigid horizontal tow bar. The car and the trailer are travelling along a horizontal straight road. The resistance to the motion of the trailer is $400\ \text{N}$ and the resistance to the motion of the car is $900\ \text{N}$. Find both the tension in the tow bar and the driving force of the car in each of the following cases.

a The car and the trailer are travelling at constant speed.

b The car and the trailer have acceleration $0.6\ \text{m s}^{-2}$.

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[Question part reference style adapted]

14 Particles P and Q, of masses $0.45\ \text{kg}$ and $m\ \text{kg}$ respectively, are attached to the ends of a light inextensible string which passes over a small smooth pulley. The particles are released from rest with the string taut and both particles $0.36\ \text{m}$ above a horizontal surface. Q descends with acceleration $0.98\ \text{m s}^{-2}$. When Q strikes the surface, it remains at rest.

a Calculate the tension in the string while both particles are in motion.

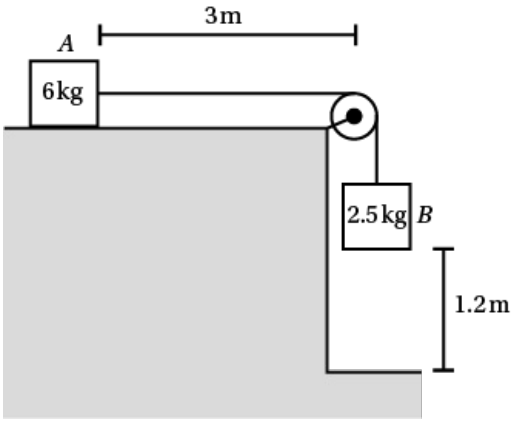
b Find the value of m.

c Calculate the speed at which Q strikes the surface.

d Calculate the greatest height of P above the surface. (You may assume that P does not reach the pulley.)

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15 Box A of mass $6\ \text{kg}$ is held at rest at one end of a rough horizontal table. The box is attached to one end of a light inextensible string which passes over a smooth pulley fixed to the other end of the table. The length of that part of the string extending from A to the pulley is $3\ \text{m}$. Box B, of mass $2.5\ \text{kg}$, is attached to the other end of the string and hangs $1.2\ \text{m}$ above the ground.



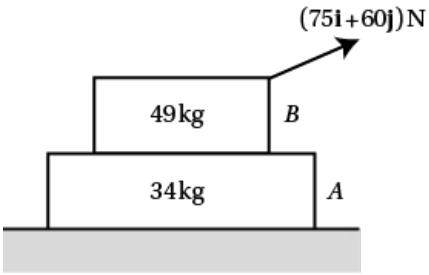
The system is released from rest and moves with acceleration $0.3\ \text{m s}^{-2}$.

a Find the magnitude of the friction force between box A and the table.

b Box B reaches the floor and remains at rest. The magnitude of the friction force between

box A and the table remains unchanged. Will box A reach the pulley?

16 Box A, of mass $34\ \text{kg}$, rests on rough horizontal ground. Box B, of mass $49\ \text{kg}$, rests on top of box A. A string is attached to Box B and the tension in the string is $(75\mathbf{i} + 60\mathbf{j})\ \text{N}$. The system is in equilibrium.



Find:

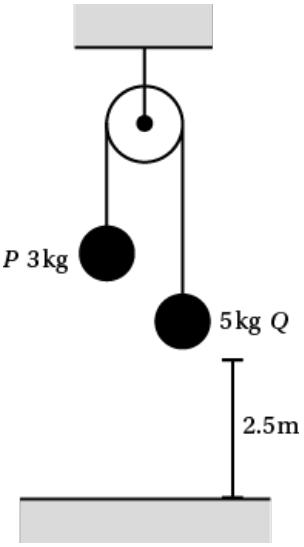
a the magnitude of the normal reaction force between box A and the ground

b the magnitude of the friction force between box A and the ground.

The tension in the string is now changed to $(75k\mathbf{i} + 60k\mathbf{j})\ \text{N}$ and the value of k is increased from 1. The maximum possible friction force between box A and Box B is $120\ \text{N}$ and the maximum possible friction force between box A and the ground is $180\ \text{N}$.

c Describe how the equilibrium is broken.

17 Particles P and Q, of masses $3\ \text{kg}$ and $5\ \text{kg}$, are connected by a light inextensible string. The string passes over a smooth pulley and the particles hang in the vertical plane with Q $2.5\ \text{m}$ above the ground.



At time $t = 0$ the system is released from rest with the string taut.

a Find the time required for Q to hit the ground.

Once Q is on the ground, P continues to move. Assume that in subsequent motion, neither particle reaches the pulley.

b Find the greatest height of P above its start point.

c Find the time when the string becomes taut again.

Worksheet

For a selection of more challenging problems involving pulleys see Extension sheet 22.

FOCUS ON … PROOF 4

Using mechanics to derive proofs

You have often used algebra and calculus to derive formulae in mechanics: for example, various forms of the constant acceleration formulae. But you don't normally think of using mechanics to prove algebraic results. This section looks at two examples of such proofs.

1. **Prove this statement about positive real numbers $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$.**

   If $\frac{\mathbf{a}}{\mathbf{b}} < \frac{\mathbf{c}}{\mathbf{d}}$ then $\frac{\mathbf{a}}{\mathbf{b}} < \frac{\mathbf{a}+\mathbf{c}}{\mathbf{b}+\mathbf{d}} < \frac{\mathbf{c}}{\mathbf{d}}$

PROOF 10

Consider a particle moving in a straight line. Suppose it travels at a constant speed v_1 for b seconds covering the distance of a metres, and then at a different constant speed v_2 for d seconds covering the distance of c metres.

Then $v_1 = \frac{a}{b}$ and $v_2 = \frac{c}{b}$, so the inequality $\frac{a}{b} < \frac{c}{d}$ says that $v_1 < v_2$.

The expression $\frac{a+c}{b+d}$ represents the total distance divided by total time, so this is the average speed for the whole journey. This has to be somewhere between the smaller and the larger speeds, so:

$$v_1 < \frac{a+c}{b+d} < v_2$$

which proves the required inequality.

2. **Prove that, for any two positive numbers $\mathbf{a}$ and $\mathbf{b}$, $\frac{\mathbf{a}+\mathbf{b}}{\mathbf{2}} \leqslant \sqrt{\mathbf{ab}}$.**

PROOF 11

Consider again an object moving in a straight line, and let s be the total distance travelled.

a. Suppose the first object travels for half the time with speed a and then for the other half of the time with speed b. Find an expression for the total time travelled in terms of a, b and s.

b. The second object travels half the distance with speed a and half the distance with speed b. Find an expression for its total time in terms of a, b and s.

c. Draw displacement–time graphs to see which object takes longer to travel the distance s. Make sure you consider both possibilities $a < b$ and $a > b$. Can you explain this without referring to the graph?

d. Use this to complete the proof that $\frac{a+b}{2} \leqslant \sqrt{ab}$.

Explore

The result in the second proof is called the AM-GM inequality. There are several similar inequalities, and they are particularly important in probability and statistics. Find out about

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their applications in probability and statistics.

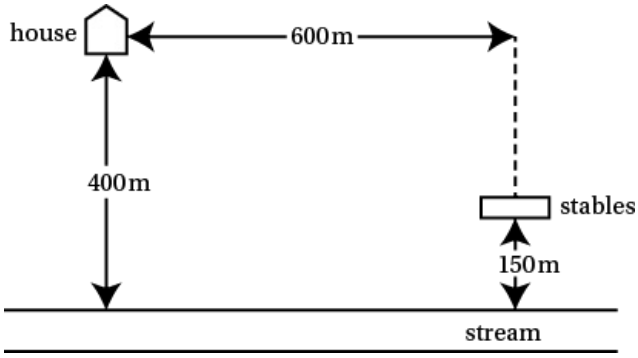
FOCUS ON … PROBLEM SOLVING 4

Alternative representations

It is easy to categorise problems in mathematics by topic, labelling things as a ‘mechanics problem’ or ‘geometry problem’. But sometimes unexpected links provide elegant solutions to otherwise difficult problems.

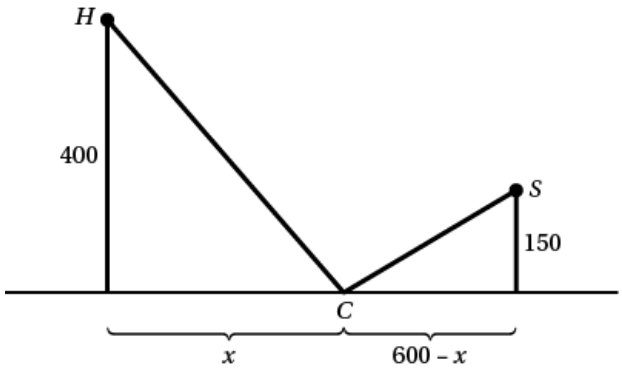
Consider the following problem:

A farmer wants to build a straight path from his house to the stream and then from the stream to the stables. The positions of the two buildings are shown in the diagram.



What is the shortest possible length of the path?

This looks like a ‘calculus problem’ you encountered in Chapter 14. You could set up a coordinate system, write an expression for the total distance in terms of x and use differentiation to find the minimum.



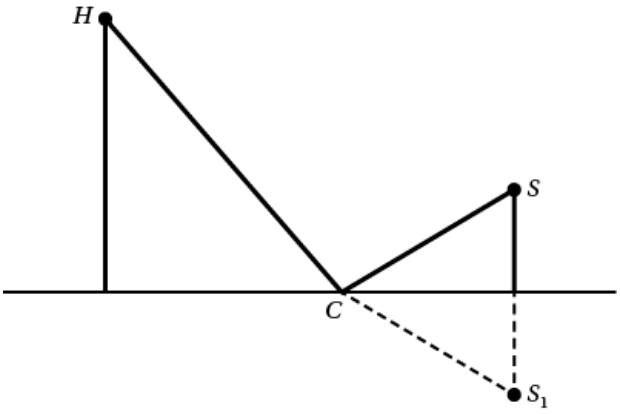
$$d = \sqrt{x^2 + 400^2} + \sqrt{(600 - x)^2 + 150^2}$$

Fast forward

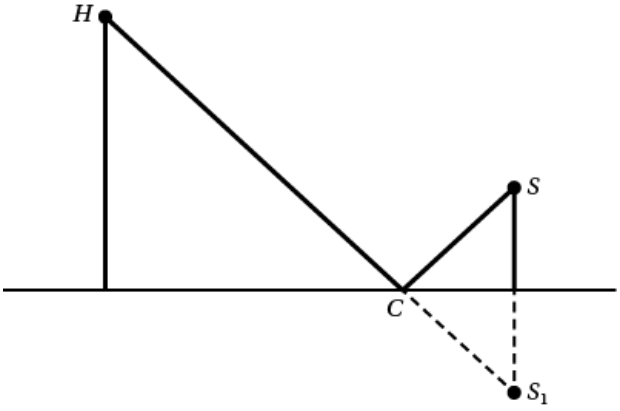
A You don’t know how to differentiate this expression for d yet, but you will learn how to do so in Student Book 2.

However, there is a much simpler solution if you use an idea from geometry: the shortest distance between two points is along a straight line. Of course, the straight line between the house and the stables does not go to the stream. The trick is to find a point on the other side of the stream that is the same distance from the stream as the stables. This is achieved by reflecting the point which represents the stables in the line which represents the stream.

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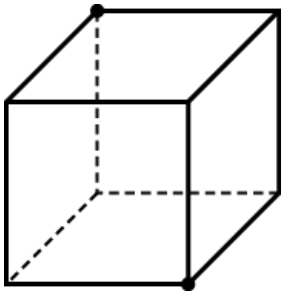
Suppose the path goes from the house to the point C, crosses the stream and then goes in a straight line to point S_1. Then the total length of the path is $d = HC + CS = HC + CS_1$. But the shortest distance between H and S_1 is along a straight line, so C should be the point where the line HS_1 crosses the stream.



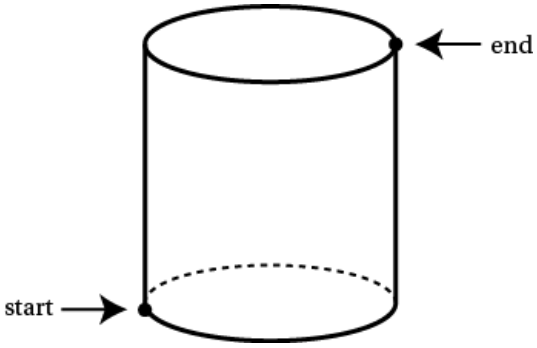
Questions

1. Find the length of the shortest possible path in the above problem.
2. A cube has side of length $1\ \mathrm{m}$. An ant starts at one corner of the cube and crawls to the opposite corner. The ant can only move on the surface of the cube. Find the shortest possible path and calculate its length.

   How does this compare to the length of the shortest path if the ant could pass through the cube?



3. Repeat question 2 for an ant moving on the surface of a cuboid with sides of lengths $1\ \mathrm{cm}$, $2\ \mathrm{cm}$ and $3\ \mathrm{cm}$.
4. A caterpillar crawls on the surface of a cylinder with radius r and height h. It starts at a point on the edge of the bottom base and crawls to the point on the top base that is diametrically opposite the starting point. Find the length of the shortest possible path it can take.



Explore

What is the shortest possible path between two points if you are moving on the surface of a sphere?

FOCUS ON … MODELLING 4

Investigating the effect of modelling assumptions

Throughout the mechanics chapters you have made various modelling assumptions that enabled you to write relatively simple equations to describe the motion of an object. You will now look at how changing some of these assumptions affects the results predicted by the model.

Changing the value of g

You usually assume that the value of gravitational acceleration is constant on the surface of the Earth, and take $g = 9.8\,\mathrm{m\,s^{-2}}$. However, the value of g, actually varies with the latitude and height above sea level.

Questions

A student conducts an experiment to measure the height of a building by dropping a small stone from the top and timing how long it takes to hit the ground.

The stone takes 1.6 seconds to fall.

1. Taking $g = 9.8\,\mathrm{m\,s^{-2}}$, estimate the height of the building. Give your answer to $3\,\mathrm{s.f.}$
2. The building is actually located in Greenland, where $g = 9.825\,\mathrm{m\,s^{-2}}$. What is the percentage error in the estimate in question 1?
3. If, instead, the building is located in Denver, where $g = 9.796\,\mathrm{m\,s^{-2}}$, find the percentage error in the estimate of the height.
4. The experiment is repeated with a much taller building, and the stone takes 10.2 seconds to fall. Repeat all the calculations from questions 1, 2 and 3.

Effect of air resistance

Another common modelling assumption is that there is no air resistance. Air resistance depends on many different factors, including the size and shape of the object, the material of which it is made, and on the density of air (which varies with height and temperature). It also varies with the speed of the object. There are two common models for air resistance.

At low speeds, air resistance can be modelled as being proportional to the speed of the object.

Questions

A cyclist starts moving from rest and applies a constant driving force of $180\,\mathrm{N}$. The air resistance has magnitude $40\mathrm{V}$, where v is the speed. Assume all other forces can be ignored. The cyclist and her bike have a mass of $120\,\mathrm{kg}$.

5. Write the Newton's second law equation for the motion of the cyclist. Verify that $v = 4.5\left(1 - e^{-\frac{t}{3}}\right)$ satisfies this equation.
6. Write down an equation for $v(t)$ if air resistance is ignored.
7. Use each model to predict how long it would take the cyclist to reach the speed of:
   - **a** $0.5\,\mathrm{m\,s^{-1}}$
   - **b** $2\,\mathrm{m\,s^{-1}}$
   - **c** $5\,\mathrm{m\,s^{-1}}$

Would you say that ignoring air resistance is a good modelling assumption in this problem?

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At higher speeds, a better model is to assume that air resistance is proportional to the square of the speed. In this model, the constant of proportionality depends on drag coefficient b and the mass of the object. The model leads to the following expression for the velocity:

$$v = A\left(\frac{1 - \mathrm{e}^{-kt}}{1 + \mathrm{e}^{-kt}}\right)$$

where $A = \sqrt{\frac{mg}{b}}$ and $k = \sqrt{\frac{4bg}{m}}$.

Questions

For a parachutist of mass $80\,\mathrm{kg}$, $\mathrm{b} = 0.27$:

8 **a** For a model without air resistance, what is the velocity of the parachutist after 1 second? How does this compare to the velocity predicted by the given model?

b How do the velocities after 10 seconds compare in the two models?

For a 2 pence coin of mass $7.2\mathrm{g}$, $\mathrm{b} = 2 \times 10^{-4}$:

9 **a** For a model without air resistance, what is the velocity of the coin after 1 second? How does this compare to the distance predicted by the given model?

b How do the velocities after 10 seconds compare in the two models?

c For a model without air resistance, find how long it takes for an object to reach the ground from a height of:

i $5\,\mathrm{m}$

ii $500\,\mathrm{m}$

d Hence investigate whether ignoring air resistance is a suitable modelling assumption for a parachutist and a coin falling from the height of $5\,\mathrm{m}$ and $500\,\mathrm{m}$.

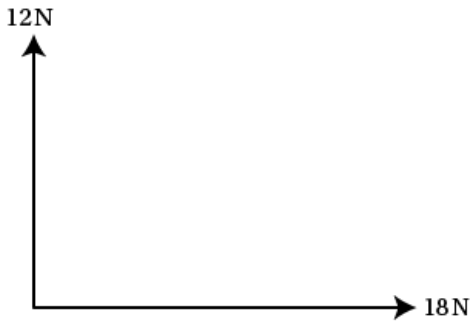
Explore

Models with air resistance predict that the object will reach a **terminal velocity**, as the resistance force increases with speed and eventually balances the weight. How does the weight of an object affect its terminal velocity, and the time it takes to reach it?

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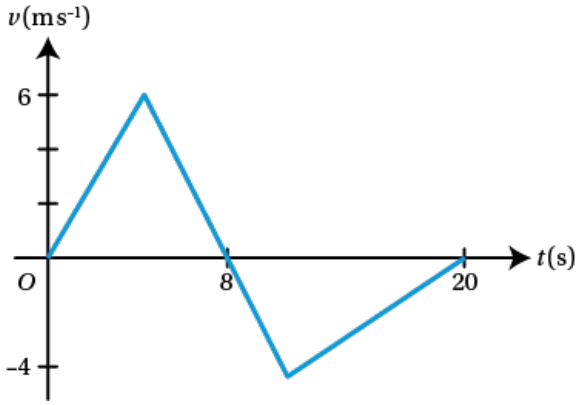
CROSS-TOPIC REVIEW EXERCISE 4

1 A particle of mass $3\,\text{kg}$ moves under the action of two perpendicular forces, as shown in the diagram.



Find, in vector form, the acceleration of the particle

2 The diagram shows the velocity–time graph for a particle moving in a straight line.



Find the average speed of the particle during the 20 seconds.

3 A particle of mass $4.2\,\text{kg}$ starts from rest and accelerates uniformly under the action of a constant force of magnitude $13.6\,\text{N}$. Find the speed of the particle after it has travelled the distance of $6.2\,\text{m}$.

4 Two perpendicular forces have magnitudes $x\,\text{N}$ and $3x\,\text{N}$ (see diagram). Their resultant has magnitude $6\,\text{N}$.

a Calculate x.

b Find the angle the resultant makes with the smaller force.



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[Question part reference style adapted]

5 A particle P is projected vertically upwards and reaches its greatest height $0.5\ \mathrm{s}$ after the instant of projection. Calculate:

a the speed of projection of P

b the greatest height of P above the point of projection.

It is given that the point of projection is $0.539\ \mathrm{m}$ above ground.

c Find the speed of P immediately before it strikes the ground.

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[Question part reference style adapted]

6 A person of mass $60\ \mathrm{kg}$ stands in a moving lift of mass $420\ \mathrm{kg}$ The normal reaction force between the person's feet and the floor of the lift is $312\ \mathrm{N}$.

a Find the magnitude and direction of the acceleration of the lift.

The lift is suspended by a cable, which can be modelled as light and inextensible.

b Find the tension in the cable.

c Explain how you have used the assumption that the cable is light.

7 A car is travelling in a straight line along a horizontal road, with constant acceleration $a\,\mathrm{m\,s^{-2}}$ and It passes point A with speed $u\,\mathrm{m\,s^{-1}}$, reaches point B five seconds later and point C two seconds after that.

a Given that the distance $AB = 95\,\mathrm{m}$ and $BC = 80\,\mathrm{m}$, find the values of u and a.

b The driving force of the car has magnitude $7200\,\mathrm{N}$. The resistance to the motion of the car is $2500\,\mathrm{N}$ Find the mass of the car.

8 A train consists of a locomotive and two carriages. The mass of the locomotive is $5600\,\mathrm{kg}$ and the mass of each carriage is $3200\,\mathrm{kg}$. The train is moving with a speed of $40\,\mathrm{km\,h^{-1}}$ when a driver applies the brakes. The train comes to rest after travelling $360\,\mathrm{m}$ The resistance forces throughout this motion are constant, $700\,\mathrm{N}$ on the locomotive and $400\,\mathrm{N}$ on each carriage.

Find the force in the coupling between the first carriage and the locomotive. Is it a tension or a thrust?

9 A particle P moves in a straight line, passing the point O with speed $35\,\mathrm{m\,s^{-1}}$. At time t seconds after leaving O the acceleration $a\,\mathrm{m\,s^{-2}}$ is given by:

$$a = 6t - 22 \qquad 0 \leqslant t \leqslant 5$$

a i Find an expression for the velocity at time t.

ii Find the times at which P is at rest.

iii Find the maximum speed of the particle in its 5-second journey.

b Find the total distance travelled by P.

10 A particle has velocity given by $v = t^2 + 1$ for $t \geqslant 0$. Velocity is measured in $\mathrm{m\,s^{-1}}$ and time ins seconds. The average velocity of the particle from $t = 0$ to $t = T$ is $4ms^{-1}$. Find the value of T.

11 A car is travelling at $13\,\mathrm{m\,s^{-1}}$ along a straight road when it passes a point A at time $t = 0$, where t is in seconds. For $0 \leqslant t \leqslant 6$, the car accelerates at $0.8t\,\mathrm{m\,s^{-2}}$.

a Calculate the speed of the car when $t = 6$.

b Calculate the displacement of the car from A when $t = 6$.

c Three (t, x) graphs are shown, for $0 \leqslant t \leqslant 6$.

Figure 1

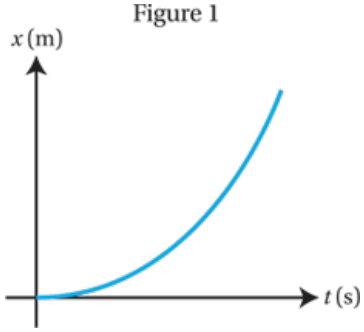


Figure 2

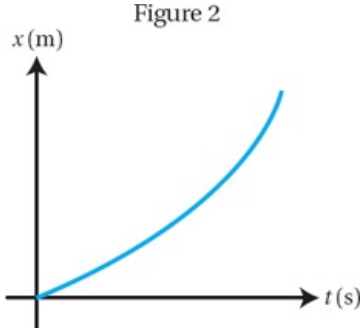
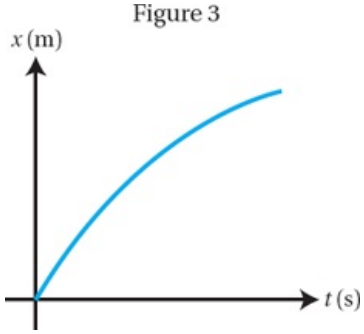


Figure 3



i State which of the three graphs is most appropriate to represent the motion of the car.

ii For each of the two other graphs give a reason why it is not appropriate to represent the motion of the car.

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[Question part reference style adapted]

12 A particle is projected vertically upwards with velocity $5\ \mathrm{m\ s^{-1}}$ from a point $2.5\ \mathrm{m}$ above the ground.

a Calculate the speed of the particle when it strikes the ground.

b Calculate the time after projection when the particle reaches the ground.

c Sketch on separate diagrams:

i the (t, v) graph

ii the (t, x) graph

representing the motion of the particle.

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[Question part reference style adapted]

13 A swimmer C swims with velocity $v\ \mathrm{m\ s^{-1}}$ in a swimming pool. At time t s after starting, $v = 0.006t^2 - 0.18t + k$, where k is a constant. C swims from one end of the pool to the other in 28.4 s.

a Find the acceleration of C in terms of t.

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b Given that the minimum speed of C is $0.65\ \mathrm{m\,s^{-1}}$, show that $k = 2$.

c Express the distance travelled by C in terms of t, and calculate the length of the pool.

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[Question part reference style adapted]

14 A particle P is projected from a fixed point O on a straight line. The displacement x m of P from O at time t s after projection is given by $x = 0.1t^3 - 0.3t^2 + 0.2t$.

a Express the velocity and acceleration of P in terms of t.

b Show that when the acceleration of P is zero, P is at O.

c Find the values of t when P is stationary.

At the instant P first leaves O, a particle Q is projected from O. Q moves on the same straight line as P and at time t s after projection the velocity of Q is given by $\left(0.2t^2 - 0.4\right)\mathrm{m\,s^{-1}}$. P and Q collide first when $t = T$.

d Show that T satisfies the equation $t^2 - 9t + 18 = 0$, and hence find T.

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[Question part reference style adapted]

Pure mathematics and statistics

90 minutes, 75 marks

Section A: Pure mathematics

1 Find the exact value of $\int_2^3 2 - \frac{4}{x^2} dx$

[4 marks]

2 Find the exact solution of the equation $\log_3(2x+1) = \log_3(x-2) + 2$.

[5 marks]

3 Points M, N and P have position vectors $\mathbf{m} = 6\mathbf{i} - 3\mathbf{j}$, $\mathbf{n} = 2\mathbf{j}$ and $\mathbf{p} = \mathbf{j} - 5\mathbf{i}$.

a Point Q is such that MNQP is a parallelogram. Find the position vector of Q

b Find the exact magnitude of the vector $v = \overrightarrow{MN} + \overrightarrow{MP}$.

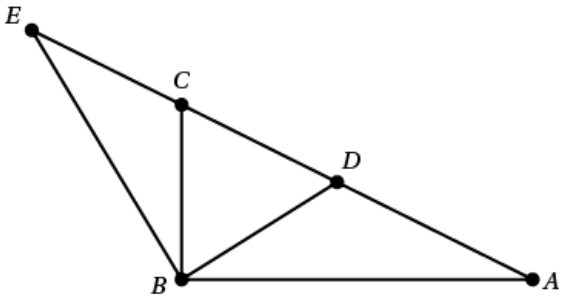
[8 marks]

4 Differentiate $3x^2$ with respect to x from first principles.

[4 marks]

5 In the diagram, ABC is a right angle, $AC = 1$ and angle $BAG = \alpha$.

Points D and E are on the line AC such that $CD = CE = CB$.



a Express the lengths AB and BC in terms of α.

b Hence prove that $\frac{1}{AD} + \frac{1}{AE} = \frac{2}{AB^2}$.

[8 marks]

6 **a** Find the coordinates of the intersection points of the graphs of $y = 12x - 2x^2$ and $2y = 6 - x$.

b Illustrate the region determined by the inequalitieson $y \leqslant 12x - 2x^2$ and $x + 2y \geqslant 6$ on a graph. Leave the required region unshaded.

c The inequality $x^2 + px + q \geqslant 0$ is satisfied for $x \in (-\infty, -3] \cup [5, \infty)$. Find the values of p and q.

[11 marks]

7 Points A and C have coordinates $(-8, 0)$ and $(0, 20)$ respectively. Point B lies on the a-axis and CB is perpendicular to AC. Find the equation of the circle that passes through points A, B and C.

[11 marks]

Section B: Statistics

8 Elena is sometimes late for school, but never more than three times in a week. For any week, the number of days she is late has the following probability distribution:

Days late	0	1	2	3
Probability	0.50	0.25	0.15	0.10

Find the probability that, in two randomly chosen weeks, Elena is late for school a total of four times.

[4 marks]

9 Pre-election polls suggest that 35% of voters in a certain town are undecided about how they are going to vote. Following a televised debate, a survey is carried out to find out whether the proportion of undecided voters has decreased.

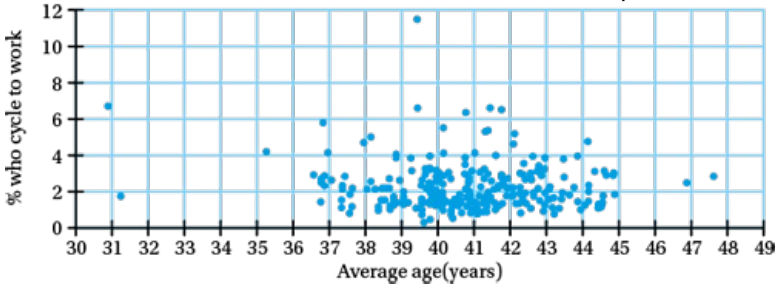
a Describe briefly how to select a simple random sample of 200 registered voters in this town.

In a random sample of 200 voters, 62 are still undecided about how they are going to vote.

b Test, at a 5% significance level, whether the proportion of undecided voters in this town has decreased.

[9 marks]

10 The scatter graph shows data about average age and the percentage of people who cycle to work. The information is from a survey done in 2015. Each data point represents one local authority. The data shown is for all the 250 local authorities that provided information.



a Describe the correlation between average age and the percentage of people who cycle to work.

The data for the percentage of people who cycle to work, $p\%$ is summarised as follows:

$$\sum p = 561 \quad \sum p^2 = 1699$$

b Calculate the mean percentage of people who cycle to work, and show that the standard deviation is 1.33.

c Hence identify the outliers in the 'percentage who cycle to work' data. Show your method and circle the outliers on the graph. [In this question, outliers are defined as data values that are more than 3 standard deviations from the mean.]

d One local authority has a particularly high percentage of people who cycle to work. Is this local authority more likely to be in an urban or a rural area? Explain your answer.

[11 marks]

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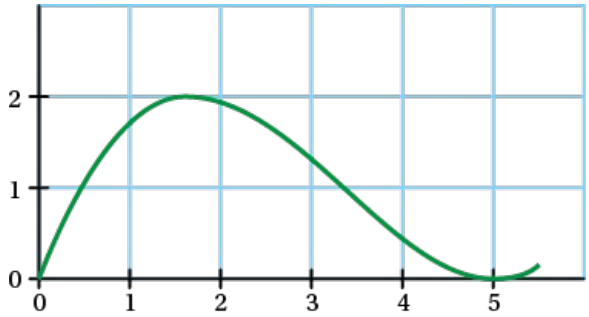
Pure mathematics and mechanics

90 minutes, 75 marks

Section A: Pure mathematics

1 Find the exact solutions of the equation $2\cos(2x) = 1$ for $x \in [0^\circ, 180^\circ]$. **[3 marks]**

2 The diagram shows the graph of $y = \mathrm{f}(x)$.



On separate diagrams sketch graphs of:

a $y = \mathrm{f}(2x)$

b $y = \mathrm{f}'(x)$. **[5 marks]**

3 Solve the equation $3^{2x+1} - 28 \times 3^x + 9 = 0$. **[5 marks]**

4 **a** Find the first four terms, in ascending powers of x, in the expansion of $\left(1 - \frac{1}{2}x\right)^8$.

b Find the coefficient of x^3 in the expansion of $(2 - x)\left(1 - \frac{1}{2}x\right)^8$. **[7 marks]**

5 A polynomial is defined by $\mathrm{f}(x) = x^3 - x^2 + ax + 12$. It is given that $(x - 2)$ is a factor of $\mathrm{f}(x)$.

a Find the value of a and factorise $\mathrm{f}(x)$ completely.

b Hence sketch the graph of $y = \mathrm{f}(x)$ giving coordinates of any axis intercepts. **[8 marks]**

6 **a** A student says: 'When you square a prime number, the sum of the digits of the answer is either a prime number or a square number'. (For example, $5^2 = 25$ and $2 + 5 = 7$; $13^2 = 169$, $1 + 6 + 9 = 16$.) Give a counter example to disprove the student's statement.

b Prove that the difference of squares of two odd numbers is always a multiple of 4. **[6 marks]**

7 A scientist models a population of insects using the equation $N = Ae^{kt}$ where N thousand is the number of insects t days after the observations started. Initially, there are 30 000 insects and the population is increasing at a rate of 12 000 insects per day.

a Find the values of A and k.

b According to this model, how long will it take for the population to reach 1 million? **[7 marks]**

8 **a** Sketch the graph of $y = \frac{5}{x^2}$.

b Show that the curve with equation $y = x^2 - x + \frac{5}{x}$ has only one stationary point. **[8 marks]**

Section B: Mechanics

9 A particle of mass $5\,\mathrm{kg}$ moves in a horizontal plane under the action of forces $\mathbf{F_1} = (23\mathbf{i} - 16\mathbf{j})\,\mathrm{N}$, $\mathbf{F_2} = (7\mathbf{j})\,\mathrm{N}$ and $\mathbf{F_3} = (11\mathbf{i} - 8\mathbf{j})\,\mathrm{N}$. Find the acceleration of the particle in the form $(p\mathbf{i} + q\mathbf{j})\ \mathrm{m\,s^{-2}}$.

[4 marks]

10 A ball is projected vertically upwards with a speed of $12.6\,\mathrm{m\,s^{-1}}$. Assume that the air resistance can be ignored.

a Find the speed and the direction of motion of the ball 2.1 seconds after projection.

b How long does the ball take to travel a distance of 15 metres?

c How would your answer to part **b** change if air resistance was included? Give a reason for your answer?

11 A box of mass $12\,\mathrm{kg}$ is attached to the roof of a lift by a light inextensible string. The mass of the lift is $450\,\mathrm{kg}$ and it is supported by a cable that is modelled as light and inextensible. The tension in the cable is $5300\,\mathrm{N}$. Find the tension in the string. **[8 marks]**

12 A particle moves in a straight line. Its velocity, $v\,\mathrm{ms^{-1}}$, at time t seconds is given by the equation: **[5 marks]**

$$v\left(t\right)=\begin{cases}9.6+4.2t-0.3t^{2}\text{ for }0\leqslant t\leqslant 7\\52.65-4.05t\text{ for }t>7\end{cases}$$

a Calculate the acceleration of the particle when $t=9$.

b At what time does the particle change direction?

c How long does it take for the particle to travel $190\,\mathrm{m}$?

[9 marks]

FORMULAE

The following formulae will be given on the AS and A Level assessment papers.

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + ... + {}^nC_r a^{n-r}b^r + ... + b^n \ \ (\mathrm{n} \in \mathrm{N}),$$

where ${}^nC_r = \binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$

Differentiation from first principles

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

Standard deviation

$$\sqrt{\frac{\sum (x-\bar{x})^2}{n}} = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} \ or \ \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$$

The binomial distribution

If $X \sim B(n,p)$ then $P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$, Mean of X is np. Variance of X is $np(1-p)$

Kinematics

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$s = \frac{1}{2}(u+v)t$$

$$v^2 = u^2 + 2as$$

$$s = vt - \frac{1}{2}at^2$$

Answers

Chapter 1

Before you start...

1 3

2 $(2x+1)(2x-1)$

3 **a** 180°

b 36°

4 π and $\sqrt{2}$

5 15

Exercise 1A

1 **a** **i** 5

ii 10

b **i** 21

ii 16

c **i** x^2+1

ii y^2+1

d **i** $4x^2+1$

ii $9y^2+1$

e **i** x^2+2x+2

ii x^2-4x+5

2 **a** **i** $\Leftarrow$

ii $\Leftarrow$

b **i** $\Rightarrow$

ii $\Rightarrow$

c **i** $\Leftrightarrow$

ii $\Leftrightarrow$

d **i** $\Leftarrow$

ii $\Leftrightarrow$

e **i** $\Leftarrow$

ii $\Rightarrow$

f **i** $\Leftrightarrow$

ii $\Leftarrow$

g **i** None

ii None

3 $a=2, b=3$

4 $a=2, b=8$

5 Equals sign is used to match up non-equal values

6 Not all lines can be connected with $\Leftrightarrow$

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7 **a** $\sqrt{x^2+9} = 3x - 7$

$\Rightarrow x^2 + 9 = 9x^2 - 42x + 49$

$\Leftrightarrow 0 = 8x^2 - 42x + 40$

$\Leftrightarrow 0 = 4x^2 - 21x + 20$

$\Leftrightarrow 0 = (4x - 5)(x - 4)$

b $x = \frac{5}{4}$ is the solution to $\sqrt{x^2+9} = -(3x - 7)$, which gives the same second line of working

8 **a** $\Rightarrow, \Leftrightarrow, \Leftrightarrow, \Leftrightarrow$

b Multiplying through by x^2 introduced the false solution $x = 0$

9 **a** $\Leftrightarrow, \Leftarrow$

b Alternative solution to the second line is $x = -3$

10 In maths, the statement is false; A OR B means A is true or B is true or both are true. In spoken (informal) English, it depends on context.

11 All is fine until cancelling $(a - b)$. Since $a = b$, this is equivalent to dividing both sides by zero.

Exercise 1B

1 **a** **i** $\{x : x > 7\}$; $(7, \infty)$

ii $\{x : x < 6\}$; $(-\infty, 6)$

b **i** $\{x : x \leqslant 10\}$; $(-\infty, 10]$

ii $\{x : x \geqslant 5\}$; $[5, \infty)$

c **i** $\{x : x > 0\} \cap \{x : x \leqslant 1\}$; $(0, 1]$

ii $\{x : x > 5\} \cap \{x : x < 7\}$; $(5, 7)$

d **i** $\{x : x > 5\} \cup \{x : x \leqslant 0\}$; $(-\infty, 0] \cup (5, \infty)$

ii $\{x : x \geqslant 10\} \cup \{x : x < 2\}$; $(-\infty, 2) \cup [10, \infty)$

2 **a** **i** $1 \leqslant x < 4$

ii $2 < x \leqslant 8$

b **i** $1 \leqslant x \leqslant 3$

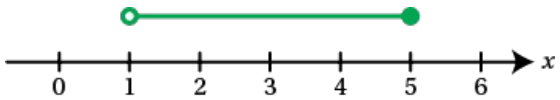
c **i** $x < 5$

ii $x \geqslant 12$

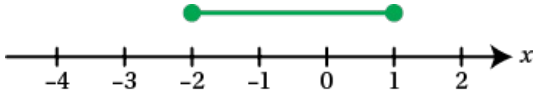
d **i** $8 \leqslant x < 10$

ii $3 \leqslant x < 4$

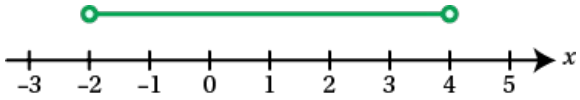
3 **a** **i**



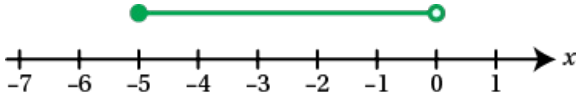
ii



b **i**

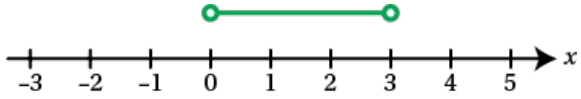


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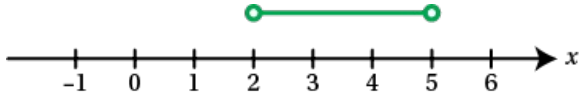


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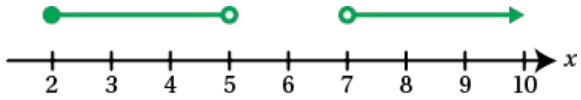
c i



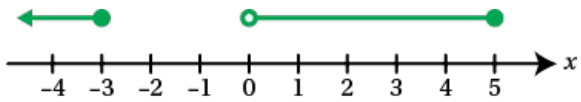
ii



d i



ii



4 a $x \in [1, 3] \cap (2, \infty) = (2, 3]$

b $x \in (-\infty, -1] \cap (1, 3) = \emptyset$

c $x \in (-\infty, -2] \cap [2, \infty)$

d $x \in [1, 5) \cap [7, \infty)$

5 a i $[10, \infty)$

ii $(6, \infty)$

b ii $(4, 6]$

c i $(9, \infty)$

ii $\left(-\infty, -\frac{16}{11}\right]$

Exercise 1C

1 e.g. $x = 4$

2 e.g. $x = 45^\circ$

3 e.g.$x = -1$

4 e.g.2×3

5 e.g. 4

6 e.g. $-\pi + \pi$

7 e.g. $x = -4$

8 e.g. $n = -41$

9 No – consider skew lines

Exercise 1D

1 Use $n = 2m + 1$

2 Use $a = 2n$ and $b = 2m + 1$

3 Use n, $n + 1$, $n + 2$

4 a Use $5n$, $5(n + 1)$

b Use $5n$, $5(n + 1)$

5 Consider areas

6 Either use an argument with exterior angles or divide the hexagon into triangles

7 Use $n + 3m + 2$

8 a $x^2 + 4x + 4$

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b Write as $y = (x + 2)^2 + 6$

9 Label the exterior angle x and use knowledge of angles on a straight line and interior angles of a triangle

10 Factorise to $(n + 1)(n + 2)$

11 **a** Position value

b Consider $n - (a + b + c + d)$

c Consider $n - (a - b + c - d)$

12 Consider rationality of $\sqrt{2}^{\sqrt{2}}$

Exercise 1E

1 Use proof by exhaustion for all possible factors up to square root of 11

2 Use proof by exhaustion for all possible factors up to square root of 83

3 Use proof by exhaustion for all polygons from triangles to hexagons

4 Use proof by exhaustion for squares of single-digit integers

5 Use proof by exhaustion for all single-digit positive integers

6 Use proof by exhaustion for all integers between 1 and 5 inclusive

7 Use proof by exhaustion for even or odd n

8 Use proof by exhaustion for $n = 5m + i$ for $i = 0, 1, 2, 3, 4$

9 Factorise and use proof by exhaustion for $n = 3m + i$ for $i = 0, 1, 2$

10 Consider cases a, b both positive; a, b both negative; a, b different signs

Mixed practice 1

1 Use $a = 2m + 1$, $b = 2n + 1$

2 Use $n = 2m$

3 Use a non-reduced fraction

4 Consider, for example, $3 + (-2)$ and $3 - (-2)$

5 Use definition of rational numbers

6 Subdivide the n-gon as for Exercise 1E question 6

7 $a = 1$, $b = -1$, $c = 1$

8 Factorise and use an exhaustive proof

9 Construct further lines to find isosceles triangles; construct a proof using knowledge of angles in triangles

10 Consider n^3 where $n = 3k$ or $n = 3k \pm 1$

11 **a** This does not work in the forward direction, e.g. $a = 4$, $b = 0.5$

b This does not work in either direction, e.g. $a = \sqrt{2}$, $b = \sqrt{2}$ so $ab = 2$ **or** $ab = \pi$, $a = \pi$, $b = 1$

12 Use $a = n - 1$, $b = n$ and $c = n + 1$

13 Use $a = 2m + 1$ and $b = 2n + 1$

14 **a** e.g. $n = 1601$

b factorise n^{2-1}

15 Factorise and use an exhaustive proof considering whether $a + b$ is odd or even

Answers

Chapter 2

Before you start...

1 24

2 3

3 **a** $2x^{-3}$

b $x^{-\frac{2}{3}}$

4 $2 - y + 2x - xy$

5 $4a^2 - b^2$

Work it out 2.1

Solution 1 is correct.

Exercise 2A

1 **a** **i** 4

ii 3

b **i** $\frac{1}{8}$

ii $\frac{1}{25}$

c **i** $\frac{9}{4}$

ii 64

d **i** 9

ii 27

e **i** $\frac{1}{25}$

ii $\frac{1}{8}$

f **i** $\frac{27}{8}$

ii $\frac{4}{9}$

g **i** 0.5

ii 4

h **i** 2

ii $\frac{1}{3}$

2 **a** **i** $4^{\frac{1}{2}}$

ii $4^{\frac{1}{3}}$

b **i** 4^2

ii 4^0

c **i** $4^{\frac{1}{2}}$

ii $4^{-\frac{1}{2}}$

d **i** 4^{-1}

ii $4^{-\frac{1}{2}}$

e **i** $4^{\frac{3}{2}}$

ii $4^{\frac{5}{2}}$

f **i** $4^{\frac{4}{3}}$

ii $4^{\frac{7}{10}}$

g **i** $4^{\frac{1}{4}}$

ii $4^{\frac{3}{4}}$

3 **a** **i** x^3

ii x^{12}

b **i** $2x^5$

ii $\dfrac{1}{2x^4}$

c **i** $\dfrac{3x^3}{4}$

ii $\dfrac{y^{12}}{x^6}$

4 **a** **i** $x = \dfrac{1}{4}$

ii $x = \dfrac{4}{9}$

b **i** $x = 2$

ii $x = \dfrac{9}{2}$

5 **a** **i** $3x^{\frac{1}{2}}$

ii $5x^{\frac{1}{3}}$

b **i** $x^{\frac{4}{3}}$

ii $x^{\frac{5}{2}}$

c **i** $3x^{-\frac{1}{2}}$

ii $5x^{-\frac{1}{3}}$

d **i** $\dfrac{1}{2}x^{-2}$

ii $\dfrac{1}{3}x^{-2}$

e **i** $\frac{2}{3}x^{-\frac{1}{2}}$

ii $\frac{5}{2}x^{-\frac{1}{3}}$

6 **a** **i** $3x^{-\frac{1}{2}} + 2x^{\frac{1}{2}}$

ii $4x^{-1} - 3x^{-\frac{1}{2}}$

b **i** $\dfrac{1}{2}x - \dfrac{3}{2}x^{-1}$

ii $2x^{-1} - \frac{3}{2}x^{-2}$

c **i** $\frac{1}{2}x^{\frac{3}{2}} + \frac{1}{4}x^{-\frac{1}{2}}$

ii $\frac{2}{3}x^{-1} - 3x^{-\frac{1}{2}}$

7 $\frac{1}{10x^2}$

8 x

9 $2p^2q^{\frac{1}{3}}$

10 $x^{-\frac{1}{3}} - x^{\frac{8}{3}}$

11 5×10^{-4}

12 **a** $\frac{1}{3}$

b $16\,\text{cm}^2$

13 8 cm

14 $\frac{2}{3x^3}$

15 $a = 0.75b$

16 $a = 9, b = 1, c = -6,\ n = 1.5$

17 $\frac{\ln b}{(2a-3)\ln 2}$

18 1

19 16

Work it out 2.2

Solution 2 is correct.

Exercise 2B

1 **a** **i** $5\sqrt{5}$

ii $2\sqrt{5}$

b **i** $5\sqrt{5}$

ii $7\sqrt{5}$

c **i** $2\sqrt{5}$

ii $26\sqrt{5}$

2 **a** **i** $\sqrt{32}$

ii $\sqrt{300}$

b **i** $\sqrt{63}$

ii $\sqrt{80}$

c **i** $\sqrt{108}$

ii $\sqrt{72}$

3 **a** **i** $3 + 5\sqrt{3}$

ii $2\sqrt{3}$

b **i** $-4 + 3\sqrt{3}$

ii $5 + 3\sqrt{3}$

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c **i** $4+2\sqrt{3}$

ii $7-4\sqrt{3}$

4 **a** **i** $\sqrt{7}$

ii $\frac{2\sqrt{5}}{5}$

b **i** $\frac{\sqrt{6}-2}{2}$

ii $\frac{\sqrt{6}+3\sqrt{2}}{3}$

c **i** $1+\sqrt{2}$

ii $\frac{\sqrt{7}-\sqrt{5}+\sqrt{35}-1}{6}$

5 $\frac{2}{1-n}$

6 $a=5, b=3$

7 $a=19, b=13$

8 $\left(3\sqrt{2}\right)^2=18$ and $\left(2\sqrt{3}\right)^2=12$

9 $x=-4$

10 $\frac{2\sqrt{n}+3}{4n-9}$

11 $15n^2+5-10n\sqrt{3}$

12 **a** $-ab+\left(b^2-a^2\right)\sqrt{2}$

b $\sqrt{3\left(a^2+b^2\right)}$

13 **a** $\sqrt{48}$

b $\sqrt{27}-\sqrt{20}>\sqrt{5}-\sqrt{3}$

14 **a** $a^2+2b^2+2\sqrt{2}ab$

b Proof

15 **a** Proof

b $\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}$

16 No; not if $x<0$

Mixed practice 2

1 $n^2+5+2n\sqrt{5}$

2 $z=9x^3$

3 $\sqrt{2}+\sqrt{7}$

4 $\frac{1}{16}x^2$

5 $x=0.8$

6 **a** x^2

b $1500y^2$

7 **a** 3^{-2}

b $3^{\frac{1}{3}}$

c 3^{280}

8 $\frac{3}{4x^2}$

9 $\frac{1+n+2\sqrt{n}}{n-1}$

10 **a** $a^2+b^2+2ab\sqrt{5}$

b Proof

c Proof

d **i** Upper bound because $3-\sqrt{5}>0$ so an odd power is also bigger than zero; worse because $\sqrt{5}$ is closer to 2 than to 3

ii Because $4-\sqrt{5}>1$, large powers get further away from zero or one

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Answers

Chapter 3

Before you start...

1 $6x^2 - 7x - 3$

2 **a** $x = -5, 4$

b $x = -8, \frac{1}{2}$

c $x = 0, \frac{3}{5}$

d $x = \pm\frac{3}{2}$

3 **a** $2 \pm \sqrt{2}$

b $x = \frac{5 \pm \sqrt{35}}{2}$

4 $x > 2$

Exercise 3A

1 **a** **i** $2, -\frac{3}{2}$

ii $5, \frac{2}{3}$

b **i** $\frac{4}{3}$

ii $-\frac{5}{4}$

c **i** $5, -4$

ii $\frac{5}{2}, -3$

d **i** $6, -\frac{1}{2}$

ii $\frac{4}{3}, 1$

2 **a** **i** $x = \frac{3 \pm \sqrt{5}}{2}$

ii $x = \frac{1 \pm \sqrt{5}}{2}$

b **i** $x = \frac{-3 \pm \sqrt{11}}{2}$

ii $x = \frac{9 \pm \sqrt{21}}{10}$

c **i** $x = -2 \pm \sqrt{6}$

ii $x = -1, \frac{4}{3}$

d **i** $x = \frac{3 \pm \sqrt{6}}{2}$

ii $x = 2 \pm \sqrt{7}$

3 No real solution

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4 $x = \frac{4}{3}, -\frac{1}{2}$

5 $x = \frac{3 \pm \sqrt{5}}{2}$

6 $x = 2k, 4k$

7 $x = -\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{5}$

8 $x = \frac{-\sqrt{10} \pm \sqrt{22}}{2}$

9 $x = \frac{-b \pm \sqrt{b^2 - 4a(c-y)}}{2a}$

10 $k = \pm 9$

Exercise 3B

1 **a** **i** A

ii C

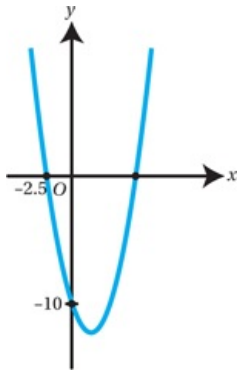
iii B

b **i** C

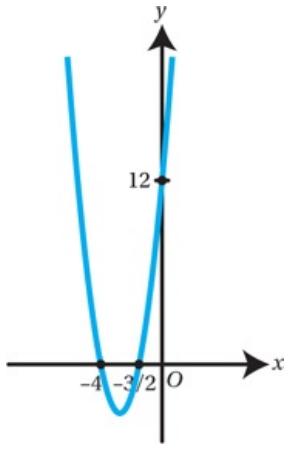
ii A

iii B

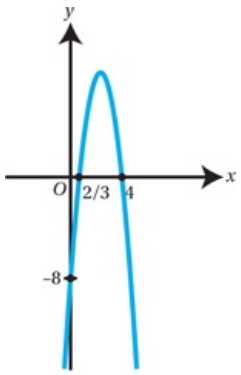
2 **a** **i**



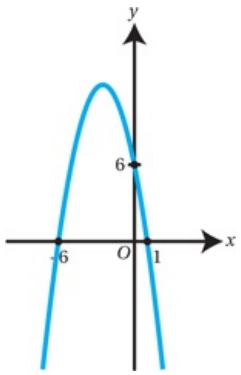
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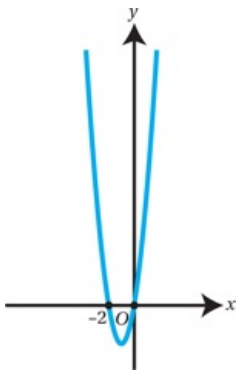
b i



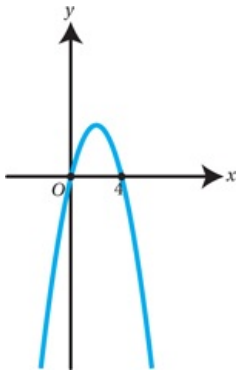
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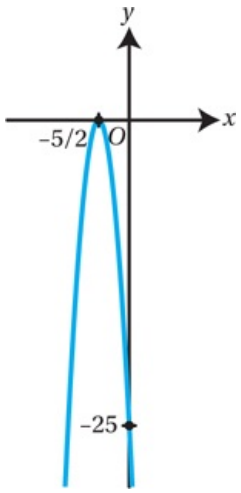
c i



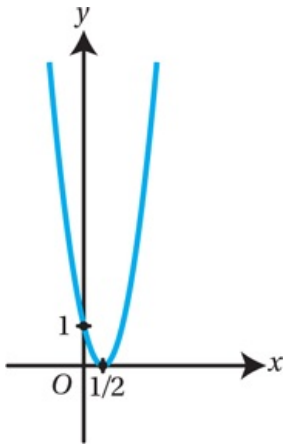
ii



d i



ii



3 a i $y = 3x^2 - 15x + 12$

ii $y = 4x^2 + 4x - 8$

b i $y = -2x^2 - 2x + 4$

ii $y = -x^2 - 6x - 5$

Work it out 3.1

Solution 2 is correct.

Exercise 3C

1 a i $(3, 4)$

ii $(5, 1)$

b i $(7, -1)$

ii $(1, -5)$

c i $(-1, 3)$

ii $(-7, -3)$

d i $(-2, -4)$

ii $(-1, 5)$

2 a i $(x - 3)^2 - 5$

ii $(x - 5)^2 - 4$

b i $(x + 2)^2 - 3$

ii $(x + 3)^2 - 12$

c i $2(x - 3)^2 - 13$

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ii $3(x+1)^2+7$

d **i** $-(x-1)^2-4$

ii $-(x+2)^2+5$

e **i** $(x+1.5)^2-1.25$

ii $(x-2.5)^2+3.75$

f **i** $2(x+1.5)^2+10.5$

ii $2(x-1.25)^2-4.125$

3 **a** **i** $2(x-1.25)^2-4.125$

ii $y=3(x+1)^2-5$

b **i** $y=-2(x+1)^2+3$

ii $y=-5(x-2)^2+1$

4 **a** $y=(x-3)^2+2$

b 2

5 **a** $b=-3, c=6$

b $a=2$

6 **a** $y=5-(x-4)^2$

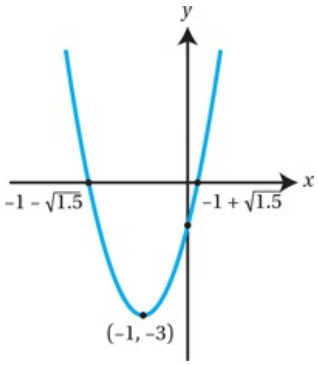
b $(4,-5)$

c $y\leqslant -5$

7 **a** $2(x+1)^2-3$

b $x=-1\pm\sqrt{1.5}$

c



d $x=-1$

8 **a** $a(x+3)^2-9$

b $p\geqslant -9$

9 **a** $3\left(x+\frac{1}{3}\right)^2+\frac{2}{3}$

b Proof

c $x=-\frac{1}{3}$

10 **a** Proof

Exercise 3D

1 **a** **i** $-2\sqrt{2}<x<2\sqrt{2}$

ii $-\sqrt{5}<x<\sqrt{5}$

b **i** $x < -\sqrt{6}$ or $x > \sqrt{5}$

ii $x \leqslant -2\sqrt{3}$ or $x \geqslant 2\sqrt{3}$

c **i** $x < -1$ or $x > 4$

ii $-\frac{2}{3} < x < \frac{5}{2}$

d **i** $x < -1$ or $x > 3$

ii $2 < x < 4$

e **i** $x < 3$ or $x > 12$

ii $-2 < x < 2$

2 **a** **i** $x \in (2,\ 3)$

ii $x \in (-3,\ 2)$

b **i** $x \in \cup\,(-\infty, -2]\,[6, \infty)$

ii $x \in (-\infty, -6] \cup [-1, \infty)$

c **i** $x \in (-\infty, -2) \cup (1.5, \infty)$

ii $x \in \left(-\infty, -\frac{5}{3}\right) \cup (2, \infty)$

d **i** $x \in [-2.5, 1]$

ii $x \in [-1, -0.2]$

3 $x < -2$ or $x > 1.5$

4 $-\frac{5}{3} \leqslant x \leqslant 2$

5 0.904 s

6 **a** **i** $x < 2.5$

ii $0.5 < x < 5$

b $0.5 < x < 2.5$

7 $x \leqslant -1$ or $1 \leqslant x < 2$ or $x \leqslant 3$

8 $-0.4 \leqslant x \leqslant 0\, or 2 \leqslant x \leqslant 3$

9 $5 \leqslant n \leqslant 38$

Exercise 3E

1 **a** **i** 36

ii 68

b **i** -47

ii -119

c **i** 0

ii 0

d **i** 49

ii 49

2 **a** Two

b None

c One

d Two

3 **a** **i** $k < \frac{1}{24}$

ii $k > -\frac{25}{12}$

b **i** $k = \frac{3}{5}$

ii $k = -\frac{17}{24}$

c **i** $k \geqslant -\frac{13}{4}$

ii $k \leqslant \frac{1}{16}$

d **i** $k > \frac{9}{24}$

ii $k < -\frac{25}{12}$

e **i** $k = \frac{17}{4}$

ii $k = \frac{55}{32}$

f **i** $k = 1$

ii $k = \frac{1}{32}$

g **i** $k < 0$

ii $k < 0$

4 $m = \pm\sqrt{2}$

5 $k = \frac{11 \pm 2\sqrt{30}}{2}$

6 $c \geqslant \frac{1}{16}$

7 $0 < k < 6$

8 $-9 < k < -1$

9 $m \leqslant -8$ or $m \geqslant 0$

10 $m < -\frac{9}{16}$

Exercise 3F

1 **a** **i** $a = \pm\sqrt{3},\ \pm\sqrt{7}$

ii $x = \pm 2,\ \pm\sqrt{3}$

b **i** $x = -\sqrt[3]{5},\ \sqrt[3]{1.5}$

ii $a = -2, 1$

c **i** $x = \pm\sqrt{2+\sqrt{6}}$

ii $x = \pm\sqrt{6}$

d **i** $x = 4,\ 16$

ii $x = 16,\ 36$

e **i** $x = 1,\ 2$

ii $x = 0,\ 4$

2 $x = 9$

3 $x = \pm 3, \pm 1$

4 $x = 0, -\frac{4}{15}$

5 $x = 4, 1$

6 $x = 0, 2$

7 $x = 0, 1$

8 $x = -2, 1.5$

9 **a** $a\left(x^2 - 2x\right)^2 - 11x^2$

b $x = -2, -1, 3, 4$

10 $x = -1, 3$

11 $x = 16$

Mixed practice 3

1 $x = k + 2$

2 $x = 1, 6$

3 $x = \pm 1, \pm 2$

4 **a** Minimum

b $a = 3,\ b = 7$

5 $a = -3, b = 2, c = 48$

6 **a** $p = 1, q = 4$

b $x = 1.5$

7 **a** $16 - 4k^2$

b $k = \pm 2 \text{ or } -2$

8 $x \geqslant 3 or x < 0$

9 a, c negative, b positive, $b^2 - 4ac = 0$

10 **a** $(x - 5)^2 + 10$

b $\frac{1}{1000}$

11 $3 - 2\sqrt{2} < k < 3 + 2\sqrt{2}$

12 $x = 1,\ 16$

13 $x = 1$

14 **a** $2\left(x\ -\ \frac{3}{2}\right)^2\ +\ \frac{9}{2}$

b $\left(\frac{3}{2},\ \frac{9}{2}\right)$

c Zero

15 **a** $P = 14x + 4$

b Proof

c $\frac{5}{2}\ \leqslant\ x < 3$

16 $x = 3,\ 2$

17 $k = \pm 5$

18 **a** $k - 1,\ 1$

b $k = -3,\ 5$

19 Proof

20 a Proof

b $6\ \text{km}$

Answers

Chapter 4

Before you start...

1 x^6

2 $2x^2 - 5x - 3$

3 $(x - 3)(x - 5)$

4 $x = -2 \pm \sqrt{2}$

Exercise 4A

1 **a** Degree $3, 3$

b Degree $5, -1$

c No

d No

e No

f No

g Degree $7, 2$

h Degree $0, 1$

2 **a** **i** $6x^3 + 8x^2 - 29x + 14$

ii $3x^3 + 16x^2 + 23x + 6$

b **i** $2x^4 - 15x^3 + 4x^2 + 4x - 1$

ii $2x^4 - 7x^3 - 30x^2 + 6x + 15$

c **i** $b^4 + b^3 - 3b^2 + 14b - 4$

ii $r^4 - 11r^3 + 33r^2 - 62r + 14$

d **i** $-x^6 + 2x^5 + 5x^4 - 10x^3 - x^2 + 5$

ii $-x^6 + 2x^4 + x^3 - x^2 - x$

3 Discussion

4 **a** Yes

b No

Exercise 4B

1 **a** **i** $x^2 - 8x - 3$

ii $x^2 - 2x + 3$

b **i** $x^2 - 2$

ii $x^2 + 3$

c **i** $x^2 - 2x + 2$

ii $x^2 - 5x + 2$

d **i** $x^3 - 5x^2 + 3x - 2$

ii $2x^3 + 3x + 5$

2 **a** **i** $x - 6$

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ii $x + 7$

b **i** $x^2 + 4x + 2$

ii $x^2 + 7x - 1$

Work it out 4.1

Solution 3 is correct.

Exercise 4C

1 **a** **i** No

ii Yes

b **i** Yes

ii No

c **i** Yes

ii No

d **i** Yes

ii No

e **i** No

ii No

2 **a** **i** $(x+1)(x-1)(x+2)$

ii $(x+1)(x-2)(x+2)$

b **i** $(x-2)^2(x-3)$

ii $(x+2)^3$

c **i** $(x-1)\left(x^2-2x+10\right)$

ii $(x-3)\left(x^2+x+5\right)$

d **i** $(x-1)(2x-1)(3x-1)$

ii $(x+2)(4x+3)(3x-5)$

3 **a** **i** $x = -3, 1, 4$

ii $x = -1, -3, 5$

b **i** $x = 2, \dfrac{3 \pm \sqrt{5}}{2}$

ii $x = 1, \dfrac{5 \pm \sqrt{17}}{2}$

4 **a** **i** $x = 1, 2, 3$

ii $x = -2, 1, 3$

b **i** $x = 2, \dfrac{3 \pm \sqrt{5}}{2}$

ii $x = 1, \dfrac{5 \pm \sqrt{17}}{2}$

c **i** $x = 1, -1$

ii $x = -3, 2, 4$

5 **a** $\mathrm{p}(2) = 0$

b $(x-2)(x-4)(x+3)\ x = -3, 2, 4$

6 **a** $\mathrm{p}(3) = 0$

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b Proof

7 $c = 14, d = 8$

8 **a** $a = 2, b = 59$

b $(x + 8)$

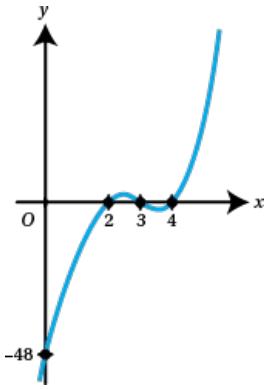
9 $\mathrm{x} = 0, 4$

10 $k = -\frac{1}{2}$

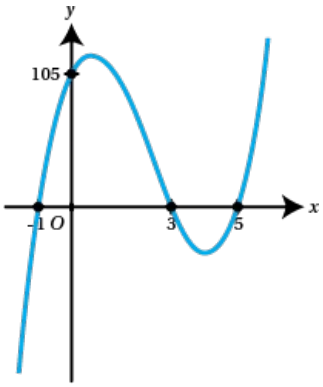
11 $a = 37, b = -30$

Exercise 4D

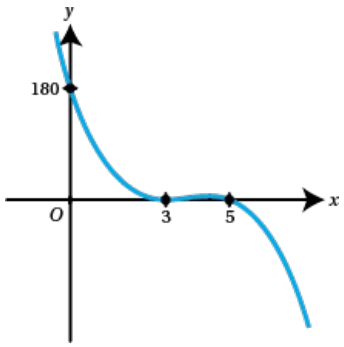
1 **a** **i**



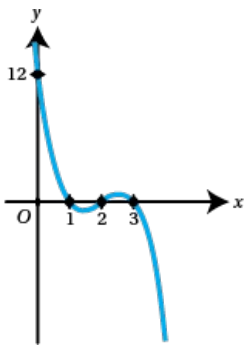
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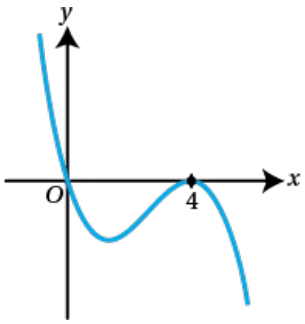
b **i**



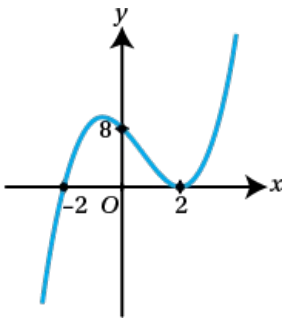
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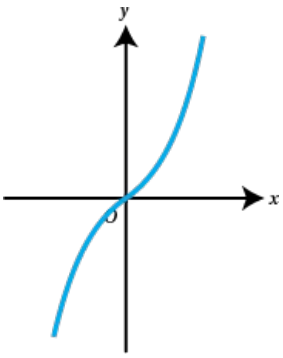
c i



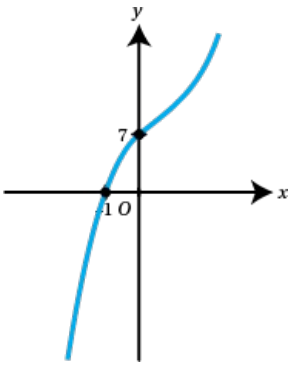
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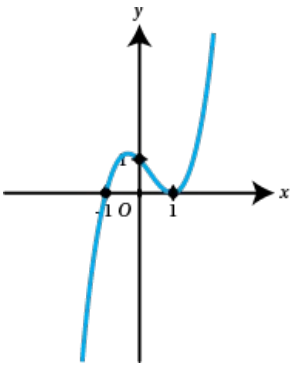
d i



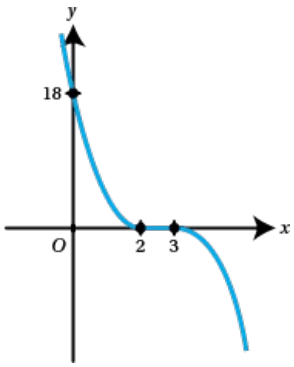
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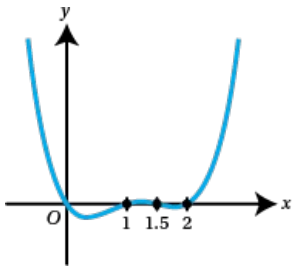
e i



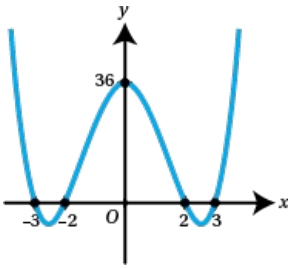
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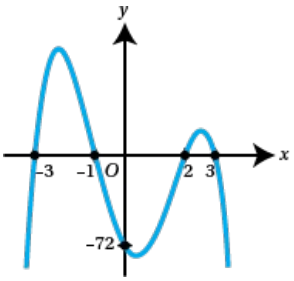
2 a i



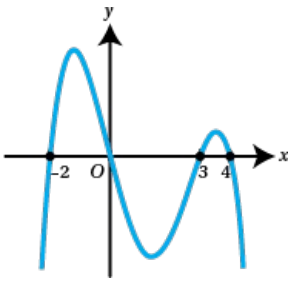
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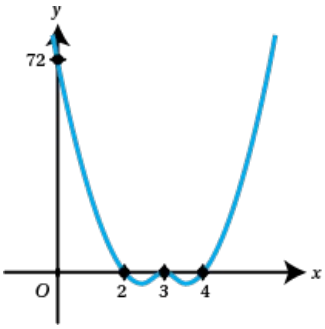
b i



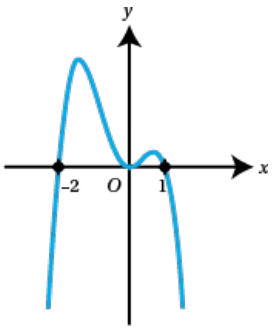
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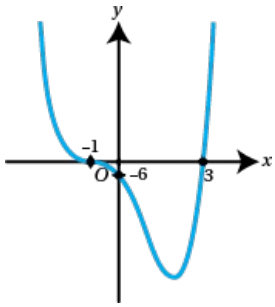
c i



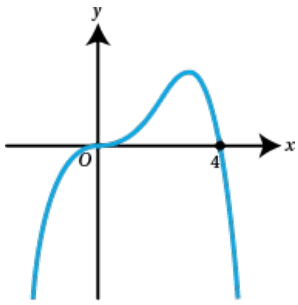
ii



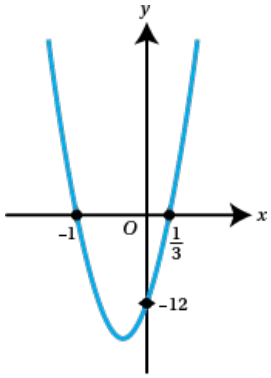
d i



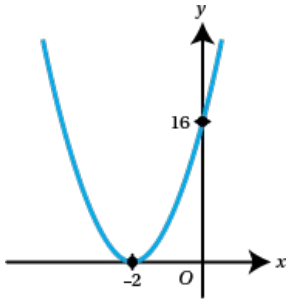
ii



e i



ii



3 a i $y = 2(x-1)(x-4)(x+2)$

ii $y = 6x(x+2)(x-3)$

b i $y = -5x(x-1)(x+1)$

ii $y = -(x-1)(x+2)(x+4)$

c i $y = -(x+1)(x-2)^2$

ii $y = (x+1)^2(x-2)$

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d **i** $y = x(x+2)(x+3)(2x-1)$

ii $y = -2(x-3)(x-4)(x+1)(x-2)$

e **i** $y = -3x^2(x-1)(x-3)$

ii $y = 5x(x-2)^2(2x+1)$

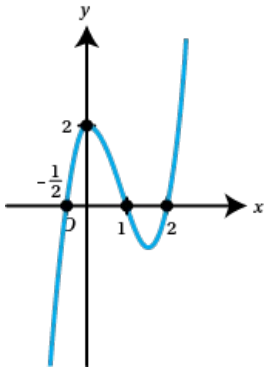
f **i** $y = 2(x-3)^2(x+1)^2$

ii $y = -(x-3)^2(x-1)^2$

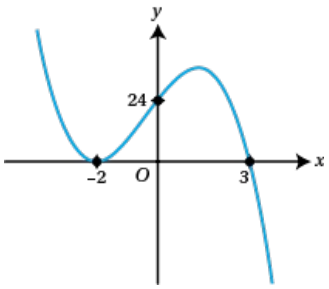
4 **a** $f(2) = 0$

b $(x-2)(x-1)(2x+1)$

c



5

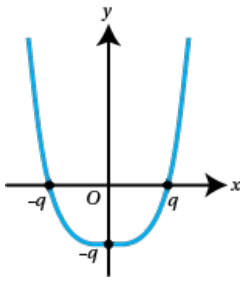


6 **a** $p = 2, q = -8, r = -6, s = 36$

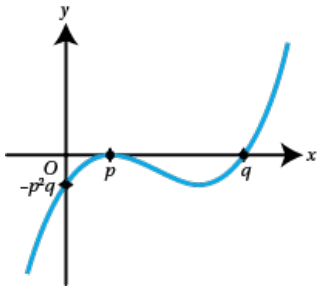
b $p = -1, q = 3, r = 0, s = 0$

7 **a** $(x-q)(x+q)(x^2+q^2)$

b



8 **a**



b One

Mixed practice 4

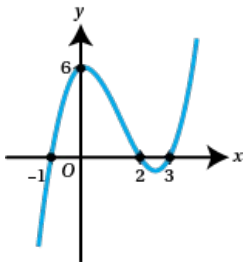
1 $a = 1, b = 2, c = -12, d = -18, e = 27$

2 $b = 0, c = -3$

3 **a** $\mathrm{f}(2) = 0$

b $\mathrm{f}(x) = (x-2)(x+1)(x-3)$

c

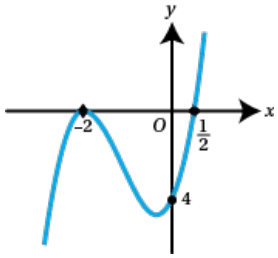


4 **a** $b = 3$

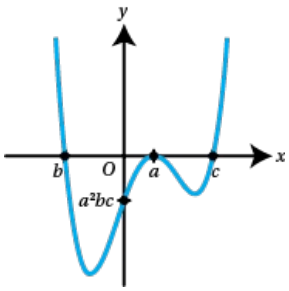
b $\mathrm{f}(x) = (x-2)(x+3)(x-1)(x+3)$ is also a factor of $\mathrm{g}(x)$.

5 **a** $a = 7, b = -4$

b



6



7 **a** $(x-2)$

b $2, \dfrac{-3 \pm \sqrt{29}}{2}$

8 $a = -10,\ b = -18$

Answers

Chapter 5

Before you start...

1 $x = \dfrac{-1 \pm \sqrt{5}}{2}$

2 One

3 $x = 1, y = 2$

4 $x = 3$

5 $x > 4$

6 $V = \dfrac{48}{r^2}$

Work it out 5.5

Solution 3 is correct.

Exercise 5A

1 **a** **i** $(-2, -3), (1, 0)$

ii $(3, 0)$

b **i** $(-3, -9), (4, 5)$

ii No intersection

2 **a** **i** $(-2.2, -1.6), (3, 1)$

ii $(-3, 3), (5, -1)$

b **i** $(1, 3), (3, 1)$

ii $(-3, -5), (-5, -3)$

c **i** $(-1, 6), (2, 3)$

ii $(1, -3), (-1, -5)$

3 $\left(\dfrac{\sqrt{2}}{2}, \sqrt{2}\right), \left(-\dfrac{\sqrt{2}}{2}, -\sqrt{2}\right)$

4 $x = 2$

5 $x = 3, y = 8$

6 **a** $4x^2 - 32x + 39 = 0$

b $1.5, 6.5$

7 $(0, 2), (0, -2)\ (-\sqrt{3}, -1), \ (\sqrt{3}, -1)$

8 $m = 0$

Exercise 5B

1 Proof

2 $-1 \pm 2\sqrt{6}$

3 $\pm 6\sqrt{2}$

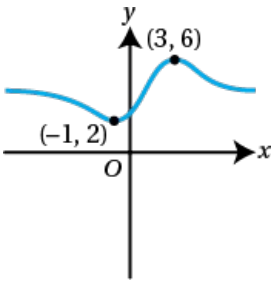
4 $a < 0$ or $a > 2$

5 Proof; that $k2 + 12 > 0$ for all k

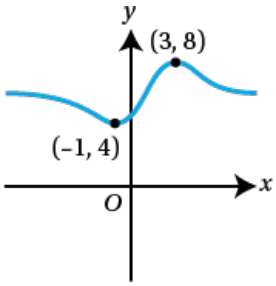
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Exercise 5C

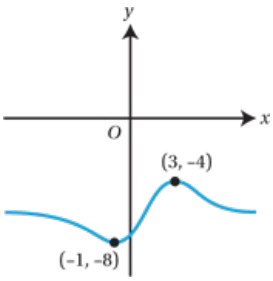
1 a i



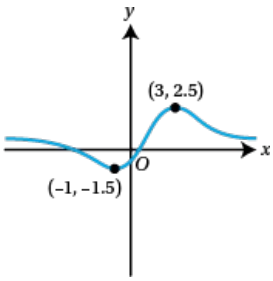
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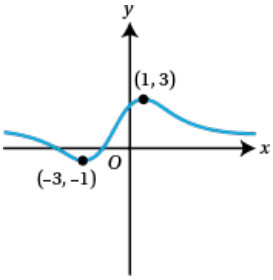
b i



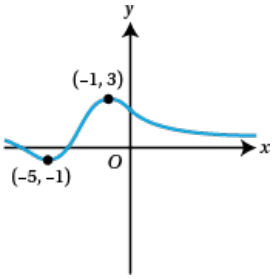
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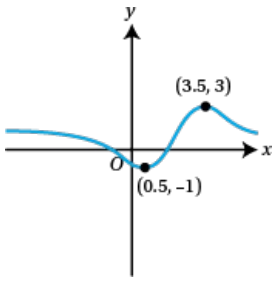
c i



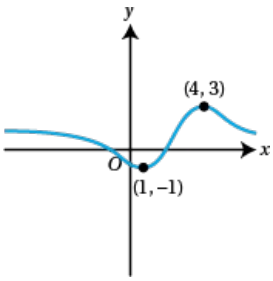
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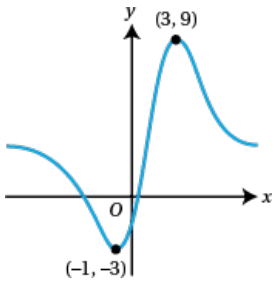
d i



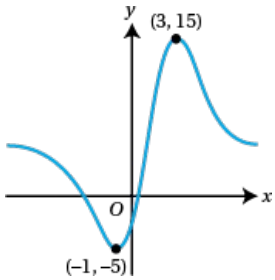
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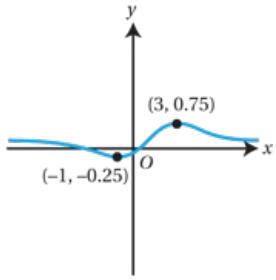
e i



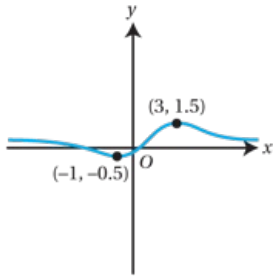
ii



f i

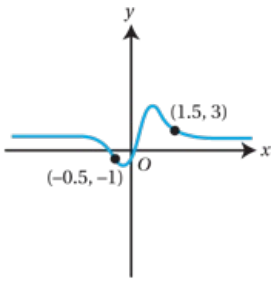


ii

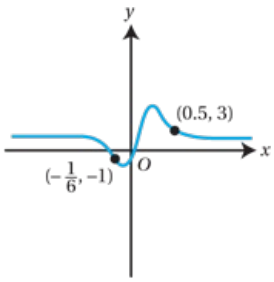


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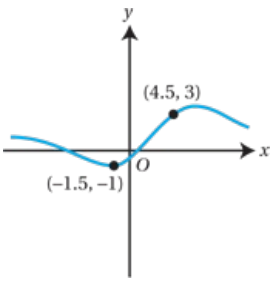
g i



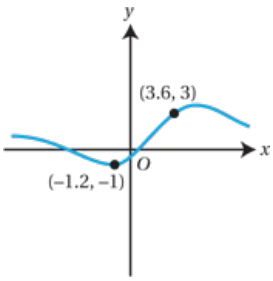
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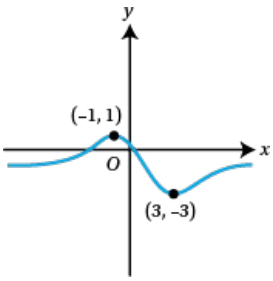
h i



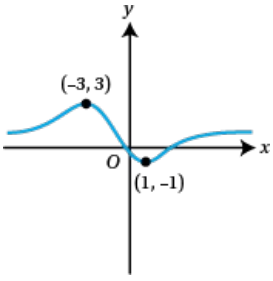
ii



i i



ii



2 a i $y = 3x^2 + 3$

ii $y = 9x^3 - 7$

b i $y = 7x^3 - 3x + 4$

ii $y = 8x^2 - 7x + 6$

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c **i** $y = 4(x-5)^2$

ii $y = 7(x+3)^2$

d **i** $y = 3\big(x+4\big)^3 - 5\big(x+4\big)^2 + 4$

ii $y = \big(x-3\big)^3 + 6\big(x-3\big) + 2$

3 **a** **i** Vertically down 5 units

ii Vertically down 4 units

b **i** Left 1 unit

ii Left 5 units

c **i** Left 3 units

ii Right 2 units

4 **a** **i** $y = 21x^2$

ii $y = 18x^3$

b **i** $y = \dfrac{1}{3}\big(7x^3 - 3x + 6\big)$

ii $y = \dfrac{4}{5}\big(8x^2 - 7x + 1\big)$

c **i** $y = x^2$

ii $y = 7\Big(\dfrac{x}{5}\Big)^2$

d **i** $y = 3\big(2x\big)^3 - 5\big(2x\big)^2 + 4$

ii $y = \left(\dfrac{3x}{2}\right)^3 + 6\left(\dfrac{3x}{2}\right) + 2$

5 **a** **i** Vertical stretch, scale factor 4

ii Vertical stretch, scale factor 6

b **i** Horizontal stretch, scale factor $\dfrac{1}{3}$

ii Horizontal stretch, scale factor $\dfrac{1}{4}$

c **i** Horizontal stretch, scale factor $\dfrac{1}{3}$ (or vertical stretch, scale factor $\sqrt{3}$)

ii Horizontal stretch, scale factor 2

6 **a** **i** $y - 3x^2$

ii $y - 9x^3$

b **i** $y - 7x^3 + 3x - 6$

ii $y - 8x^2 + 7x - 1$

c **i** $y = 4x^2$

ii $y - 7x^3$

d **i** $y - 3x^3 - 5x^2 + 4$

ii $y = -x^3 - 6x + 2$

7 **a** **i** Reflection in the x-axis

ii Reflection in the x-axis

b **i** Reflection in the y-axis

ii Reflection in the y-axis

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c **i** Reflection in the y-axis

ii Reflection in the y-axis

Exercise 5D

1 **a** Translation d to the right

b $x = d, y = 0$

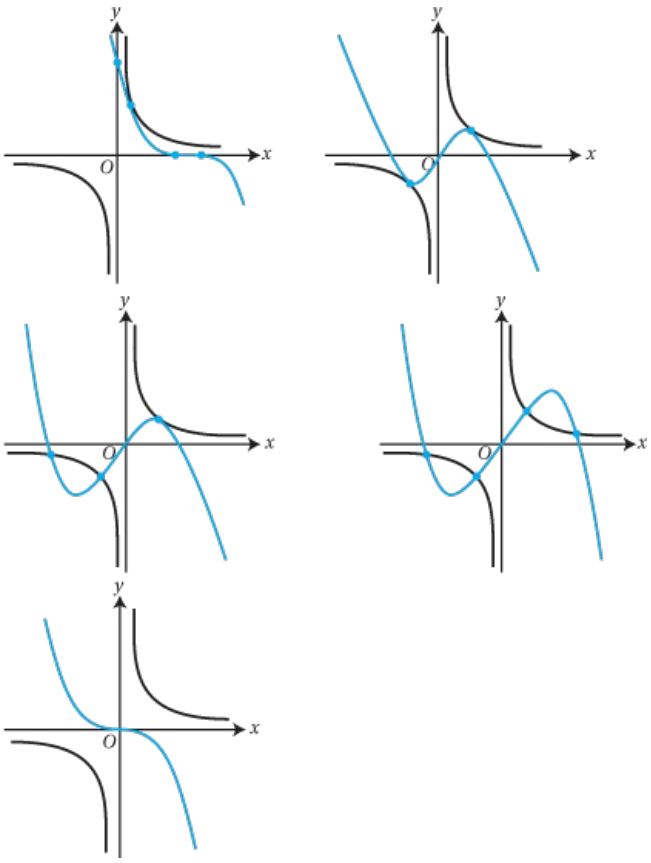
2 $\sqrt{a}$

3 **a** $\left(\frac{b}{a}, \frac{a^2}{b}\right)$

b Area $= a$

4 $m = -\frac{c^2}{4a}$

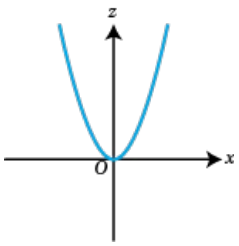
5



Exercise 5E

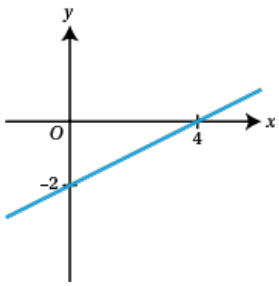
1 **a** 48

b



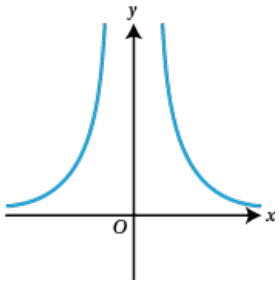
2 **a** 2

b



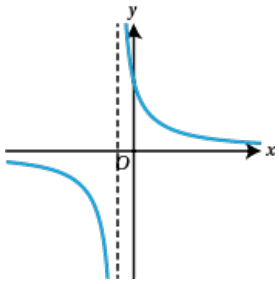
3 **a** 1.25

b



4 **a** 6

b



5 **a** $y_S = 8p - 80$

b It is the number of items that would be sold if they were free. It is likely that more would be 'sold' if they were given away for free

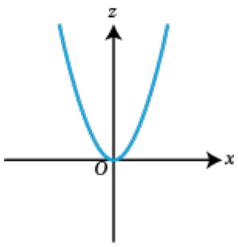
c $£39.87$

d $£25$

6 **a** $C_A = 65 + 0.03m, C_B = 0.05m$

b After 3250 minutes

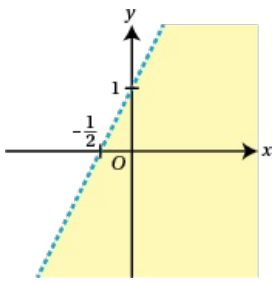
7



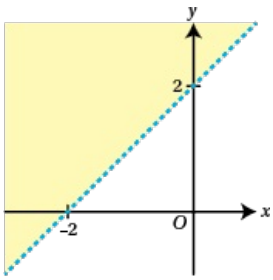
8 17.4%

Exercise 5F

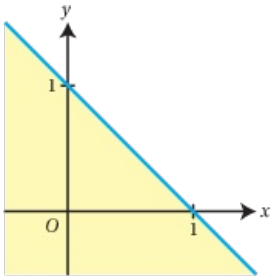
1 **a** **i**



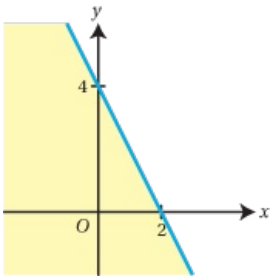
ii



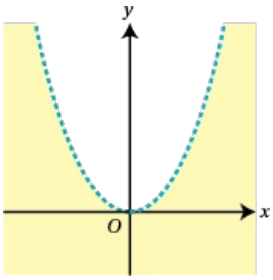
b i



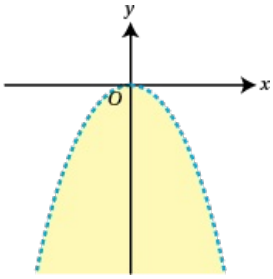
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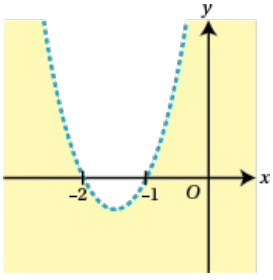
c i



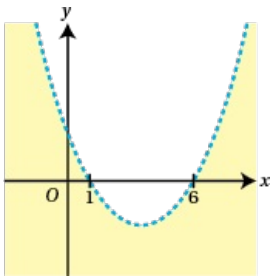
ii



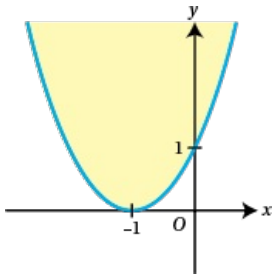
d i



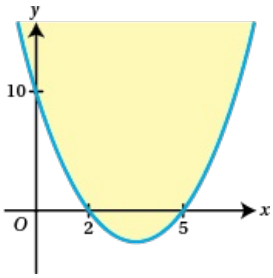
ii



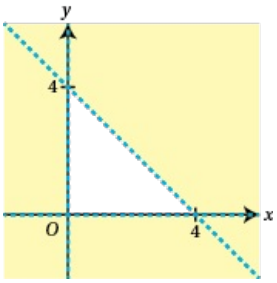
e i



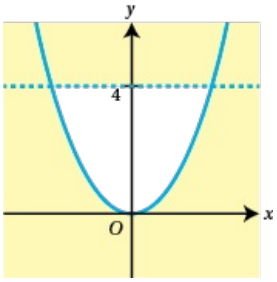
ii



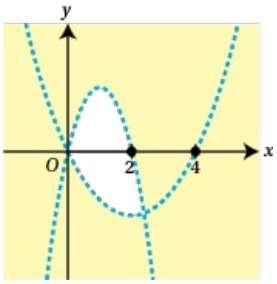
2



3



4



5 $y > x$

$x + y \leqslant 4$

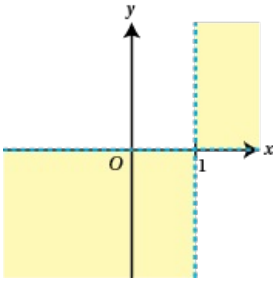
6 $y > (x-1)(x-2)$

$y < 2$

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7 54

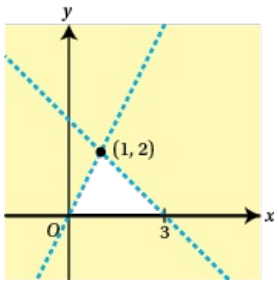
8



Mixed practice 5

1 $(3, 4)$ and $(4, 3)$

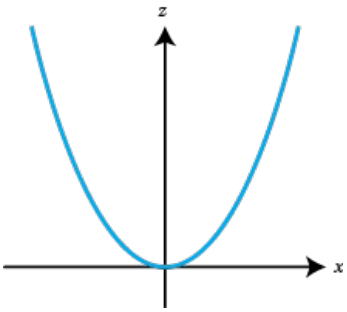
2 **a**



b 2

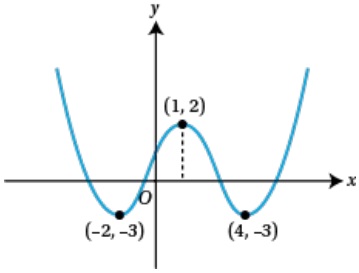
3 Translation left 3 units

4



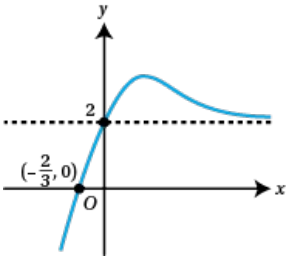
5 $3.75\,\text{km}$

6 **a**



b $(1, 2)$

7



8 **a** $y = (x - 2)^2$

b $y = 4 - x^3$

9 **a** Proof

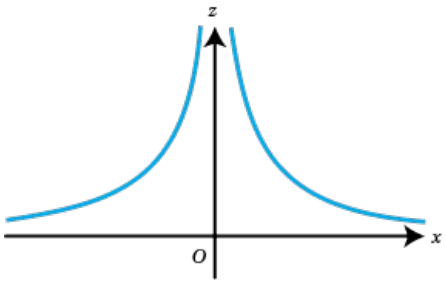
b $M = 3.63$ represents the weight at birth, $0.233\,\text{kg}$ is the weight gain every week

c Not appropriate (predicts $M = 15.7\,\text{kg}$)

10 **a** $x = -\frac{1}{2}, y = -3$

b The line is tangent to the curve.

11



12 **a** $x = \pm 2, \pm 3$

b $-3 \leqslant x \leqslant -2$ or $2 \leqslant x \leqslant 3$

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Answers

Chapter 6

Before you start...

1 **a** $y = 2x - 1$

b $y = -2x + 9$

c $y = 3x$

2 11

3 $x = \frac{45}{11}, y = \frac{29}{11}$

4 $(x-2)^{2-}7$

5 **a** $(2, 4), (-3, 9)$

b Proof

6

	Parallelogram	Rectangle	Rhombus	Square
opposite sides parallel	✔	✔	✔	✔
opposite sides equal	✔	✔	✔	✔
all four sides equal			✔	✔
sides perpendicular		✔		✔
diagonals equal		✔		✔
diagonals perpendicular			✔	✔
diagonals bisect each other	✔	✔	✔	✔

7 **a** $w = 70^\circ, x = 5, y = 90^\circ, z = 20^\circ$

b $a = 50^\circ, b = 90^\circ, c = 130^\circ$

c $d = 4, e = 3$

Exercise 6A

1 **a** **i** 13

ii 5

b **i** $\sqrt{5}$

ii $\sqrt{10}$

c **i** $2\sqrt{5}$

ii $2\sqrt{2}$

d **i** $\sqrt{10}$

ii $\sqrt{17}$

2 **a** **i** $(2.5, 6)$

ii $(1.5, 2)$

b **i** $(1.5, 5)$

ii $(2.5, 3.5)$

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c **i** $(1,\ 3)$

ii $(-2, 2)$

d **i** $(-2.5, -1.5)$

ii $(-1.5, -3)$

3 $3a\sqrt{5}$

4 $(2-a, 2-b)$

5 **a** Both are 5

b It does not lie on the line AB

6 $1 \pm \dfrac{2\sqrt{5}}{5}$

7 $y = \dfrac{x^2+1}{2}$

8 Proof

9 **a** **i** $2\sqrt{5}\sqrt{a^2+1}$

ii $(-1+2a, 2+a)$

b It is $2{:}1$

10 $5\sqrt{5}m$

Exercise 6B

1 **a** **i** $y+1 = 3\,(x-4)$

ii $y-2 = 5\,(x+3)$

b **i** $y-1 = -\dfrac{1}{2}(x+3)$

ii $y-3 = \dfrac{4}{3}(x-1)$

c **i** $y-7 = 4\,(x-3)$

ii $y-1 = 3\,(x-4)$

d **i** $y+1 = -\dfrac{6}{7}(x-3)$

ii $y+7 = \dfrac{9}{4}(x+1)$

2 **a** **i** $2x+y-1=0$

ii $5x+y-13=0$

b **i** $x-3y-20=0$

ii $3x+2y+4=0$

c **i** $5x-3y+11=0$

ii $9x+2y-37=0$

d **i** $6x+5y-17=0$

ii $7x+4y+13=0$

3 **a** **i** $3, \left(-\dfrac{1}{3}, 0\right), \left(0,\ 1\right)$

ii $-2, (1.5, 0), (0,\ 3)$

b **i** $1.5,\ \left(-\dfrac{5}{3}, 0\right),\ (0,\ 2.5)$

ii $-0.8, (0.25, 0),\ (0,\ 0.2)$

c **i** $1,\ \left(\frac{5}{3}, 0\right),\ \left(0, -\frac{5}{3}\right)$

ii $4, (-1.5, 0),\ (0,\ 6)$

d **i** $\frac{4}{3}, (1.75, 0),\ \left(0, -\frac{7}{3}\right)$

ii $-2.5, (0.6, 0),\ (0,\ 1.5)$

e **i** $2, (-0.5, 0), (0,\ 1)$

ii $3,\ \left(\frac{16}{3}, 0\right),\ (0, -16)$

4 **a** **i** $(-8, -7)$

ii $(3,\ 4)$

b **i** $\left(\frac{7}{3}, -\frac{11}{9}\right)$

ii $\left(\frac{16}{17}, -\frac{3}{17}\right)$

c **i** $\left(-\frac{5}{7},\ \frac{11}{14}\right)$

ii $\left(\frac{15}{14},\ \frac{23}{14}\right)$

5 **a** $4x + 9y - 11 = 0$

b $4x + 9y - 2 = 0$

c $k = \frac{2}{21}$

6 $\frac{361}{12}$

7 **a** $x - y = -2, 3x + 2y = -24$

b $\left(-\frac{28}{5}, -\frac{18}{5}\right)$

c $\frac{54}{5}$

8 **a** $k = 3$

b $2x - 3y = -17$

9 $\frac{\sqrt{65}}{4}$

10 $\text{area } = \frac{4}{3}$

Work it out 6.6

Solution 2 is correct.

Exercise 6C

1 **a** **i** $-\frac{3}{5}$

ii -6

b **i** $-\frac{1}{2}$

ii $-\frac{2}{5}$

c **i** $\frac{1}{2}$

ii $\frac{1}{3}$

d **i** $-\frac{7}{3}$

ii -5

2 **a** **i** Perpendicular

ii Parallel

b **i** Parallel

ii Parallel

c **i** Perpendicular

ii Neither

d **i** Neither

ii Neither

3 **a** Proof

b $2x + 5y - 22 = 0$

4 Right angle at A, $\text{area} = 6.5$

5 **a** $\left(2,\ \frac{9}{2}\right)$

b $y = 1.2x + 2.1$

6 **a** $2x + 3y = -5$

b $\frac{11}{2}$

7 $\left(6 \pm 2\sqrt{2},\ 0\right)$

8 $18x - 4y - 1 = 0$

9 **a** Proof

b $q = \frac{14}{3}$, $\text{area} = \frac{53}{3}$

10 **a** $y = -2x + 4$

b $2\sqrt{5}$

11 **a** Proof

b $k = 2$

Exercise 6D

1 **a** **i** $(x-3)^2 + (y-7)^2 = 16$

ii $(x-5)^2 + (y-1)^2 = 36$

b **i** $(x-3)^2 + (y+1)^2 = 7$

ii $(x+4)^2 + (y-2)^2 = 5$

2 **a** **i** Centre $(2, -3)$, radius $\frac{3}{2}$

ii Centre $(-1, -5)$, radius $\frac{2}{5}$

b **i** Centre $\left(3,\ \frac{1}{2}\right)$, radius $\sqrt{6}$

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ii Centre $\left(-\frac{3}{4}, \frac{1}{5}\right)$, radius $\sqrt{3}$

3 **a** **i** Centre $(-2, 3)$, radius 3

ii Centre $(4, -1)$, radius 3

b **i** Centre $(1, -3)$, radius 3

ii Centre $(5, -2)$, radius $\sqrt{30}$

c **i** Centre $\left(-\frac{5}{2}, \frac{1}{2}\right)$, radius $\frac{3\sqrt{2}}{2}$

ii Centre $\left(\frac{3}{2}, -\frac{7}{2}\right)$, radius $\frac{\sqrt{70}}{2}$

d **i** Centre $\left(0, \frac{5}{2}\right)$, radius $\frac{\sqrt{73}}{2}$

ii Centre $\left(-\frac{3}{2}, 0\right)$, radius $\frac{7}{2}$

4 **a** **i** On circumference

ii On circumference

b **i** Outside circle

ii Inside circle

5 **a** $(x+6)^2 + (y-3)^2 = 117$

b $(0, -6), (0, 12)$

6 **a** $\left(\frac{5}{2}, -\frac{1}{2}\right)$, $r = \frac{\sqrt{38}}{2}$

b Outside

7 $4\sqrt{6}$

8 **a** Proof

b $\sqrt{442}$

c $\left(x - \frac{5}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{221}{2}$

9 **a** $\sqrt{17}$

b Outside the circle

10 **a** 90°

b Proof

Exercise 6E

1 **a** **i** $y = 4$

ii $y = \frac{3}{5}x + \frac{19}{5}$

b **i** $y = -\frac{2}{3}x + \frac{5}{3}$

ii $y = \frac{2}{3}x - \frac{4}{3}$

c **i** $y = -2x + 9$

ii $y = -x + 6$

2 **a** **i** $\left(\frac{\pm 7 + \sqrt{509}}{10}, \frac{\pm 2 + \sqrt{509}}{5}\right)$

ii $\left(\frac{7+\sqrt{389}}{10}, \frac{2+\sqrt{389}}{5}\right)$

b **i** $(2,\ 7)$

ii $(4,\ 4)$

c **i** Inside

ii No intersection

3 **a** **i** Intersect

ii Intersect

b **i** Disjoint

ii Intersect

c **i** Intersect

ii Tangent

4 $\left(\frac{33}{29}, -\frac{19}{29}\right)$

5 **a** Proof

b $(1, -2)$

c $y = x + 5$

6 **a** $y = \frac{1}{2}x + 1$

b $\left(\frac{18}{5}, \frac{14}{5}\right)$

c $\frac{49}{5}$

7 **a** $y = 8 - x,\ y = -\frac{1}{6}x + \frac{25}{4}$

b $(2.1, 5.9)$

c $y = \frac{3}{2}x + \frac{11}{4}$

d $\frac{\sqrt{962}}{10}$

8 **a** $(x-2)^2 + (y-5)^2 = 29$

b Proof

c 7.43

9 **a** 7 or 21

b $(-2.39, -1.99)$ and $(-2.39, 12.0)$

10 $\frac{200}{3}$

11 $1.04, 6.16$

12 4.11

13 **a** $(2, -1)$

b Proof

14 **a** Proof

b Proof

c $150 - 25\pi$

15 $a^2 + b^2 = (1 \pm r)^2$

Mixed practice 6

1 $k = 13$

2 **a** $k = -\frac{5}{2}$

b $p = 6$

c $3x - 2y - 6 = 0$

d $(2, 0)$

3 **a** $(1,\ 5)$

b proof

c $(3.4,\ 3.8)$

4 **a** Centre $(-3,\ 2)$, radius $\sqrt{17}$

b $\left(\frac{1}{5}, \frac{23}{5}\right)$ and $(-2, -2)$

5 $k = -19$

6 **a** Proof

b $(x - 6)^2 + y^2 = 13$

c $3x + 2y - 5 = 0$

7 $-6 - 5\sqrt{5} < k < -6 + 5\sqrt{5}$

8 **a** $\frac{256}{3}$

b $\frac{104}{3}$

9 $x^2 - 2x + y^2 - 6y - 22 = 0$

10 $k = \pm\sqrt{\frac{1}{8}} = \pm\frac{\sqrt{2}}{4}$

11 **a** $x^2 + y^2 - 4x - 2y - 95 = 0$

b $k = 1 + \sqrt{91}$

c Inside

d $y = -\frac{3}{4}x + 15$

12 $\frac{24\sqrt{13}}{13}$

13 Proof; $\sqrt{80} > 7$

14 **a** $p = 8; (x - 5)^2 + (y - 7)^2 = 58$

b $21 + 5\sqrt{33}$

Answers

Chapter 7

Before you start...

1 **a** False

b False

c True

d False

2 **a** 81

b $\frac{1}{27}$

3 **a** $x^{2.5}$

b $x^{-0.5}$

4 $x = \frac{7}{3}$

5 **a** $x = 2, -4$

b $x = \pm 1$

Exercise 7A

1 **a** **i** 3

ii 2

b **i** 1

ii 1

c **i** 0

ii 0

d **i** -1

ii -3

e **i** $\frac{1}{2}$

ii $\frac{1}{3}$

f **i** $\frac{1}{2}$

ii $\frac{1}{2}$

g **i** $\frac{2}{3}$

ii $\frac{3}{4}$

h **i** $\frac{3}{2}$

ii $\frac{5}{4}$

i **i** $\frac{3}{4}$

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ii $\frac{9}{4}$

j **i** $-\frac{1}{2}$

ii $-\frac{1}{2}$

2 **a** **i** 1.70

ii -0.602

b **i** -2.30

ii 2.30

c **i** 1.95

ii -0.317

d **i** -0.683

ii 2.29

3 **a** **i** $5\log x$

ii $5\log x$

b **i** $\log x \log y - \log y + 3\log x - 3$

ii $(\log x)^2 + 4\log x + 4$

c **i** $\frac{1}{\log b} + \frac{1}{\log a}$

ii $\log a + 1$

4 **a** **i** $x = 3^y$

ii $x = 16^y$

b **i** $x = a^{y+1}$

ii $x = a^{y^2}$

c **i** $x = \sqrt[3]{3y}$

ii $x = \sqrt{y}$

d **i** $x = e^{y-2}$

ii $x = e^2 y$

5 **a** **i** $x = 32$

ii $x = 16$

b **i** $x = 0.4$

ii $x = 0.25$

c **i** $x = 6$

ii $x = 100$

6 $x = 111$

7 $x = -3$

8 $x = \frac{e^2 + 1}{3}$

9 $x = 10^{1.5} = 31.6$

10 $x = 9,\ \frac{1}{9}$

11 $x = 81, y = 25$

12 It is very close to e.

Work it out 7.1

Solution 2 is correct.

Exercise 7B

1 **a** **i** 4

ii $\frac{1}{2}$

b **i** 6

ii $\frac{3}{2}$

2 **a** **i** $7y$

ii $2x + y$

b **i** $x + 2y - z$

ii $2x - y - 3z$

c **i** $2 - y - 5z$

ii $1 + y + 2z$

d **i** $x - 4y$

ii $2 + 2x + y + 2z$

3 **a** **i** $x = 1$

ii $x = 4$

b **i** $x = 9$

ii $x = 2$

c **i** $x = 8$

ii $x = 4$

4 $x = 4$

5 $x = 27$

6 $x = 8$

7 **a** $2x + y$

b $2 + x - \frac{z}{2}$

8 **a** $a + 2b$

b $2(a - b)$

9 $x = \frac{1}{3}\mathrm{e}^{\frac{3}{2}}$

10 $x = \frac{5}{3}$

11 $x = 2$

12 $x = \frac{1}{3}$

Work it out 7.2

Solution 1 is correct.

Work it out 7.3

Solution 2 is correct.

Exercise 7C

1 **a** **i** $x = 2.45$

ii $x = 116$

b **i** $x = -1.83$

ii $x = 4.62$

c **i** $x = -1.71$

ii $5x = 6^{1-x}\,9.83$

d **i** $x = 1.11$

ii $x = -2.98$

2 **a** **i** $x = \dfrac{\ln 5 - \ln 2}{\ln 3}$

ii $x = \dfrac{\ln 3 - \ln 5}{\ln 7}$

b **i** $x = -\ln 4$

ii $x = -\ln 3$

c **i** $x = \dfrac{\ln 5}{2\ln 2}$

ii $x = \dfrac{\ln 7}{2\ln 10}$

d **i** $x = \dfrac{1}{3}\left(\dfrac{\ln 10}{\ln 2} - 1\right)$

ii $x = \dfrac{1}{2}\left(\dfrac{\ln 4}{\ln 5} + 3\right)$

e **i** $x = \dfrac{\ln 2}{\ln\left(\dfrac{5}{2}\right)}$

ii $x = \dfrac{2\ln 25}{\ln\left(\dfrac{5}{3}\right)}$

f **i** $x = \dfrac{\ln 3}{3\ln 2 - 1}$

ii $x = \dfrac{\ln 5}{2 - \ln 2}$

g **i** $x = \dfrac{\ln\left(\dfrac{5}{64}\right)}{\ln\left(\dfrac{8}{3}\right)}$

ii $x = \dfrac{\ln\left(\dfrac{7}{12}\right)}{\ln\left(\dfrac{9}{2}\right)}$

h **i** $x = \dfrac{\ln\left(\dfrac{27}{7}\right)}{\ln\left(\dfrac{27}{8}\right)}$

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ii $x = \dfrac{\ln\left(\dfrac{1}{30}\right)}{\ln\left(\dfrac{25}{8}\right)}$

3 $x = -0.232$

4 $x = 5 - \log_3 4 \left(\text{or } 5 - \dfrac{\text{In}4}{\text{In}3}\right)$

5 $x = \dfrac{(3 - \ln\ 10)}{2}$

6 $x = \frac{\log_{10}\ 5}{1 - \log_{10}\ 8}$

7 $x = \dfrac{\ln\ 50}{\ln\ 40}$

8 Proof

Exercise 7D

1 **a** **i** $x = 1,\ \log_2 3$ or $\dfrac{\log 3}{\log 2}$

ii $x = \log_3 2,\ \log_3 4$

b **i** $x = 2$

ii $x = \log_5 3$

c **i** $x = \ln\ 4$

ii $x = \ln\ 4, \ln 5$

d **i** $x = 1,\ \dfrac{1}{\log 5}$

ii $x = 2,\ \dfrac{\log 3}{\log 2}$

e **i** $x = 1, \sqrt{2}$

ii $x = 3,\ 9$

2 **a** Proof

b $1.26,\ 1.46$

3 $x = 1,\ 3$

4 $x = 2\ln 2, -\ln 2$

5 $x = \log_5 3$

6 $x = \ln\ 3, \ln 5$

Mixed practice 7

1 $x = \pm 24$

2 D

3 **a** $2a + \dfrac{b}{2} - c$

b $\dfrac{a - 1}{2}$

c $\dfrac{b - c}{2}$

4 $x = 0.367$

5 $x = 3$

6 $x = 1.17$

7 $y = \dfrac{a^4}{7}$

8 $x = \dfrac{\log_3 4 + 1}{2}$

9 $x = \mathrm{e}^{\frac{4}{3}} = 3.79,\ y = \mathrm{e}^{\frac{10}{3}} = 28.0$

10 $b = \dfrac{x^2}{3}$

11 11

12 $x = 1 \pm \sqrt{1 - \mathrm{e}^y}$

13 $x = 1.46$

14 $x = \dfrac{1}{3}$

15 $x = 146$

16 **a** **i** $p + q$

ii $2 + 3p - q$

b **i** $\log_{10}\left(\dfrac{x^2 - 10}{x}\right)$

ii $x = 10$

17 $x = \dfrac{\ln 3}{\ln 2}$

Solution 2 is correct.

18 $x = \dfrac{\ln\left(\dfrac{4}{5}\right)}{\ln\ 36}$

19 $x = \ln\left(2 \pm \sqrt{3}\right)$

20 $x = 3,\ 9$

Answers

Chapter 8

Before you start...

1 **a** $2 + \ln 5$

b $5\mathrm{e}$

2 $2 + 3 \log x$

3 Horizontal stretch, scale factor $\frac{1}{2}$

4 $-\frac{2}{3}$

5 $y = 3 - \frac{2}{3}x$

Exercise 8A

1 **a** **i** C

ii B

iii A

b **i** C

ii B

iii A

c **i** A

ii B

2 **a** **i** $3.2\mathrm{e}^{3.2x}$

ii $0.6\mathrm{e}^{0.6x}$

b **i** $-1.3\mathrm{e}^{-1.3x}$

ii $-\mathrm{e}^{-x}$

3 **a** 57.6

b 0.0893

c -0.0160

d -0.784

4 **a** 25.5

b 2.4

c -2.1

d -0.5

5 **a** 0.0643

b 18

6 **a** -1.15

b $x = -0.0363$

7 **a** $k = 3.25$

b 0.126

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8 $x = -11.3$

9 **a** $k = \ln 8$

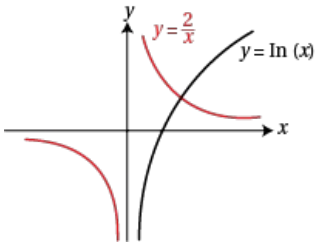
b 0.735

10 **a** $p = \ln 0.3$

b -0.0783

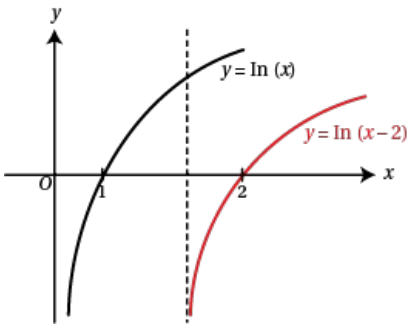
Exercise 8B

1 **a**



b One

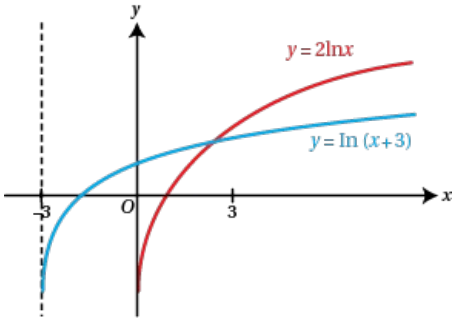
2



3 Proof

4 translation by $\begin{pmatrix} 0 \\ \ln 3 \end{pmatrix}$ or horizontal stretch with scale factor $\frac{1}{3}$

5 **a**



b $x = \dfrac{1+\sqrt{13}}{2}$

6 **a** Proof

b Vertical stretch with scale factor $\dfrac{1}{\ln 10}$

Work it out 8.1

Solution 2 is correct.

Exercise 8C

1 **a** **i** $C\left(1+\dfrac{p}{100}\right)$

ii $C\left(1+\frac{p}{200}\right)^2$

iii $C\left(1+\frac{p}{400}\right)^4$

b Proof

c It tends to (or approaches) e.

d $V = Ce^x$

2 **a** 100

b $48\,300$

c $2.24\,\mathrm{h}$

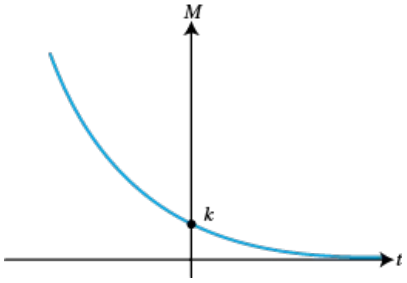
d 1030 cells per hour

3 $13.31\,\mathrm{m}^2$

4 **a** 3 billion

b 6.44 billion

5 **a**



b 2.31 min

6 **a** $N_0 = 450, a = 4.8$

b 3600 per day

c 7.46 hours

7 $N = 500\mathrm{e}^{-0.4t}$

8 **a** $V = 6800\mathrm{e}^{-0.2t}$

b £920

9 **a** $P = 70\,000\,000 \times 1.02^n$

b e.g. Population growth rate might change due to changing economic conditions. Immigration is not being taken into account.

10 **a** $21.2^\circ\mathrm{C}$ above room temperature

b **i** No change

ii a would get larger, but not above 1

11 **a** $0\,\mathrm{m\,s^{-1}}$

b $40\,\mathrm{m\,s^{-1}}$

12 **a** e^{-2}

b Proof

Exercise 8D

1 **a** $A = 2.01, b = 3.32$

b $A = 22.2, b = 0.549$

c $A = 0.0183, b = 9.97$

2 **a** $C = 3.32, n = 0.7$

b $C = 0.00910, n = 2.1$

c $C = 7.39, n = -0.9$

3 **a** B

b $A = 150$

c $k = 0.165$

d Not suitable, as it predicts indefinite growth.

4 **a** The initial number of bacteria

b $b = \frac{1}{3}$

c 19.5 hours

5 **a** $\ln(M) = -0.16t + 2.0$

b $K = 7.4, c = 0.85$

c 13 seconds

6 **a** $y = 1.2x + 2$

b $C = 7.4, n = 1.2$

c 5.2 years

7 **a** $N = 330\mathrm{e}^{0.16t}$

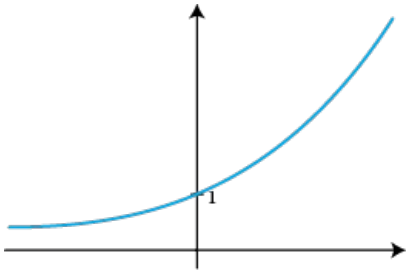
b 190 mice per week

8 **a** $\ln\left(\frac{1-\mathrm{N}}{\mathrm{N}}\right)$ against t will have gradient $-b$ and intercept $\ln J$.

b The logistic function predicts that the population will level off (to one unit), which is more realistic than the exponential model, which predicts unlimited population growth.

Mixed practice 8

1 **a**



b 8.82

c One

2 **a** $32\,\mathrm{g}$

b $-2.94\,\mathrm{g\,s^{-1}}$

c $4.95\,\mathrm{s}$

3 One

4 **a** $0.4\,\mathrm{cm}^3$

b 10 weeks

c Model is for algae in a jar, which limits volume; extrapolation beyond model's validity

5 **a** $A = 18, k = 0.0141$

b 64

c $1.58\,\mathrm{p.m.}$

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6 **a** $12\,\mu\mathrm{U/ml}$

b No – long-term level is $2\mu\mathrm{U/ml}$

7 **a** **i** Proof

ii $k = 37\,000,\ a = 0.949$

iii 2700

iv 2039

8 **a** $\ln(N) = kt + \ln(A)$

b $4.13,\ 4.26$

c $\ln(N) = 0.0137t + 3.56;\ N = 35.2\mathrm{e}^{0.0137t}$

d 1240

e Size of the lake limits indefinite growth; seasonal variation

9 $p = -2.5, a = 10^{10}$

10 **a** 17.3 years

b 38.0 years

11 **a** **i** $40,320$

ii $k = 0.0330$

b $2.4\mathrm{g\,h^{-1}}$

12 $0.002\,67\,\mathrm{s}$

13 **a** $0\,\mathrm{m\,s^{-1}}$

b $42\,\mathrm{m\,s^{-1}}$

c $3.71\,\mathrm{s}$

14 **a** $C = 76, k = 0.027$

b $11.3\ \mathrm{min}$

Answers

Chapter 9

Before you start...

1 18

2 **a** $16x^{12}$

b x^{11}

c x^9

3 $4x^2 + 12x + 9$

4 $x = -4, -1$

Work it out 9.1

Solution 1 is correct.

Exercise 9A

1 **a** **i** 35

ii 36

b **i** 1

ii 1

c **i** 8

ii 45

2 **a** **i** $32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$

ii $729 + 1458x + 1215x^2 + 540x^3 + 135x^4 + 18x^5 + x^6$

b **i** $16 - 96x + 216x - 216x^3 + 81x^4$

ii $8x^3 - 84x^2 + 294x^2 - 343$

c **i** $243x^5 + 405x^4y + 270x^3y^2$

ii $16c^4 - 32c^3d + 24c^2d^2$

d **i** $8x^6 - 36x^5 + 54x^4 - 27x^3$

ii $\frac{8}{x^3} + \frac{60y}{x^2} + \frac{150y^2}{x} + 125y^3$

e **i** $8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3}$

ii $x^4 - 12x^2 + 54 - \frac{108}{x^2} + \frac{81}{x^4}$

3 **a** 4

b 35

c 7

d 56

4 **a** **i** -216

ii 20

b **i** $560x^3y^4$

ii $-280x^3y^4$

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c **i** -5

ii $78\,030$

5 $27 + 27x + 9x2 + x3$

6 **a** $80y^4$

b $-80y^3$

7 $-10\,500$

8 $20412x^2$

9 $y^6 + 18y^7 + 135y^8 + 540y^9$

10 **a** $1 + 8x + 24x2 + 32x3 + 16x^4$

b $1 + \frac{8}{x} + \frac{24}{x^2} + \frac{32}{x^3} + \frac{16}{x^4}$

11 56

12 **a** $n = 6$

b $a = \pm 2$

13 $16z^8 + 96z^5 + 216z^2 + 216z^{-1} + 81z^{-4}$

14 **a** $n = 3$

b $27x^6y^3 + 135x^5y + 225x^4y^{-1} + 125x^3y^{-3}$

15 -672

16 $-945x^5$

17 $792\,00\,000$

Exercise 9B

1 **a** **i** 21

ii 10

b **i** 120

ii 220

c **i** 8

ii 15

d **i** 1

ii 1

e **i** 1

ii 1

f **i** 7

ii 12

2 **a** **i** $n = 10$

ii $n = 13$

b **i** $n = 10$

ii $n = 12$

3 **a** $n = 5$

b $a = 80$

4 **a** $n = 10$

b $a = 180$

5 **a** $n = 12$

b $a = 6$

6 $n = 2k + 1$

7 **a**, **b** Student's own (reasonable) answers.

Exercise 9C

1 **a** $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$

b $3 + 29x + 110x^2 + 200x^3 + 160x^4 + 16x^5 - 32x^6$

2 **a** $128 - 2240x + 16\,800x^2 - 70\,000x^3$

b $-36\,400$

3 **a** $81 - 540x + 1350x^2$

b 80.461

4 **a** $32\,768 + 40\,960x + 20\,480x^2$

b $33\,179.648$

5 **a** $128 + 1344x + 6048x^2$

b **i** 322.28

ii 142.0448

c **ii** Smaller value of x means higher order terms much smaller and therefore less important, so the error is less for 2.03^7.

6 $16\,384 - 28\,672x + 21\,504x^2; -30\,720$

7 **a** $\mathrm{e}^5 + 10\mathrm{e}^3 + 40\mathrm{e} + \dfrac{80}{\mathrm{e}} + \dfrac{80}{\mathrm{e}^3} + \dfrac{32}{\mathrm{e}^5}$

b $2\mathrm{e}^5 + 80\mathrm{e} + \dfrac{160}{\mathrm{e}^3}$

8 **a** $\left(1 - x^2\right)^n$

b $1 - 10x^2 + 45x^4$

9 126

Mixed practice 9

1 $n = 8$

2 $-101\,376$

3 $8x^{-3} + 60x^{-2}y\ + 150x^{-1}y^2 + 125y^3$

4 720

5 $232 - 164\sqrt{2}$

6 -32

7 $x^8 - 8x^5 + 24x^2 - 32x^{-1} + 16x^{-4}$

8 $-\dfrac{1}{3}$

9 5733

10 -5

11 **a** $16x^4 + 160x^3 + 600x^2 + 1000x + 625$

b $k = 2000$

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c $x = 0.5, -2.5$

12 **a** $n = 5$

b $a = 0.5$

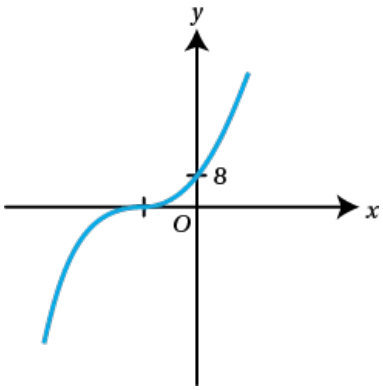
c $32x^5 + 40x^4y + 20x^3y^2 + 5x^2y^3$

d $3\ 240\ 200$

13 80

14 $a = 2, n = 5$

15 **a**



b $x^3 + 6x^2 + 12x + 8$

c 8.120601

d $x = -4$

16 **a** proof

b $k = 4, a = \frac{1}{2}$

c 2

Focus on … 1

Focus on … Proof 1

1 Discussion

2 Proof

3 Proof

4 **a** Proof

b No

Focus on … Problem solving 1

1 **a**, **b** $4\sqrt{6}$ Accept student's own (reasonable) answers.

2 $\frac{60}{13}$

3 **a**, **b** $k = \pm\frac{\sqrt{2}}{4}$

c Discussion

4 $r = 4$

Focus on … Modelling 1

1 **a**, **b**, **d**, **e** Student's own (reasonable) answers

c $k \approx 0.24, A \approx 10.5$

2 **a** A

b Discussion

c $k \approx 0.27, A \approx 10, C \approx 460$

3 **a**, **b** Discussion

4 **a** A

b, **c** Discussion

5 **a** $A = 7\,430\,000, k = 0.005\,59$

b 10.3 per 1000 people

c **i** About $40\,250$

ii They have the same difference between the birth and death rate as the indigenous population.

Cross-topic Review exercise 1

1 **a** 2

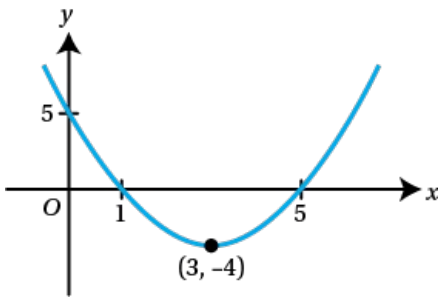
b $z + 2y - 2x$

c $K = -\dfrac{e^2}{c}$

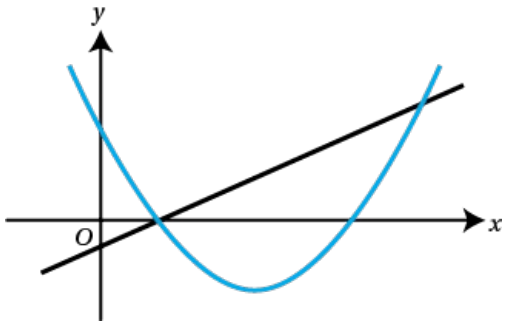
2 **a** **i** $(x-3)^2 - 4$

ii Translation by $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

iii

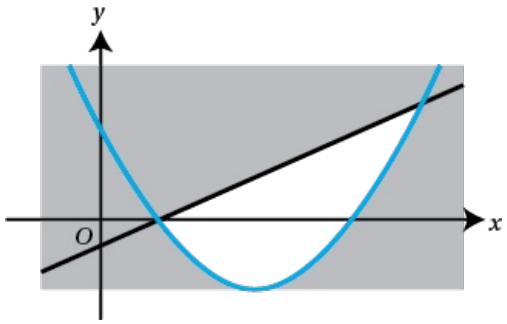


b **i**



ii $x = 1, 6$

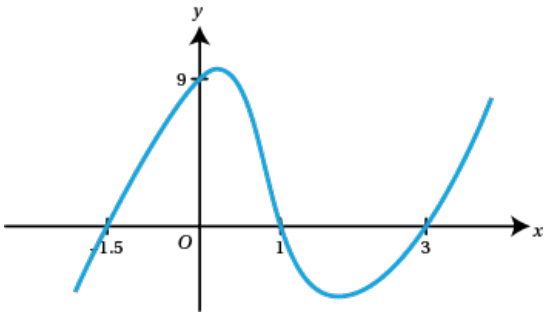
iii



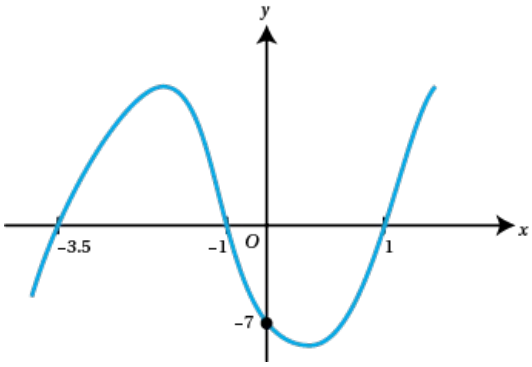
3 **a** $p(3) = 0$

b $(x-3)(2x+3)(x-1)$

c



d



4 84

5 $x = -\ln 3, \ln 2$

6 **a** $A = 5, C = 2$

b $k = \dfrac{\ln 5}{2}$

7 **a** $2 + 12x^2 + 2x^4$

b 34

8 **a** Proof

b **i** $x^2 + 2 + \dfrac{1}{x^2}, x^3 + 3x + \dfrac{3}{x} + \dfrac{1}{x^3}$

ii 7, 18

9 **a** $\dfrac{1}{k}$

b $\begin{pmatrix} 0 \\ \ln k \end{pmatrix}$

10 $n = \dfrac{1 + \sqrt{1 + 8k}}{2}$

11 **a** $A = 42, k = \dfrac{1}{9}, m = 0.0770$

b 11.3

c 47.9

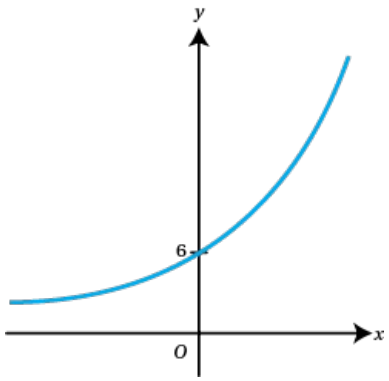
12 Proof

13 **a** $x = \ln\left(\dfrac{y \pm \sqrt{y^2 - 4}}{2}\right)$

b **i** $y > 2$

ii Proof that the sum of the two values of x is zero

14 a



b Proof

Answers

Chapter 10

Before you start...

1 3.86 cm

2 21.8°

3 13

4 $x = 1$

5 $x = \frac{1}{e}$ or e^2

Exercise 10A

1 **a** **i** 0.669

ii -0.978

b **i** -0.766

ii -0.682

2 **a** **i** 1

ii 0

b **i** 1

ii -1

c **i** -1

ii 0

3 **a** 0.766

b 0.766

c -0.766

d -0.766

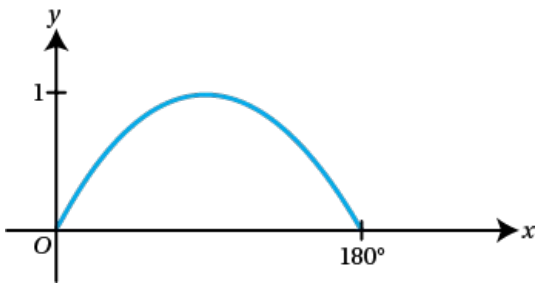
4 **a** 0.766

b 0.766

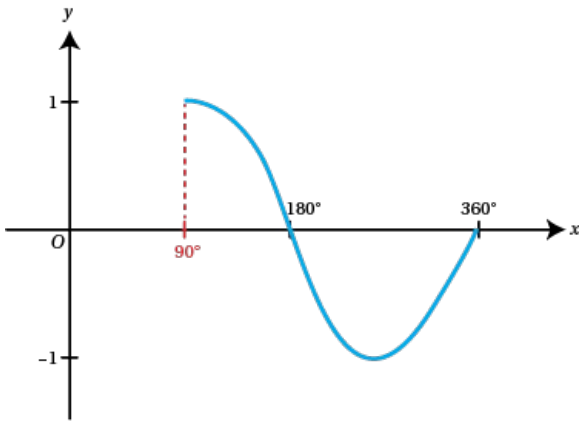
c -0.766

d -0.766

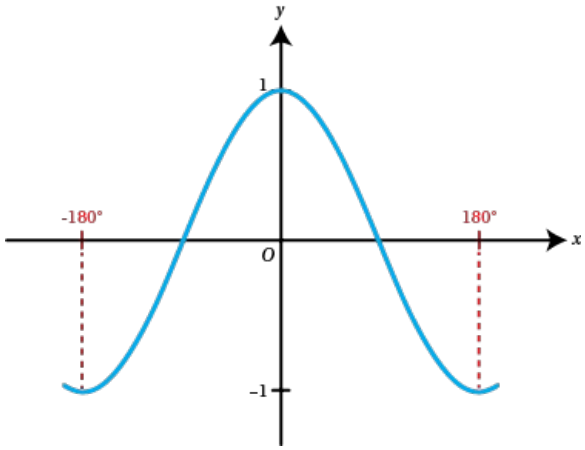
5 **a** **i**



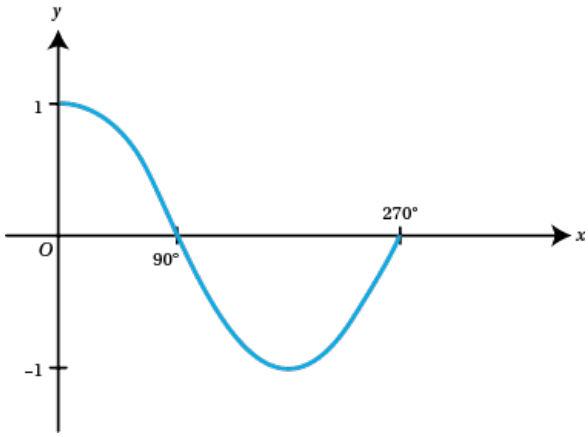
ii



b i



ii



6 0

7 $k = -2$

8 $k = -1$

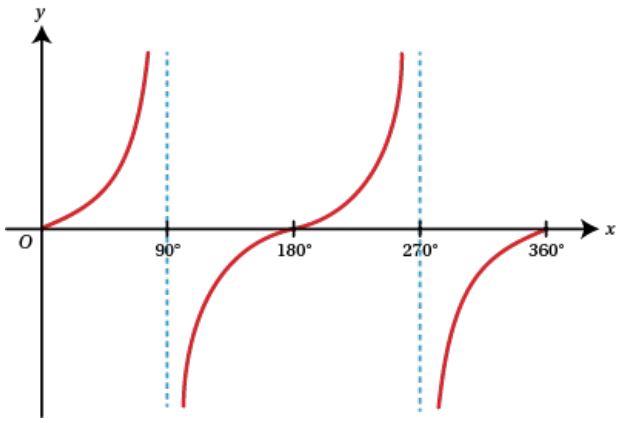
9 $\sin x$

10 $90 - \ln 3, \ln 3 - 90$

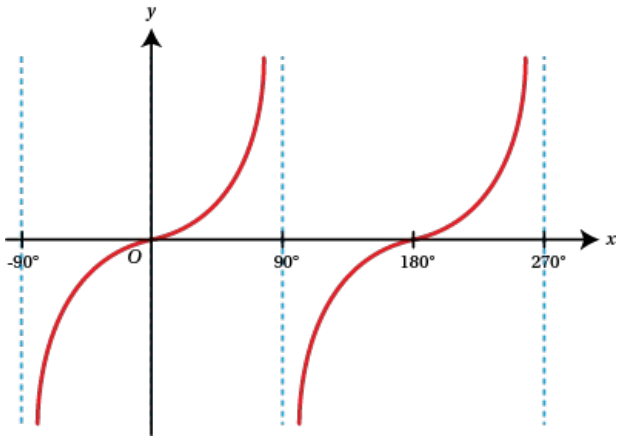
$-270 - \ln 3, 270 + \ln 3$

Exercise 10B

1 **a**



b



2 **a** **i** 0.625

ii -0.213

b **i** 0

ii 1.28

3 **a** 0.364

b 0.364

c -0.364

d -0.364

4 **a** $\dfrac{1}{\tan x}$

b $-\dfrac{1}{\tan x}$

c $\dfrac{1}{\tan x}$

d $-\tan x$

5 $2\tan x$

6 e, e $-$ 180

7 $-t^2$

Exercise 10C

1 **a** $\dfrac{\sqrt{2}}{2}$

b $\dfrac{\sqrt{2}}{2}$

c $-\dfrac{\sqrt{2}}{2}$

d 1

e -0.5

f $-\frac{\sqrt{3}}{2}$

g $\frac{1}{\sqrt{3}}$

h $\frac{\sqrt{3}}{3}$

2 **a** 0.75

b $\frac{4}{3}$

c $\frac{\sqrt{2}+\sqrt{3}}{2}$

d $\frac{1-\sqrt{3}}{2}$

3 **a** Proof

b Proof

c Proof

d Proof

4 **a** Proof

b 120°

5 **a** $\frac{1-\sqrt{3}}{2}$

b $A = B = 0^\circ$

6 $a = 2, b = -2$

Exercise 10D

1 **a** **i** $\cos x = \frac{2\sqrt{2}}{3}, \tan x = \frac{\sqrt{2}}{4}$

ii $\cos x = 0.6, \tan x = \frac{4}{3}$

b **i** $\sin\theta = -\frac{\sqrt{8}}{3}, \tan\theta = 2\sqrt{2}$

ii $\cos\theta = -\frac{\sqrt{7}}{4}, \tan\theta = \frac{3\sqrt{7}}{7}$

c **i** $\cos x = -\frac{2\sqrt{6}}{5}$

ii $\cos x = \frac{\sqrt{3}}{2}$

d **i** $\tan x = -\frac{4}{3}$

ii $\tan x = 0$

e **i** $\pm\frac{3\sqrt{13}}{13}$

ii $\pm\frac{\sqrt{5}}{5}$

2 **a** 3

b 1

c -2

d $\frac{3}{2}$

3 **a** $4 - \sin^2 x$

b $2\cos^2 x - 1$

4 **a** **i** Proof

ii Proof

b **i** Proof

ii Proof

5 **a** $-\frac{\sqrt{5}}{3}$

b $-\frac{\sqrt{5}}{2}$

6 $\pm\frac{3\sqrt{10}}{10}$

7 $-\sqrt{1 - s^2}$

8 **a** $5 - \frac{2}{\cos^2 x}$

b $\cos^2 x$

9 1

10 $k = -2$

11 **a** $\frac{1}{1 + t^2}$

b $\frac{t^2}{1 + t^2}$

c $\frac{1 - t^2}{1 + t^2}$

d $\frac{2 + 3t^2}{t^2}$

12 **a** Proof

b Proof

13 Proof

Exercise 10E

1 **a** **i** 44.4°

ii 17.5°

b **i** 128.3°

ii 138.6°

c **i** 81.1°

ii -82.0°

2 **a** **i** $x = 30^\circ, 150^\circ$

ii $x = 45^\circ, 135^\circ$

b **i** $x = 60^\circ, 300^\circ$

ii $x = 30^\circ, 330^\circ$

c **i** $x = 45^\circ, 225^\circ$

ii $x = 60^\circ, 240^\circ$

d **i** $x = 240^\circ, 300^\circ$

ii $x = 210^\circ, 330^\circ$

e **i** $x = 135^\circ, 225^\circ$

ii $x = 180^\circ$

f **i** $x = 150^\circ, 330^\circ$

ii $x = 135, 315^\circ$

3 **a** **i** $x = 26.7^\circ, 153.3^\circ$

ii $x = 44.4^\circ, 135.6^\circ$

b **i** $x = -138.6^\circ, 138.6^\circ$

ii $x = 101.5^\circ,\ -101.5^\circ$

c **i** $\theta = 18.4^\circ, 198.4^\circ, 378.4^\circ, 558.4^\circ$

ii $\theta = 53.1^\circ, 233.1^\circ, 413.1^\circ, 593.1^\circ$

d **i** $t = -138.2^\circ, -41.8^\circ, 221.8^\circ, 318.2^\circ$

ii $t = -165.5^\circ, -14.5^\circ, 194.5^\circ, 345.5^\circ$

4 **a** **i** $\theta = 5.74^\circ, 174^\circ$

ii $\theta = -14.5^\circ, 194^\circ$

b **i** $x = 63.6^\circ, 296^\circ$

ii $x = 57.3^\circ, 303^\circ$

c **i** $t = -121^\circ, 59.0^\circ$

ii $t = 65.6^\circ, 246^\circ$

5 $x = -150^\circ, -30^\circ$

6 $x = 121^\circ, 301^\circ$

7 $x = \pm 30^\circ, \pm 330^\circ$

8 $\theta = 0^\circ, 48.2^\circ, 180^\circ, 312^\circ, 360^\circ$

9 $23.6^\circ, 135^\circ, 156^\circ, 315^\circ$

10 e.g. $x = 1$

11 e.g. $x = 180^\circ$

Work it out 10.1

Solution 3 is correct.

Exercise 10F

1 **a** **i** $x = 22.2^\circ, 67.8^\circ, 202^\circ, 248^\circ$

ii $x = 63.8^\circ, 116^\circ, 184^\circ, 236^\circ, 304^\circ, 356^\circ$

b **i** $x = 37.9^\circ, 82.1^\circ, 158^\circ, 202^\circ, 278^\circ, 322^\circ$

ii $x = 0^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ$

c **i** $x = 14.1^\circ, 59.1^\circ, 104^\circ, 149^\circ, 194^\circ, 239^\circ, 284^\circ, 329^\circ$

ii $x = 58.3^\circ, 148^\circ, 238^\circ, 328^\circ$

2 **a** **i** $\theta = -25.5^\circ, 126^\circ$

ii $\theta = -106^\circ, -23.6^\circ$

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b **i** $\theta = -156^\circ, -104^\circ$

ii $\theta = -7.5^\circ, 153^\circ$

c **i** $\theta = -71.6^\circ, -108^\circ$

ii $\theta = -132^\circ, 48.4^\circ$

3 $x = 20^\circ, 80^\circ, 140^\circ$

4 $x = -120^\circ, -60^\circ, 60^\circ, 120^\circ$

5 $x = -276^\circ, -83.6^\circ$

6 $\theta = 205^\circ$

7 $x = -255^\circ, -225^\circ, -75^\circ, -45^\circ, 105^\circ, 135^\circ, 285^\circ, 315^\circ$

8 $x = -88.3^\circ, 1.67^\circ, 31.7^\circ$

9 $x = \pm 30^\circ, \pm 150^\circ$

10 $x = \pm 5.48^\circ, \pm 12.2^\circ$

Exercise 10G

1 **a** **i** $x = 54.7^\circ, 125^\circ, 235^\circ, 305^\circ$

ii $x = 52.2^\circ, 128^\circ, 232^\circ, 308^\circ$

b **i** $x = 71.6^\circ, 117^\circ, 252^\circ, 297^\circ$

ii $x = 48.2^\circ, 180^\circ, 312^\circ$

c **i** $x = 41.4^\circ, 319^\circ$

ii $x = 53.1^\circ, 127^\circ$

d **i** $x = 0^\circ, 180^\circ, 199^\circ, 341^\circ, 360^\circ$

ii $x = 0^\circ, 129^\circ, 180^\circ, 309^\circ, 360^\circ$

2 **a** **i** $\theta = \pm 180^\circ, \pm 66.4^\circ, 0^\circ$

ii $\theta = -127^\circ, -90^\circ, -53.1^\circ, 90^\circ$

b **i** $\theta = -90^\circ, 14.5^\circ, 90^\circ, 166^\circ$

ii $\theta = \pm 180^\circ, \pm 53.1^\circ, 0^\circ$

3 $x = -330^\circ, -270^\circ, -210^\circ, -90^\circ, 30^\circ, 90^\circ, 150^\circ, 270^\circ$

4 **a** -0.5

b $x = 210^\circ, 330^\circ$

5 $x = -104.0^\circ, -71.6^\circ, 76.0^\circ, 108.4^\circ$

6 $\theta = 90^\circ, 180^\circ, 270^\circ$

7 $36.4^\circ, 83.5^\circ, 156^\circ$

Work it out 10.2

Solution 2 is correct.

Exercise 10H

1 **a** **i** $x = 33.7^\circ$

ii $x = 59.0^\circ$

b **i** $x = 66.8^\circ$

ii $x = 78.7^\circ$

2 **a** **i** $x = 45^\circ, 135^\circ, 225^\circ, 315^\circ$

ii $x = 54.7^\circ, 125^\circ, 235^\circ, 305^\circ$

b **i** $x = 45^\circ, 135^\circ, 225^\circ, 315^\circ$

ii $x = 0^\circ, 180^\circ, 360^\circ$

3 $x = 67.5^\circ, 158^\circ, 248^\circ, 338^\circ$

4 $\theta = 26.6^\circ, -153^\circ$

5 $x = 0^\circ, 135^\circ, 180^\circ, 315^\circ, 360^\circ$

6 $x = 0^\circ, \pm 180^\circ$

7 $\theta = \pm 41.8^\circ, \pm 138.2^\circ$

8 $\theta = 30^\circ, 150^\circ, 270^\circ$

9 $-14.5^\circ, -90^\circ, -166^\circ$

10 **a** $\frac{1}{3}$

b $t = 70.5^\circ, 289^\circ$

11 **a** $\frac{2}{3}, -\frac{1}{2}$

b $x = 48.2^\circ, 120^\circ, 240^\circ, 312^\circ$

12 **a** Proof

b $x = -153^\circ, -135^\circ, 26.6^\circ, 45^\circ$

Mixed practice 10

1 $-a$

2 $x = -31.8^\circ, 148.2^\circ$

3 $\theta = -135^\circ, -45^\circ, 225^\circ, 315^\circ$

4 $x = 570^\circ, 690^\circ$

5 $\theta = \pm 48.6^\circ, \pm 131^\circ$

6 **a** Proof

b $60^\circ, 300^\circ$

7 **a** Proof

b $\theta = 90^\circ, 120^\circ, 240^\circ, 270^\circ$

8 Eight

9 $a = 5, b = 45^\circ$

10 $x = 48.2^\circ, 120^\circ, 240^\circ, 312^\circ$

11 Proof

12 **a** Proof

b $x = 48.2^\circ, 120^\circ, 240^\circ, 312^\circ$

13 Proof

14 **a** Proof

b $x = 63.4^\circ, 108^\circ, 288^\circ, 243^\circ$

15 Proof

16 $x = -69.1^\circ, -20.9^\circ, 15^\circ, 75^\circ$

17 **a** $k = \pm 4$

b Proof

c **i** One

ii $\theta = \pm 60^\circ, \pm 300^\circ$

iii $k = 5$

iv Seven

Answers

Chapter 11

Before you start...

1 41.8°

2 110°

3 $x = -1, -4$

4 $8.63^\circ, 171^\circ$

Work it out 11.1

Solution 1 is correct.

Exercise 11A

1 **a** **i** $x = 6.04$

ii $y = 14.4$

b **i** $a = 10.6 \text{ cm}$

ii $b = 23.3 \text{ cm}$

2 **a** **i** $x = 49.7^\circ$

ii $y = 59.2^\circ$

b **i** $z = 74.6^\circ$ or 105°

ii $u = 62.0^\circ$ or 118°

c **i** $a = 50.9^\circ$

ii $b = 54.4^\circ$

3 $21.0^\circ, 29.0^\circ, 8.09 \text{ cm}$

4 $10.4 \text{ cm}, 49.9^\circ, 95.1^\circ$ or $2.69 \text{ cm}, 130^\circ, 14.9^\circ$

5 9.94 cm

6 23.3 m

7 Proof

Exercise 11B

1 **a** **i** $x = 5.37$

ii $y = 3.44$

b **i** $z = 8.00$

ii $u = 11.3$

2 **a** **i** $\text{A} = 60.6^\circ$

ii $\text{B} = 120^\circ$

b **i** $\text{C} = 81.5^\circ$

ii $\text{D} = 99.9^\circ$

3 **a** 106°

b 36.2°

4 7.08 km

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5 25.6 km

6 $x = 7.95\text{ cm}$

7 $x = 4.4$

8 $x = 8$

9 $2\sqrt{2} + \sqrt{41}$

Exercise 11C

1 **a** **i** 10.7 cm^2

ii 24.3 cm^2

b **i** 27.6 cm^2

ii 26.2 cm^2

2 **a** $x = 81.7^\circ, 98.3^\circ$

b $y = 60.9^\circ, 119^\circ$

3 $17.7\text{ cm}, 29.7\text{ cm}^2$

4 10 cm

5 $x = 5$

6 $x = 8$

7 $4\sqrt{3}\text{ cm}^2$

Mixed practice 11

1 $58.7^\circ, 121^\circ$

2 **a** 8.09 m

b 6.58 m

3 **a** 11.6 cm

b 49.2cm^2

c 6.49 cm

4 **a** 115°

b 20.3 cm^2

5 $2\sqrt{43}\text{ cm}$

6 7.23 cm^2

7 **a** $\dfrac{23}{32}$

b $\dfrac{3\sqrt{55}}{32}$

c $\dfrac{15\sqrt{55}}{4}$

8 **a** $\dfrac{x^2}{4} + 25 - 5x\cos\theta$

b Proof

c $\theta = 41.4^\circ$

9 **a** 009° or 071°

b 15.5 km

10 10.8 cm

11 a Proof

b $\left[-1, -\frac{\sqrt{3}}{2}\right] \cup \left[\frac{\sqrt{3}}{2}, 1\right]$

c $0^\circ < \theta \leqslant 30^\circ$

Answers

Chapter 12

Before you start...

1 $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

2 $5.39, 21.8^\circ$

3 Two

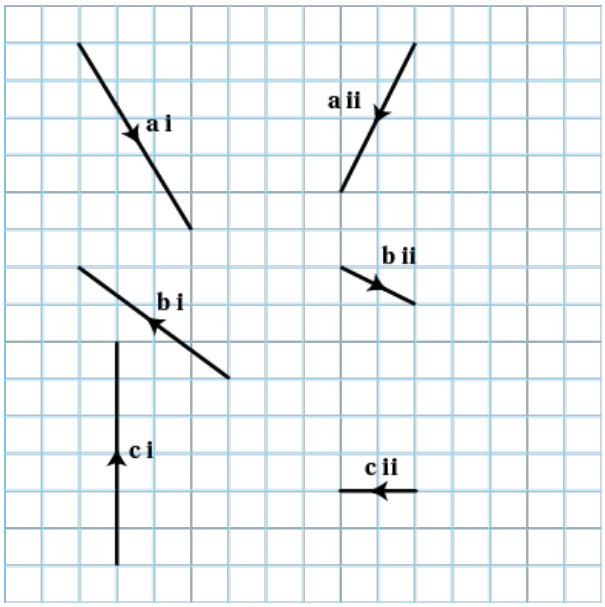
Work it out 12.1

Solution 2 is correct.

Exercise 12A

1 $\mathbf{a} = 2\mathbf{i} + \mathbf{j}, \mathbf{b} = -3\mathbf{j}, \mathbf{c} = -2\mathbf{i} + \mathbf{j}, \mathbf{d} = \mathbf{i} - \mathbf{j}$

2



3 A a i 5.83

ii 4.47

b i 5

ii 2.24

c i 6

ii 2

B a i -59.0°

ii -117°

b i 143°

ii -26.6°

c i 90°

ii 180°

C a i -149°

ii 153°

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b **i** 53.1°

ii -63.4°

c **i** 0°

ii 90°

4 **a** No, magnitude $= \frac{\sqrt{13}}{6}$

b Yes

c Yes

d No, magnitude $= 3$

5 $\pm\frac{1}{\sqrt{29}}$

6 $\pm\frac{1}{\sqrt{13}}$

7 $t = -1$ or -3

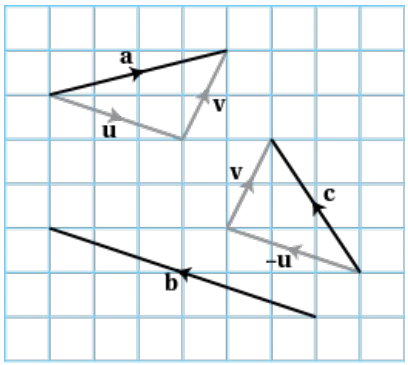
8 **a** 63°

b $5.45\mathbf{i} + 10.7\mathbf{j}$

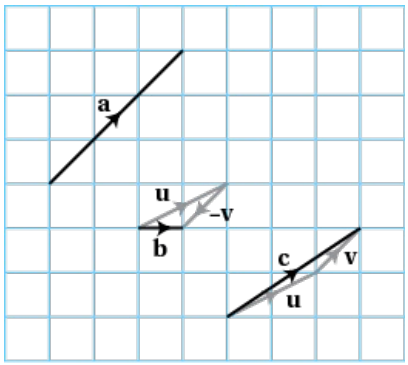
9 $\begin{pmatrix} -3 \\ \sqrt{3} \end{pmatrix}$

Exercise 12B

1 **a**



b



2 **a** **i** $\begin{pmatrix} 21 \\ 3 \end{pmatrix}$

ii $\begin{pmatrix} 20 \\ -8 \end{pmatrix}$

b **i** $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

ii $\begin{pmatrix} 6 \\ -1 \end{pmatrix}$

c **i** $\begin{pmatrix} 11 \\ -3 \end{pmatrix}$

ii $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$

d **i** $\begin{pmatrix} 10 \\ -3 \end{pmatrix}$

ii $\begin{pmatrix} 17 \\ 6 \end{pmatrix}$

3 **a** **i** $-6\mathbf{i}$

ii $4\mathbf{i} + 8\mathbf{j}$

b **i** $\mathbf{i} - 3\mathbf{j}$

ii $2\mathbf{j}$

c **i** $4\mathbf{i}$

ii $5\mathbf{i} - 4\mathbf{j}$

4 **a** $-4\mathbf{i} + 2\mathbf{j}$

b $-\frac{8}{3}\mathbf{i} + \frac{4}{3}\mathbf{j}$

c $4\mathbf{i} - 3\mathbf{j}$

d $-\frac{1}{2}\mathbf{i} + \mathbf{j}$

5 **a** **i** Parallel

ii Parallel

b **i** Not parallel

ii Not parallel

c **i** Parallel

ii Not parallel

6 $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

7 $t = -2$

8 $p = -\frac{4}{3}$

9 $\lambda = -2$

10 $q = \frac{1}{2} - p$

11 $\frac{11}{7}$

12 $\pm \begin{pmatrix} 12 \\ -16 \end{pmatrix}$

13 $\pm \begin{pmatrix} \frac{6}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} \end{pmatrix}$

14 $\pm \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

15 a Proof

b $t = 0$

Work it out 12.2

Solution 1 is correct.

Exercise 12C

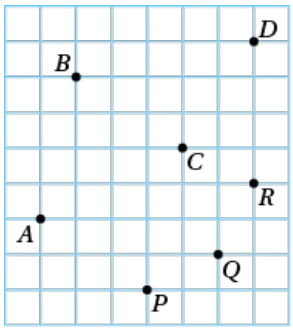
1 a i $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

ii $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

iii $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

iv $\begin{pmatrix} -6 \\ -5 \end{pmatrix}$

b i, **ii**, **iii** On grid diagram



2 a i $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

ii $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$

b i $\begin{pmatrix} -6 \\ 1 \end{pmatrix}$

ii $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

c i $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$

ii $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

3 a i $(2, 1)$

ii $(-4, -4)$

b i $(1, -2)$

ii $(2, 0)$

c i $(11, -15)$

ii $(-9, -1)$

d i $(-8, 7)$

ii $(-7, 30)$

4 a i $\sqrt{205}$

ii $\sqrt{2}$

b **i** $3\sqrt{2}$

ii $\sqrt{137}$

5 **a** $\overrightarrow{PR} = 5\mathbf{i} - \mathbf{j}$

b $\overrightarrow{SP} = -7\mathbf{i} - 4\mathbf{j}$

6 **a** $\overrightarrow{AC} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \overrightarrow{DB} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}$

b $(0,\ 3)$

7 $(3, -19)$

8 **a** $\overrightarrow{AC} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$

b $\overrightarrow{BP} = \begin{pmatrix} r-6 \\ q+3 \end{pmatrix} \overrightarrow{CP} = \begin{pmatrix} r-8 \\ q+4 \end{pmatrix}$

c $CP = 2\sqrt{5}$

Exercise 12D

1 **a** **i** $\mathbf{b}$

ii $\mathbf{a} + \mathbf{b}$

b **i** $-\mathbf{a}$

ii $-\frac{1}{2}\mathbf{a}$

c **i** $\mathbf{a} + \frac{1}{2}\mathbf{b}$

ii $\frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$

2 **a** **i** $\mathbf{a} + \frac{4}{3}\mathbf{b}$

ii $\mathbf{a} + \frac{1}{2}\mathbf{b}$

b **i** $-\frac{3}{2}\mathbf{a} + \mathbf{b}$

ii $-\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{a}$

c **i** $\frac{3}{2}\mathbf{a} - \mathbf{b}$

ii $-\frac{4}{3}\mathbf{b} + \frac{1}{2}\mathbf{a}$

3 **a** **i** Collinear; $AB : BC = 1 : 13$

ii Collinear; $AB : BC = 2 : 5$

b **i** Not collinear

ii Not collinear

c **i** Not collinear

ii Collinear; $AB : BC = 1 : 2$

d **i** Collinear; $AB : BC = 1 : 1 - 2a$

ii Not collinear

4 $2\mathbf{i} + 12\mathbf{j}$

5 $AB : BC = 1 : 2$

6 **a** $\begin{pmatrix} 8 \\ -4 \end{pmatrix}$

b No

7 **a** $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$

b $\overrightarrow{AB} = \overrightarrow{DE} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

8 Proof

9 $\begin{pmatrix} 0.8 \\ 1.8 \end{pmatrix}$

10 $\overrightarrow{MN} = \overrightarrow{QP} = \frac{1}{2}(\mathbf{c} - \mathbf{a})$

11 **a** $C(10 - 8x,\ 1 + 6x)$

b $x = \frac{6}{25}$

12 **a** $\frac{1}{2}(\mathbf{p} + \mathbf{q})$

b Proof

13 **a** Proof

b Proof

Mixed practice 12

1 $3\sqrt{2}$

2 **a** $6\mathbf{i} - 5\mathbf{j}$

b $(0,\ 1)$

3 **a** $\begin{pmatrix} 4.5 \\ -1 \end{pmatrix}$

b $\frac{3}{2}\sqrt{41}$

4 $6i - 2\mathbf{j}$

5 $\overrightarrow{MN} = \mathbf{b} - \frac{1}{3}\mathbf{a}$

6 Proof

7 $\sqrt{9 + 16\cos^2}$

8 $\pm\left(\frac{9}{\sqrt{97}}\mathbf{i} + \frac{4}{\sqrt{97}}\mathbf{j}\right)$

9 **a** $(4, -6)$

b $\overrightarrow{\mathrm{OP}} = \sqrt{65},\ \overrightarrow{\mathrm{ON}} = 2\sqrt{13},\ \overrightarrow{\mathrm{PN}} = \sqrt{13}$

10 **a** $\overrightarrow{MN} = \frac{1}{2}(\mathbf{c} - \mathbf{a}),\ \overrightarrow{PQ} = \frac{1}{2}(\mathbf{a} - \mathbf{c})$

b Parallelogram

11 **a** $\mathbf{a} + \mathbf{b}$

b $\mathbf{a} + \frac{1}{2}\mathbf{b}$

c $t\mathbf{a} + (t - 1)\,\mathbf{b};\ t = \frac{2}{3}$

12 a $\begin{pmatrix} 5-3t \\ 2+t \end{pmatrix}; \sqrt{10t^2-26t+29}$

b $\frac{11\sqrt{10}}{10}$

13 a $1:1$

b Diagonals bisect each other

Answers

Chapter 13

Before you start...

1 **a** $x^{\frac{2}{3}} - \frac{5}{2}x^{-1}$

b $2x^{-\frac{1}{2}} + x^{\frac{1}{2}}$

2 **a** $x \leqslant -3$

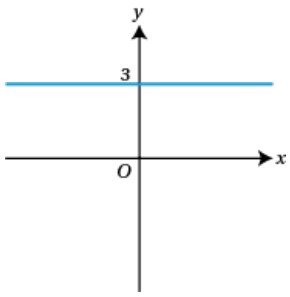
b $x \leqslant -2, x \geqslant 6$

3 $-\frac{3}{4}$

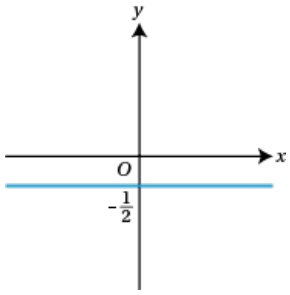
4 $x^3 + 6x^2 + 12x + 8$

Exercise 13A

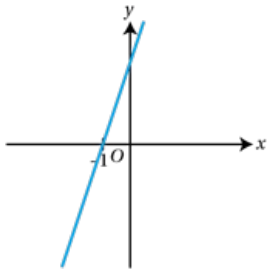
1 **a** **i**



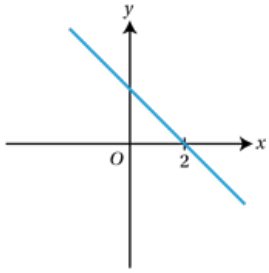
ii



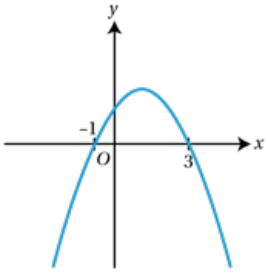
b **i**



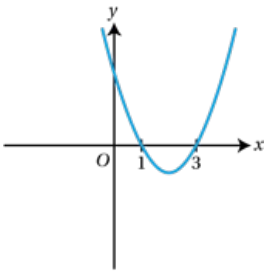
ii



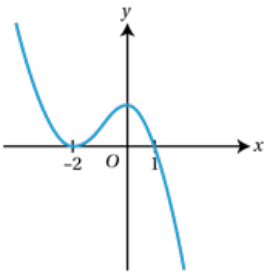
c i



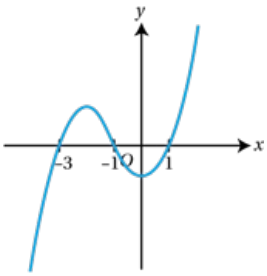
ii



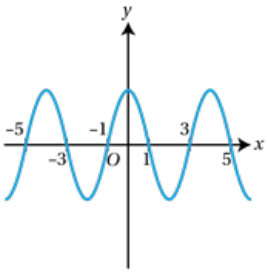
d i



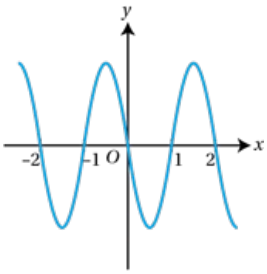
ii



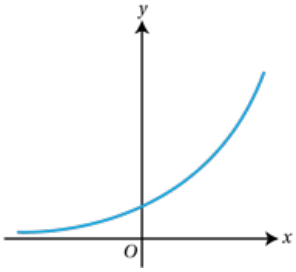
e i



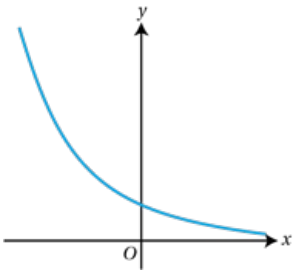
ii



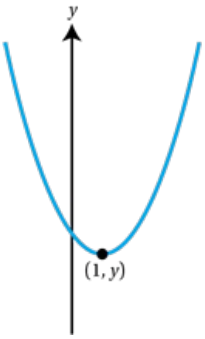
f i



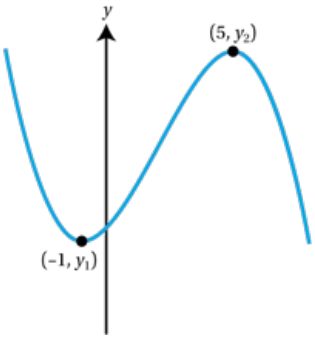
ii



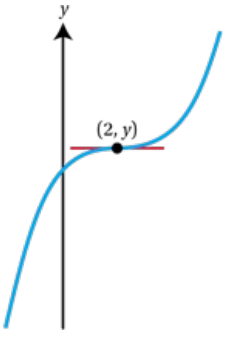
2 a



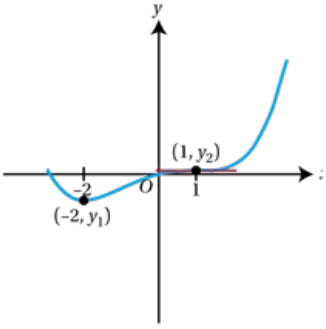
b



c



d



3 **a** False

b False

c Always true

d False

e False

f False

Exercise 13B

1 Proof

2 -4

3 $6x$

4 Proof

5 $2x - 6$

6 $2x - 3$

7 **a** $x^3 + 3x^2h + 3xh^2 + h^3$

b $3x^2$

8 $9x^2$

9 **a** $x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

b Proof

10 **a** $12a^2 + 12ah + 4h^2$

b $12a^2$

c It is the gradient of the tangent to the curve at A.

11 Proof

12 Proof

Exercise 13C

1 **a** **i** $y' = 4x^3$

ii $y' = 6x^5$

b **i** $y' = 21x^6$

ii $y' = -20x^4$

c **i** $y' = -1$

ii $y' = 3$

d **i** $y' = 0$

ii $y' = 0$

e **i** $y' = 2x^5$

ii $y' = -\frac{3}{2}x$

f **i** $y' = 12x^2 - 10x + 2$

ii $y' = 8x^3 + 9x^2 - 1$

g **i** $y' = 7 - \frac{3}{2}x^2$

ii $y' = -20x^3 + x^4$

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2 **a** **i** $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$

ii $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$

b **i** $f'(x) = 4x^{-\frac{1}{2}}$

ii $f'(x) = 8x^{\frac{1}{3}}$

c **i** $f'(x) = \frac{1}{3}x^{-\frac{1}{4}}$

ii $f'(x) = \frac{1}{2}x^{-\frac{1}{6}}$

d **i** $f'(x) = 12x^3 - 6x^{-\frac{3}{5}}$

ii $f'(x) = 3x^2 - x^{\frac{2}{3}} + \frac{2}{3}x^{-\frac{1}{2}}$

e **i** $f'(x) = -x^{-2}$

ii $f'(x) = 3x^{-4}$

f **i** $f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$

ii $f'(x) = -\frac{2}{3}x^{-\frac{5}{3}}$

g **i** $f'(x) = 8x^{-\frac{7}{3}}$

ii $f'(x) = 6x^{-\frac{7}{4}}$

h **i** $f'(x) = 5 + \frac{4}{3}x^{-\frac{7}{2}}$

ii $f'(x) = x^{-\frac{10}{7}} - 8x^{-7}$

3 **a** **i** $10.8x^{1.7}$

ii $10.5x^{2.5}$

b **i** $3x^{-0.4}$

ii $-1.2x^{-0.6}$

c **i** $1.4x^{-1.7}$

ii $-6x^{-2.5}$

4 $\frac{dy}{dx} = 2x - \frac{3}{2}x^{-\frac{1}{2}}$

5 $f'(x) = 12x^2 - 6x - 3x^{-\frac{5}{2}}$

6 $f'(x) = -4x^{-\frac{4}{3}} + \frac{1}{3}x^{-\frac{3}{5}}$

Work it out 13.1

Solution 2 is correct.

Exercise 13D

1 **a** **i** $y' = \frac{1}{3}x^{-\frac{2}{3}}$

ii $y' = \frac{1}{5}x^{-\frac{4}{5}}$

b **i** $y' = 2x^{-\frac{3}{4}}$

ii $y' = \frac{1}{6}x^{-\frac{1}{2}}$

c **i** $y' = x^{-2}$

ii $y' = -4x^{-5}$

d **i** $y' = -6x^{-3}$

ii $y' = 4x^{-11}$

e **i** $y' = -\frac{1}{2}x^{-\frac{3}{2}}$

ii $y' = -\frac{1}{3}x^{-\frac{4}{3}}$

f **i** $y' = 2x^{-\frac{6}{5}}$

ii $y' = -\frac{2}{3}x^{-\frac{5}{4}}$

2 **a** **i** $\mathrm{f}'(x) = 4x + 5$

ii $\mathrm{f}'(x) = 6x - 15$

b **i** $\mathrm{f}'(x) = 6x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}}$

ii $\mathrm{f}'(x) = \frac{4}{3}x^{\frac{1}{3}} - \frac{1}{3}x^{-\frac{2}{3}}$

c **i** $\mathrm{f}'(x) = 1 + 8x + 6x^{\frac{1}{2}}$

ii $\mathrm{f}'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-\frac{3}{4}}$

d **i** $\mathrm{f}'(x) = 2x - 2x^{-3}$

ii $\mathrm{f}'(x) = 2x + 8x^{-3}$

3 **a** **i** $\mathrm{f}'(x) = 2x^{-2}$

ii $\mathrm{f}'(x) = -\frac{1}{2}x^{-2} + 2$

b **i** $\mathrm{f}'(x) = -\frac{3}{2}x^{-\frac{5}{2}} + 6x^{-3}$

ii $\mathrm{f}'(x) = \frac{3}{2}x^{\frac{1}{2}} - 2x^{-\frac{3}{2}}$

4 $y' = 9x^2 - 8x$

5 $y' = 7x^{\frac{5}{2}} + 4x^{-\frac{1}{2}}$

6 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{5}x^{-\frac{1}{5}}$

7 $\mathrm{f}'(x) = -2x^{-\frac{7}{4}}$

8 **a** $y = 3x^3 - 2x^{-1}$

b $y' = 9x^2 + 2x^{-2}$

9 Proof

10 $\mathrm{f}'(x) = \frac{15}{2}x^{\frac{2}{3}} - \frac{1}{2}x^{-\frac{4}{3}}$

11 $y' = -2x^{-\frac{3}{2}} + 12x^{-2} - \frac{27}{2}x^{-\frac{5}{2}}$

Work it out 13.2

Solution 2 is correct.

1 **a** $\frac{\mathrm{d}z}{\mathrm{d}t}$

b $\frac{\mathrm{d}Q}{\mathrm{d}P}$

c $\frac{\mathrm{d}R}{\mathrm{d}m}$

d $\frac{\mathrm{d}V}{\mathrm{d}t}$

e $\frac{\mathrm{d}y}{\mathrm{d}x}$

f $\frac{\mathrm{d}^2z}{\mathrm{d}y^2}$

g $\frac{\mathrm{d}^2H}{\mathrm{d}m^2}$

2 **a** **i** $\frac{5}{3}x^{-\frac{2}{3}}$

ii $15q^4$

b **i** $3-7t^{-2}$

ii $1-c^{-2}$

c **i** $18+6x$

ii $6t^{-3}$

3 **a** **i** 30

ii $\frac{227}{36}$

b **i** 7

ii -29999.8

c **i** 12

ii -10

d **i** 24

ii 32

e **i** 6

ii $\frac{7}{2\sqrt{6}}$

4 **a** **i** $2ax+1-a$

ii $3x^2$

b **i** $\frac{1}{2}\sqrt{\frac{b}{a}}$

ii $6a^2v$

5 **a** **i** 54

ii 384

b **i** 8

ii $\frac{1}{108}$

c **i** 0

ii 42

6 **a** **i** $x = 2$

ii $x = 1$

b **i** $y = \pm 6$

ii $y = 6$

c **i** $x = -3$

ii $x = 2$

7 **a** **i** $x > \frac{1}{2}$

ii $x < -1$

b **i** $x < -1$ or $x > 1$

ii $-\frac{4}{3} < x < 0$

c **i** $x < 0$

ii $x > \frac{1}{6}$

8 **a** $-\frac{1}{18}$

b Increasing

9 $\frac{3}{16}$

10 $-\frac{2}{3} < x < \frac{2}{3}$

11 $x = \frac{7}{6}$

12 $x = \pm\sqrt{2}$

13 Proof

14 $(1,\, 0)$ and $(3,\, 4)$

15 $x < 0$ or $x > 4$

16 $a = -6, b = 2$

17 $x > -\frac{1}{3}$

18 $-1 - \frac{2\sqrt{3}}{3} < x < -1 + \frac{2\sqrt{3}}{3}$

19 $2 < x < 3$

20 $n!$

Mixed practice 13

1 $\frac{3}{2}x^{-\frac{1}{2}} + x^{-\frac{3}{2}}$

2 $\frac{\mathrm{d}y}{\mathrm{d}x} = -12x^2 + 24x + 1$

3 $-\frac{1}{2}$

4 **a** $\mathrm{f}'(x) = \frac{3}{2}x^{-\frac{1}{2}} + x^{-\frac{3}{2}}$

b $\frac{7}{8}$

5 $b = 8, c = -7$

6 **a** $\frac{9}{4}$

b Positive gradient; increasing

7 $x > \frac{2}{3}$

8 $\frac{109}{54}$

9 **a** $-\frac{10}{x^3} + \frac{1}{4x^2} + 1$

b $\frac{30}{x^4} - \frac{1}{2x^3}$

10 $a = -10$

11 $\frac{11}{4}$

12 D

13 **a** 2

b 1.3

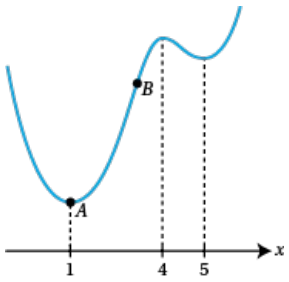
c Anything between 2 and 2.3

14 $3x^2 - 5$

15 **a** 0

b Increasing

c



16 $\left(\frac{1}{16}, \frac{7}{16}\right)$

17 -54

18 $a = 4.8, b = -1.8$

19 $x = 1, 25$

20 $\frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Answers

Chapter 14

Before you start...

1 **a** $\frac{6x}{5} + \frac{4x^{-3}}{5}$

b $-\frac{3}{4}x^{-\frac{3}{2}}$

2 $\frac{9}{16}$

3 $y = 3x + 7$

4 $y = -\frac{4}{3}x - \frac{1}{3}$

Work it out 14.1

Solution 2 is correct.

Exercise 14A

1 **a** **i** Tangent $y + 2 = 0$, normal $x - 1 = 0$

ii Tangent $4x + y - 8 = 0$, normal $x - 4y + 15 = 0$

b **i** Tangent $x + 3y + 6 = 0$, normal $3x - y + 8 = 0$

ii Tangent $3x - 2y - 9 = 0$, normal $4x + 6y + 1 = 0$

c **i** Tangent $x - 4y + 36 = 0$, normal $4x + y - 26 = 0$

ii Tangent $x + 6y - 32 = 0$, normal $6x - y - 44 = 0$

2 $11x - 4y - 4 = 0$

3 $32x + 70y - 583 = 0$

4 $(162, 0)$

5 $\frac{9}{4}$

6 $y = 5, y = 6$

7 1.55 or -0.215

8 $a = -6, b = 8$

9 $p = -1,\ q = -\frac{1}{6}$

10 $\left(\frac{1}{4}, \frac{7}{2}\right)$

11 $(-1, -4)$

12 $(0.410, 0.348)$

13 $c = -2$

14 Proof; $\text{area} = 2k$

15 Proof

Exercise 14B

1 **a** **i** $(0,\ 0)$ local maximum; $\left(\frac{10}{3}, -\frac{500}{27}\right)$ local minimum

ii $(0,\ 1)$ local minimum; $(16,\ 2049)$ local maximum

b **i** $(0,\ 0)$ local maximum; $(\pm 2, -16)$ local minima

ii $(0,\ 3)$ local maximum; $(\pm 1, 2)$ local minima

c **i** No stationary points

ii $(1.14, 11.8)$ local minimum

d **i** $\left(\frac{1}{9},\ \frac{1}{3}\right)$ local maximum

ii $(0.303, 2.73)$ local minimum

2 $(-4, 92)$ local maximum; $(2, -16)$ local minimum

3 Consider discriminant of the derivative expression

4 **a** $x = \pm\frac{1}{3}$

b $\left(\frac{1}{3},\ 6\right)$ local minimum; $\left(-\frac{1}{3}, -6\right)$ local maximum

5 $\left(\frac{1}{4}, -\frac{1}{4}\right)$ local minimum

6 **a** $(0,\ 0)$ and $(1, -1)$

b $(0,\ 0)$ local maximum; $(1, -1)$ local minimum

7 **a** $x = \frac{4}{3}$

b Local maximum at $x = -8$

c Local minimum at $x = \frac{4}{3}$

8 $a = 3, b = 12$

9 $0 \leqslant \mathrm{f}(x) \leqslant 54$

10 Proof

11 **a** $(0,\ 6), (1,\ 11), (3, -21)$

b $6 < c < 11$

12 $(0,\ 0)$ local minimum; $\left(-\frac{4}{k},\ \frac{32}{k^2}\right)$ local maximum if $k \neq 0$

Exercise 14C

1 **a** **i** 2

ii $\frac{49}{12}$

b **i** $2\sqrt{3}$

ii $5\sqrt{2}$

c **i** $5\sqrt{2}$

ii $\frac{8}{3}$

2 **a** $225m^2$

b 60 m

3 **a** Proof

b **i** Proof

ii $r = 10$ cm

4 **a** Proof

b $x = 2$

5 Proof

6 **a** $V = 225r - \pi r^3$

b $733\,\text{cm}^3$

7 **a** $V = \dfrac{\left(243 - x^2\right)x}{2}$

b $729\,\text{cm}^3$

8 **a** Proof

b $166\,\text{cm}^2$

c Proof

9 **a** Proof

b $r = 5.56, h = 7.86$

10 **a** 3 and 3

b 0 and 6

11 Maximum time 4 min; minimum time 2 min

12 **a** Proof

b Proof

13 Proof

14 $\left(\sqrt{\dfrac{7}{2}},\ \dfrac{7}{2}\right)$

15 **a** Proof

b $V = \pi(6 + x)^2\left(\dfrac{12}{x} + 2\right)$

c 486π

Mixed practice 14

1 C

2 $19x - 4y = 28$

3 $x = 2$

4 $x = 2 + \sqrt{2}$ local minimum; $x = 2 - \sqrt{2}$ local maximum

5 **a** $(0, -1)$ $(2,\ 3)(4, -1)$

b $(0, -1)$ local minimum $(2,\ 3)$ local maximum $(4, -1)$ local minimum

6 $\left(\dfrac{2\sqrt{3}}{3},\ 0\right)$

7 **a** Proof

b $50m^2$

8 $a = 3$

9 $a = 2, b = -13$

10 $(1, -2)$ local maximum

$(4, -3)$ local minimum

11 **a** $\left(-\sqrt{3},\ 6\sqrt{3}\right)$ and $\left(\sqrt{3}, -6\sqrt{3}\right)$

b Minimum: $-6\sqrt{3} \leqslant \mathrm{f}(x) \leqslant 80$

12 $\frac{3}{4}$

13 **a** 4 litres

b 41.5 litres

c 20 seconds

14 **a** Proof

b $40\sqrt{2}\,\mathrm{m}$

15 **a** $(-1, 7)$

b Minimum

16 **a** Proof

b $3a + b = 6$

c $a = 1, b = 3$

d $(-7, -192)$

17 $a = 3^{-\frac{1}{4}}$

18 $a = 1.5$

Answers

Chapter 15

Before you start...

1 **a** $x = 0, \frac{4}{3}$

b $x = 0, 4$

c $x = 0, \ \pm\sqrt{5}$

2 **a** $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x - 1$

b $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^{-\frac{1}{2}} + 2x^{-\frac{5}{3}}$

3 **a** $\frac{\mathrm{d}y}{\mathrm{d}x} = -3x^{-3}$

b $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{9}{2}x^{\frac{1}{2}}$

Exercise 15A

1 **a** **i** $y = \frac{1}{6}x^6 + c$

ii $y = \frac{1}{7}x^7 + c$

b **i** $y = x^3 + c$

ii $y = -x^5 + c$

c **i** $y = \frac{1}{2}x^4 + c$

ii $y = -\frac{3}{8}x^8 + c$

d **i** $y = \frac{1}{2}x^2 + c$

ii $y = -2x^2 + c$

e **i** $y = c$

ii $y = x + c$

f **i** $y = -\frac{1}{6}x + c$

ii $y = \frac{1}{27}x^9 + c$

g **i** $y = \frac{2x^7}{5} + c$

ii $y = \frac{5x^2}{7} + c$

h **i** $y = \frac{1}{2}x^6 - \frac{1}{2}x^2 + 4x + c$

ii $y = \frac{2}{5}x^5 + x^4 - 4x^2 - x + c$

i **i** $y = \frac{x^3}{6} + x^2 - \frac{3x}{4} + c$

ii $y = \frac{x^3}{3} - \frac{x^6}{16} + c$

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2 **a** **i** $\frac{3}{5}x^{\frac{5}{3}} + c$

ii $\frac{3}{4}x^{\frac{4}{3}} + c$

b **i** $3x^{\frac{5}{3}} + c$

ii $8x^{\frac{7}{4}} + c$

c **i** $x^{\frac{5}{4}} + c$

ii $-x^{\frac{3}{2}} + c$

d **i** $-\frac{x^{\frac{7}{3}}}{2} + c$

ii $\frac{2x^{\frac{7}{5}}}{3} + c$

e **i** $\frac{5}{2}x^2 - 2x^{\frac{7}{2}} + c$

ii $6x + \frac{x^2}{6} - 8x^{\frac{3}{2}}$

f **i** $-x^{-1} + c$

ii $-\frac{1}{2}x^{-2} + c$

g **i** $\frac{3}{2}x^{\frac{2}{3}} + c$

ii $\frac{4}{3}x^{\frac{3}{4}} + c$

h **i** $-12x^{\frac{1}{3}} + c$

ii $10x^{-\frac{1}{2}} + c$

i **i** $2x^{\frac{1}{2}} + \frac{2x^{-3}}{5} + c$

ii $-\frac{3x^{-4}}{4} + \frac{42x^{\frac{5}{6}}}{5} + c$

3 **a** **i** $3t + c$

ii $7z + c$

b **i** $\frac{q^6}{6} + c$

ii $\frac{r^{11}}{11} + c$

c **i** $\frac{15}{2}g^{\frac{8}{5}} + c$

ii $\frac{4}{3}y^{\frac{9}{2}} + c$

d **i** $20a^{\frac{1}{4}} + c$

ii $-21p^{-\frac{1}{3}} + c$

4 $\mathrm{f}(x) = \frac{1}{8}x^4 + 9x^{-\frac{2}{3}} + 2x + c$

5 $\frac{2}{5}x^{\frac{5}{2}} + 2x^{-\frac{1}{2}} + c$

6 $\frac{1}{12}x^6 + \frac{3}{4}x^{-2} + \frac{1}{4}x^2 + c$

Work it out 15.1

Student A is correct.

Exercise 15B

1 **a** **i** $\frac{2}{3}x^{\frac{3}{2}} + c$

ii $\frac{4}{5}x^{\frac{5}{4}} + c$

b **i** $5x^{\frac{6}{5}} + c$

ii $\frac{1}{8}x^{\frac{4}{3}} + c$

c **i** $-\frac{1}{2}x^{-2} + c$

ii $\frac{1}{6}x^{-6} + c$

d **i** $2x^{-4} + c$

ii $-\frac{2}{3}x^{-5} + c$

e **i** $2x^{\frac{1}{2}} + c$

ii $\frac{3}{2}x^{\frac{2}{3}} + c$

f **i** $-18x^{\frac{5}{6}} + c$

ii $\frac{3}{2}x^{\frac{3}{4}} + c$

2 **a** **i** $\frac{2}{3}x^3 + \frac{5}{2}x^2 + 3x + c$

ii $\frac{3}{4}x^4 - 2x^3 + c$

b **i** $2x^{\frac{5}{2}} + \frac{8}{3}x^{\frac{3}{2}} + c$

ii $\frac{3}{5}x^{\frac{10}{3}} + \frac{9}{4}x^{\frac{4}{3}} + c$

c **i** $\frac{2}{3}x^{\frac{3}{2}} + 3x^3 + \frac{8}{3}x^{\frac{9}{4}} + c$

ii $\frac{1}{2}x^2 + \frac{1}{5}x^5 - \frac{4}{7}x^{\frac{7}{2}} + c$

d **i** $\frac{1}{3}x^3 - x^{-1} - 2x + c$

ii $\frac{4}{3}x^3 + \frac{1}{3}x^{-3} + c$

3 **a** **i** $-7x^{-1} + x^{-2} + c$

ii $-\frac{1}{4}x^{-2} - \frac{5}{4}x^{-1} + c$

b **i** $-2x^{-\frac{1}{2}} + 3x^{-1} + c$

ii $\frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + c$

4 $4x^{\frac{5}{2}} + c$

5 $\frac{3}{10}x^{\frac{10}{3}} - 4x^3 + c$

6 $4x^{\frac{9}{4}} + c$

7 $x^{\frac{1}{3}} + c$

8 $2x^{\frac{1}{2}} - x + c$

9 $-\frac{1}{6}x^{-2} - \frac{1}{12}x^{-3} + c$

10 a $\mathrm{f}(x) = 6x^{-\frac{1}{3}} + x^{\frac{2}{3}}$

b $9x^{\frac{2}{3}} + \frac{3}{5}x^{\frac{5}{3}} + c$

11 a $a = \frac{1}{2}, b = -2$

b $\frac{1}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + c$

12 Proof

13 $\frac{1}{12}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} + c$

Work it out 15.2

Solution 3 is correct.

Exercise 15C

1 a i $y = \frac{x^2}{2} + 5$

ii $y = 2x^3 + 5$

b i $y = \frac{-1}{x}$

ii $y = -\frac{1}{x} + 4$

c i $y = \frac{3}{2}x^2 - 5x + 10$

ii $y = 3x - \frac{1}{2}x^4 + \frac{5}{2}$

d i $y = 2x^{\frac{3}{2}} - 56$

ii $y = 2\sqrt{x} + 4$

2 a i $y = x^3 - x + 5$

ii $y = 3x^{-1} + 2x - 2$

b i $\mathrm{f}(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3 + 3x - \frac{19}{3}$

ii $\mathrm{f}(x) = \frac{1}{4}x^4 - \frac{1}{6}x^3 + \frac{5}{2}x + \frac{89}{12}$

3 $y = \frac{x^2}{2} + \frac{1}{x} + \frac{3}{2}$

4 $\mathrm{f}(x) = \frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - \frac{14}{3}$

5 $\frac{149}{3}$

6 $y = 2x^2 - 16$

7 **a** $x = -2$

b Proof

8 $y = x^2 - \frac{x^4}{12} + \frac{10x}{3} - \frac{5}{4}$

9 858

10 $y = \frac{1}{x} + \frac{5}{2}$

Exercise 15D

1 **a** **i** 320

ii 420.2

b **i** 0

ii 36

c **i** 46

ii 48

d **i** 28.5

ii $\frac{33}{2}$

e **i** $\frac{76}{3}$

ii $\frac{585}{2}$

f **i** 36

ii 2

g **i** $\frac{5}{2}$

ii $-\frac{26}{3}$

2 $-60 + 28\sqrt{2}$

3 $a = 2, b = 10$

4 $2k + \frac{1}{k} - 3$

5 $8 - a^4 - a^3$

6 $a = 16$

7 $p = -\frac{10}{3}$ or $p = -1$

8 20

9 $\frac{15}{2}$

10 **a** **i** 16

ii -16

b **i** $a^2 - b^2$

ii $b^2 - a^2$

c $\int_b^a \mathrm{f}(x)\mathrm{d}x = -\int_a^b \mathrm{f}(x)\mathrm{d}x$

11 **a** $\frac{a^3}{3} - \frac{1}{3}$

b a^2

c **i** $\mathrm{f'}(a) = a^2$

ii $\mathrm{f'}(a) = a^2$ the same

d **i** $4a^2 - 2a$

ii $\dfrac{3}{a^2}$

iii $1 - 3a^2$

Work it out 15.3

Solution 3 is correct.

Exercise 15E

1 **a** **i** $\dfrac{7}{3}$

ii $\dfrac{1}{4}$

b **i** $\dfrac{2}{3}$

ii $\dfrac{22}{3}$

c **i** $\dfrac{11}{4}$

ii $\dfrac{79}{6}$

2 **a** **i** 45

ii $\dfrac{17}{3}$

b **i** $\dfrac{32}{3}$

ii $\dfrac{4}{3}$

c **i** $\dfrac{243}{2}$

ii $\dfrac{1}{2}$

3 2

4 $k = 9$

5 **a** 6

b $\dfrac{22}{3}$

6 **a** $(x - 2)(x - 5)$

b $\dfrac{9}{2}$

7 **a** $(0,\ 0)\ \ (k,\ 0)$

b $k = 2$

8 **a** 15

b 18

9 96

10 108

11 $\frac{1}{6}p\sqrt{p}$

12 $\frac{a^3}{48}$

Tip

(Question 7) Instead of evaluating both areas, you could simply write $\int_0^3 y\,\mathrm{dx} = 0$

Mixed practice 15

1 $y = \frac{3}{2}x^2 - \frac{2}{3}x^{\frac{3}{2}} - \frac{59}{3}$

2 $-\frac{1}{x} + 2\sqrt{x} + c$

3 $\frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + 2x - x^2 + \frac{26}{15}$

4 **a** $-12 + 8\sqrt{3}$

b $12 - 4\sqrt{3}$

5 **a** $\frac{1}{3}x^3 - x^2 + 5x + c$

b $\frac{1}{3}x^3 - x^2 + 5x - 4$

6 $4x + 5$

7 **a** $\frac{1}{4}x^4 - 2x^3 + 2x^2 - 24x + c$

b $\frac{28}{5}$

8 26

9 $a = 4$

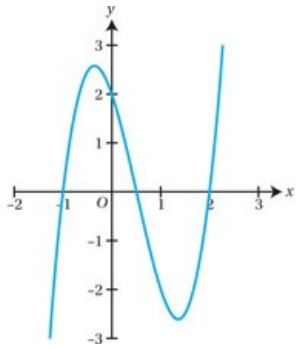
10 **a** $a = \sqrt{2}$

b $\frac{1}{2}$

11 $\frac{4k^3}{3}$

12 **a** **i** $(x-2)(2x-1)(x+1)$

ii



b $\frac{81}{16}$

13 **a** **i** a^{n+1}

ii $\frac{b^{n+1} - a^{n+1}}{n+1}$

b $n = 3$

14 a **i** $A(-a, 0), B(a, 0), C(0, a^2)$

ii $\frac{1}{3}a^3$

b $\frac{1}{3}a^3$

15 Proof

16 a A (local) minimum

b $x^3 + 3x^2 - 45x + 100$

17 $\frac{27}{2}$

Focus on ... 2

Focus on ... Proof 2

1 Proof

2 Discussion

3 Proof

Focus on ... Problem solving 2

1 $1200\ \text{kg}$

2 $20\ \text{kg}$

3 **a** $\sqrt{65}$

b 29.7°

Focus on ... Modelling 2

1 **a** $7:11\ \text{am}$

b $3:44\ \text{am}$

2 5 h35 min after local noon

3 **a** 12 h29 min

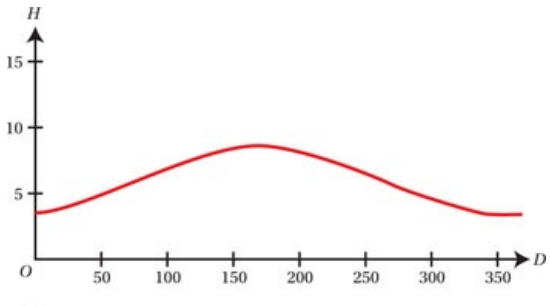
b 11 h7 min

4 13 h56 min

5 Proof

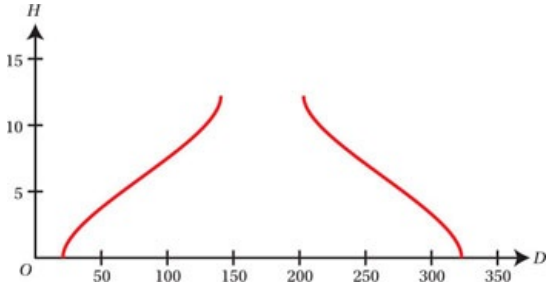
6 No sunrise or sunset on those days ($L > 66.5$ corresponds to places north of the Arctic Circle)

7 **a**

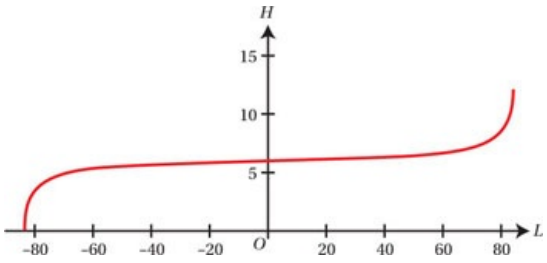


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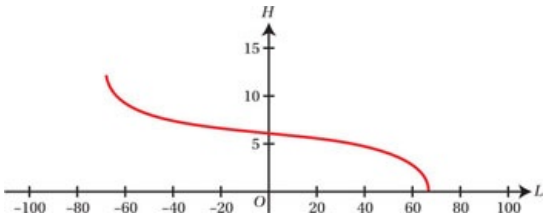
b



8 **a**



b



9 Discussion (Dcould be $59, 59.5$ or 60)

10 Discussion

11 Division by 15 rescales the answer from $[0, 180]$ to $[0, 12]$; 23.44 is the inclination of the Earth's axis; 284 makes January Day 1 (it is 284 days after the Spring Equinox); $\frac{360}{356}$ rescales from $[0, 356]$ to $[0, 360]$ so that the value is in degrees.

12 Discussion (for example: the Earth is spherical; the Sun is modelled as a point)

Cross-topic review exercise 2

1 $a = 2, b = -7$

2 180°

3 **a** $\overrightarrow{MP} = \frac{1}{3}a + \frac{3}{4}b, \overrightarrow{QN} = \frac{1}{2}b + \frac{2}{9}a$

b Proof

4 $\theta = 30^\circ, 120^\circ$

5 **a** $(x + 1)(x - 1)(x - 5)$

b $\frac{148}{3}$

6 **a** $a = \frac{3}{2},\ b = \frac{1}{2},\ c = -\frac{1}{2}$

b $y = -4x + 14$

c **i** $P\left(\frac{7}{2},\ 0\right), Q(0, 14)$

ii $\frac{49}{2}$

7 **a** Proof

b **i** Proof

ii 225°

8 $\frac{20}{3}$

9 $4+2\sqrt{7}=9.29$

10 a $4x^3h+6x^2h^2+4xh^3+h^4$

b Proof

11 a $\mathrm{f}(x)=x^2-6x+8$

b $y=\frac{x}{2}-1$

c $\left(\frac{9}{2},\frac{5}{4}\right)$

12 a $a=\frac{1}{4},p=4,q=5$

b $M\left(0,9\right),N\left(\frac{16}{3},\frac{49}{9}\right)$

c $\frac{16}{3}$

13 $\left(\frac{3}{8},\frac{631}{64}\right)$

14 a $x=1,\ \pm\sqrt{3}$

b $x=45^\circ,60^\circ,120^\circ$

15 $\frac{\sqrt{8}\ 2\sqrt{2}}{3}$

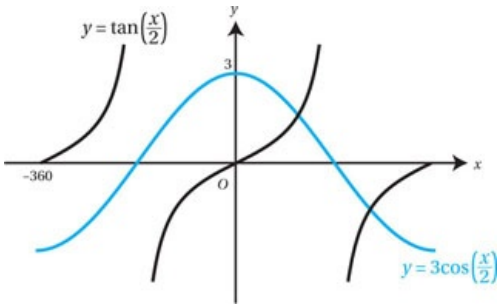
16 a Proof

b $r\ \leqslant\ \frac{1}{2a}$

17 a a^4+b^4

b Proof

18 a



b 116° or 244°

Answers

Chapter 16

Before you start...

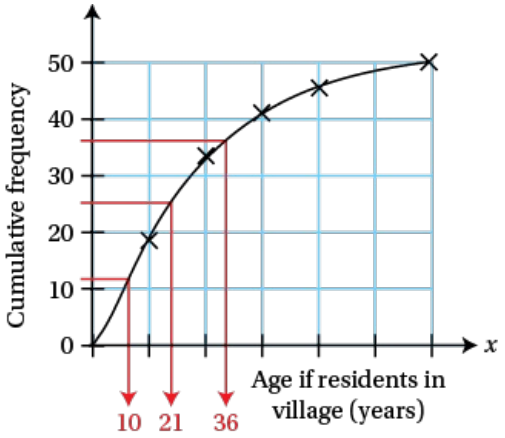
1 20%

2 Mean: 5, median: 4.5, mode: 5

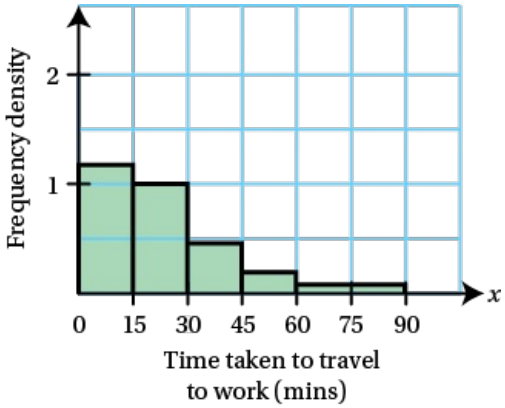
3 Range: 8 IQR: 5.5

Exercise 16A

1 A a

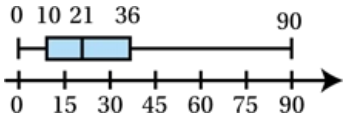


b

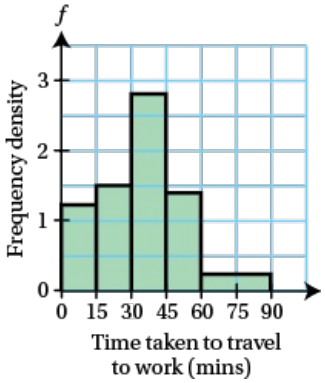


c Median $= 21$ s, IQR ≈ 26 s

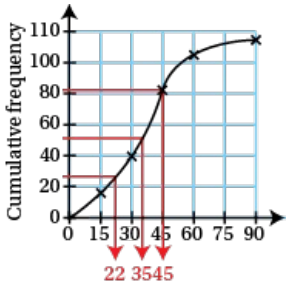
d



B a

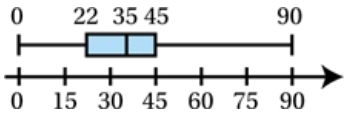


b



c $\text{Median} = 35,\ \text{IQR} \approx 23$

d



2 **a** **i** 0.3

ii $\frac{4}{15}$

b **i** 0.4

ii 0.3

c **i** $\frac{2}{15}$

ii 0.25

3 **a** $\text{Median} = 12,\ \text{IQR} = 9$

4 **a** 197s

b $c = 191,\ d = 203$

c 12s

5 **a** 53%

b 28 minutes

6 **a** Results in English were generally lower but more consistent.

b 75%

c For example, if the results were for the same group of pupils

d The number of pupils sitting each test

7 **a** 15 minutes

b Alpha Commerce, 25%

c Beta Bank, 50%

8 **a** 0.10

b A

9 **a** Median $= 23.5$ seconds, IQR $= 12.5$ seconds

b 25.2 seconds

c Group 2 has a symmetrical distribution, Group 1 doesn't. The range is the same.

10 $\text{A3}, \text{B2}, \text{C1}$

11 $\frac{2bh}{3}$

Work it out 16.1

Solution 1 is correct.

Exercise 16B

1 **a** **i** $\sigma = 7.23,\ \text{IQR} = 12.6$

ii $\sigma = 6.57,\ \text{IQR} = 13.5$

b **i** $\sigma = 11.1,\ \text{IQR} = 3$

ii $\sigma = 35.3,\ \text{IQR} = 72$

c **i** $\sigma = 1.50,\ \text{IQR} = 3$

ii $\sigma = 2.75,\ \text{IQR} = 6$

2 **a** **i** 2.07

ii 0.943

b **i** 2.20

ii 32.8

c **i** 8.78

ii 21.4

3 **a** $x = 12$

b 2.92

4 **a** 46.8

b 527

5 **a** $124.5\ \text{mph}$

b $11.8\ \text{mph}$

c Andy

6 **a** 15.2

b The Physics result is skewed by one very low score. We do not know the total mark available in the two subjects so the values might not be comparable.

7 18

8 $101 \text{ or } 141$

9 **a** Proof

b Proof

Exercise 16C

1 **a** **i** $\bar{x} = 1,\ \sigma = 0.894,\ q_2 = 1$

ii $\bar{x} = 0,\ \sigma = 1.06,\ q_2 = 0$

b **i** $\bar{x} = 12.1,\ \sigma\ = 1.90,\ q_2 = 12$

ii $\bar{x} = 0.263,\ \sigma\ = 0.137,\ q_2 = 0.2$

2 **a** **i** $\bar{x} = 26.1,\ \sigma\ = 20.4$

ii $\bar{x} = 253,\ \sigma = 151$

b **i** $\bar{x} = 6.38,\ \sigma = 5.23$

ii $\bar{x} = 102,\ \sigma = 5.78$

c **i** $\bar{x} = 6.58,\ \sigma = 5.11$

ii $\bar{x} = 15.3,\ \sigma = 9.85$

3 **a** $17,\ 21$

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b $17,\ 20$

c $16.5,\ 20.5$

d $16.01,\ 20$

> **Tip**
>
> The first group starts at 0, not -0.5

4 **a** $64,\ 15,\ 5$

b $\bar{x} = 1.68,\ \sigma = 0.875$

c Households in Mediton tend to have more cars but there is greater variation.

5 **a** $\bar{x} = 3.82,\ \sigma = 1.67$

b Use more groups; measure more accurately.

> **Tip**
>
> You might be tempted to say measure more than 50 dogs. This answers a different question − it would give you a better estimate of the mean of **all** dogs.

6 **a** 1.5

b 1.84

c 3.01

d **i** The variance is lower so the process is more consistent.

ii There were more broken eggs on average.

iii The sample size is too small for small differences to be significant − they may be due to chance.

7 **a** $\bar{x} = 9.88,\ \sigma = 7.37$

b Data is at the centre of each group

c The mean is skewed to a higher value by a small number of very large values.

8 **a** Proof

b $1 \pm 2k$

c 3

9 **a** $p = 45,\ q = 5$ and $p = 10,\ q = 40$

10 Proof

Exercise 16D

1 **a** Strong positive

b No correlation

c Strong positive

d Perfect positive

e Strong negative

f Strong positive

g Weak negative

2 **a** Strong positive

b Strong negative

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c Weak positive

d Weak negative

e No linear correlation

f Non-linear correlation

g Perfect negative

h Two separate groups, each with strong positive correlation

3 **a** Strong positive linear relationship − faster processors tend to last longer

b No − both may be due to improvements in technology

4 **a** Strong positive linear relationship − older cars have a longer braking distance

b Yes − the correlation coefficient is positive

5 **a** Strong positive linear relationship – as the age of the babies increases, the mass increases

b **i** This is the amount of mass typically gained each month.

ii This is the predicted birth mass of a baby.

c 14 years is a large extrapolation from the data. It does not take into account gender which might be an important factor.

6 **a** **i** $24.8\,\mathrm{s}$ is the predicted 100 m time of students when they start school

ii $0.609\,\mathrm{s}$ is the typical improvement in $100\,\mathrm{m}$ time each year

b The low correlation coefficient suggests that a straight line model is not appropriate. Even if it were appropriate, 60 years is a large extrapolation from the data.

7 **a** A

b D

c B

d C

8 **a** **i** $1.5\,\mathrm{m}$

ii $3\,\mathrm{m}$

b **i** Is valid as the correlation coefficient is large, but

ii Is problematic as it is extrapolating from the data.

9 **a** **i** $5\,\mathrm{Hz}$

ii Impossible to say

b The points are not in order, so there is no sense in using a line to interpolate between points.

c if the points were ordered in time − for example, vertical and horizontal displacement of a ball over time.

10 **a** false

b false

c true

d false

Exercise 16E

1 **a** **i** No

ii No

b **i** Yes

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ii Yes

2 **a** **i** No

ii No

b **i** Yes: -3

ii Yes: 64

3 There appear to be two distinct groups − he should find the correlation coefficient for each group separately.

4 **a** 5

b Proof

5 Proof

6 **a** underestimate

b close

c overestimate

7 $46.5,\ 48.7$

8 **a** 21°C

b **i** Decrease

ii Decrease

9 **a** $t = 62.9,\ \sigma = 7.40$

b Thursday

c $\bar{x} = 63,\ \sigma = 7.98$

d $0.314\,\text{km/min}$

e No − it is extrapolating from the data

10 **a** Proof

b $\bar{x} = 11.6,\ \sigma = 2.58$

11 $4.24,\ 17.0$

12 **a** Proof

b Proof

c $q = 3,\ p = 27$

d Such as $p = 9,\ q = 32$

Mixed practice 16

1 **a** 24

b 125

c 124

2 2

3 **a** $4.08\,\text{minutes}$

b $8.80\,\text{minutes}^2$

4 $x > 140$ or $x < -20$

5 **a** e.g. $1,\ 0$

b $0,$ e.g. $1,\ 1$

6 **a** **i** Mean $= 17.7,\ \mathrm{SD} = 3.84$

ii Exact data values unknown

b 4

c $16 - 20$

d **i** Increase

ii Decrease

7 **a** $\bar{t} = 10.0,\ \sigma = 6.71$

b 0.323

8 **a** 84%

b 79%

c 12.1%

9 **a** $66\ \mathrm{mph}$

b 11

c $a = 61\ \mathrm{mph},\ b = 73\ \mathrm{mph}$

d $26, 8, 4, 2$

e $\bar{x} = 67.5,\ \sigma = 11.3$

f Do not remove − they are extreme but no reason to believe they are incorrect

10 **a** Shows relationships

b Strong positive

c $200\ \mathrm{km}^2$

d It will go down

e the increase in population for each area increase of $1\ \mathrm{km}^2$

f Extrapolating from the data

11 **a** 23

b $x = w = 30$

c $38,\ 39$

d Shows all data

e Easier to find median and quartiles

12 **a** α:D, β:B, γ:C

b **i** B

ii A

iii C

13 150

14 $118\ 888.2$

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Answers

Chapter 17

Before you start...

1 $\frac{1}{20}$

2 $\frac{7}{24}$

3 **a** 5040

b 840

c 35

d 1

e 1

Exercise 17A

1 **a** **i** Mutually exclusive $-\frac{1}{2}$

ii Mutually exclusive $\frac{2}{3}$

b **i** Not mutually exclusive

ii Not mutually exclusive

c **i** Mutually exclusive $\frac{1}{3}$

ii Not mutually exclusive

d **i** Mutually exclusive $\frac{13}{25}$

ii Not mutually exclusive

e **i** Not mutually exclusive

ii Mutually exclusive $\frac{7}{15}$

2 **a** **i** Not independent

ii Not independent

b **i** Independent $\frac{1}{26}$

ii Independent $\frac{1}{52}$

c **i** Not independent

ii Not independent

d **i** Not independent

ii Independent $\frac{4}{25}$

e **i** Not independent

ii Not independent

3 **a** **i** pq

ii 0

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b Only if the probability of at least one of them is zero

4 **a** $\frac{1}{16}$

b $\frac{7}{16}$

5 $\frac{1}{9}$

6 **a** ACT, ATC, CAT, CTA, TAC, TCA

b $\frac{1}{3}$

7 **a** $\frac{1}{6}$

b Not independent

8 **a** **i** $\frac{19}{60}$

ii $\frac{61}{300}$

b Not independent

9 $\frac{1}{32}$

10 **a** $\frac{1}{8}$

b $\frac{7}{8}$

11 **a** 0.858

b Not reasonable; group behaviour, common experiences and external events

Exercise 17B

1 **a**

t	0	1	2	3
$\mathrm{P}(T = t)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b

d	0	1	2	3	4	5
$\mathrm{P}(D = d)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

c

x	1	2	3	4	6
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

d

g	1	2	3
$\mathrm{P}(G = g)$	$\frac{6}{72}$	$\frac{36}{72}$	$\frac{30}{72}$

e

c	1	2	3	4
$\mathrm{P}(C = c)$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{9}{64}$	$\frac{27}{64}$

f

x	1	2	3	4	6	8	9	12	16

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$\mathrm{P}(X=x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

2 **a** **i** $k=\frac{1}{8}$

ii $k=\frac{1}{8}$

b **i** $\frac{1}{8}$

ii $k=\frac{3}{7}$

c **i** $k=\frac{1}{10}$

ii $k=\frac{12}{25}$

3 **a** $k=\frac{1}{12}$

b $\frac{5}{12}$

4 **a** $k=0.19$

b 0.55

5 **a** 0.8

b 0.01

6 **a**

X	2	3	4	5	6	7	8
$\mathrm{P}(X=x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{3}{8}$	$\frac{3}{16}$	$\frac{2}{36}$	$\frac{1}{16}$

b $\frac{3}{8}$

7 $a=0.1,\ b=0.2$

8 **a** Proof

b 0.72

9 **a** $\frac{13}{60}$

b 0.207

10 **a** 0.11

b 0.97

11 **a** $\frac{1}{9}$

b

x	1	2	3
$\mathrm{P}(X\leqslant x)$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{9}{9}$
$\mathrm{P}(X=x)$	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{5}{9}$

12 **a** 4

b

y	0	1	4	9	16
$\mathrm{P}(Y=y)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$

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Work it out 17.1

Solution 1 is correct.

Work it out 17.2

Solution 4 is correct.

Exercise 17C

1 **a** $X \sim \mathrm{B}(20,\ 0.5)\,;\ \mathrm{P}(X = 15)$

b Not binomial, the probability may not be the same for each question (e.g. if the questions get harder)

c Not binomial, there are three possible outcomes

d Not binomial − trials are not independent

e Approximately binomial: $X \sim \mathrm{B}(100,\ 0.02)\,;\mathrm{P}(X \geqslant 1)$

2 **a** **i** 0.147

ii 0.0459

b **i** 0.944

ii 0.797

c **i** 0.0563

ii 0.0104

d **i** 0.990

ii 0.797

e **i** 0.203

ii 0.832

f **i** 0.0562

ii 0.822

g **i** 0.486

ii 0.203

h **i** 0.203

ii 0.495

3 **a** $X \sim \mathrm{B}(20,\ 0.7)\,;\ \mathrm{P}(X \geqslant 16) = 0.238$

b $X \sim \mathrm{B}(5,\ 0.35)\,;\ \mathrm{P}(X = 3) = 0.181$

c $X \sim \mathrm{B}(5,\ 0.5)\,;\ \mathrm{P}(X \geqslant 3) = 0.5$

4 **a** **i** $\frac{5}{32}$

ii $\frac{1}{32}$

b **i** $\frac{31}{32}$

ii $\frac{6}{32}$

c **i** $\frac{1}{32}$

ii $\frac{26}{32}$

5 **a** Drawing from a finite population without replacement so probability is only approximately

constant

b 0.103

6 **a** 0.0231

b 0.0273

7 **a** 0.108

b 0.0267

c The probability that a person going to the doctor has the virus is the same as for the whole country.

8 The second one $(0.0165 > 0.0154)$

9 **a** **i** 0.6

ii 0.6

iii $\frac{1}{3}$

iv $0.3; 0.288$

b **i** 0.6

ii 0.6

iii 0.36

iv 0.288

10 **a** Not independent, but constant probability

b Independent and constant probability

c Independent, but different probabilities

11 0.377

12 **a** 0.630

b His guesses are independent of each other

13 **a** Both genders are equally likely. Genders of different babies are independent of each other.

b 0.00591

14 **a** No. The probability of being late depends on, for example, the distance they live from school.

b One student being late is independent of other students. This is not a reasonable assumption; for example, friends from the same class may be travelling together.

15 $p = \frac{2}{3}$

16 $p = 0.6 \text{ or } 0.4$

17 **a** Proof

b $n = 10$

18 $n = 3$

19 $n = \frac{5 - p}{p}$

Mixed practice 17

1 0.261

2 **a** $k = 0.16$

b 0.72

c 3

3

x	0	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

4 **a** 0.142

b 0.335

5 **a** 0.2

b $p = 0.05, q = 0.15$

6 **a** 0.572

b 0.0319

c 0.537

7 **a** 0.232

b 0.894

c That the attempts are independent of each other

8 **a** 16%

b 52%

9 0.653

10 0.36

11 **a**

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\mathrm{P}(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

b $\frac{5}{12}$

12 **a** 0.64

b $p = 0.3 \text{ or } 0.7$

13 **a** 0.0296

b 2.44×10^{-4}

14 **a** $0.031\ 25$

b $0.015\ 625$

15 **a** 0.460

b $1 - (0.95)^n$

c 32

16 **a**

x	0	1	2	3	4	5
$\mathrm{P}(X = x)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

b 0.0697

17 **a** Proof

b

1	4	5

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$\frac{1}{1296}$	$\frac{175}{1296}$	$\frac{369}{1296}$

18 a 0.0781

b 0.0130

Answers

Chapter 18

Before you start...

1 **a** 0.575

b 0.274

2 $X \sim B(12, 0.7)$

Exercise 18A

1 **a** Sixth form students may be more likely to progress to university and so be more worried about fees than people who are not affected.

b People in supermarkets are more likely to be knowledgeable about food prices.

c School children will be shorter than average.

d This is a real-life example. Mostly richer people had telephones, so the poll predicted the wrong winner.

2 **a** Quota sampling

b Stratified sampling

c Cluster sampling

d Simple random sampling

e Systematic sampling

f Opportunity sampling

3 **a** 2.55

b Probably larger

c It is likely to be better, but with random fluctuations it is possible to be worse.

4 **a** This would require a complete list of all fish in the North Sea and the ability to get measure any fish required.

b Quota sampling

c It will be more representative.

5 Obtain a numbered list of all the students; use a random number generator to generate 70 different random numbers; select students with those numbers.

6 **a** Opportunity sampling

b Students who eat burgers may not be interested in healthy eating.

7 $16, 762, 537, 722, 152$

8 **a**

	Female	Male
Under 42	16	24
42 or over	16	24

b It is random therefore less biased or more generalisable.

c It is cheaper. It does not require all selected people to participate. It does not require a list of all people in the city.

9 **a** A − cluster, B − simple random, C − quota, D − stratified

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b D

c Less time-consuming − no need to travel as far; cheaper

10 Not everybody will return the questionnaire so not all possible samples are equally likely.

11 **a** 9.33×10^{-4}

b **i** 0.0441

ii 0.00996

c 4.65%

d Increase − genders may cluster together

Work it out 18.1

Hypothesis 1: acceptable.

Work it out 18.2

Solution 3 is correct.

Exercise 18B

1 **a** **i** $\mathrm{H}_0 : p = 0.6\ \mathrm{H}_1 : p > 0.6; p$ is the proportion of children who like football

ii $\mathrm{H}_0 : p = \frac{1}{3} \mathrm{H}_1 : P > \frac{1}{3}; p$ is the proportion of households with a pet

b **i** $\mathrm{H}_0 :\ p = 0.06\ \mathrm{H}_1 :\ p < 0.06;\ p$ is the proportion of faulty components produced

ii $\mathrm{H}_0 :\ p = 0.5\ \mathrm{H}_1 : p < 0.5;\ p$ is the proportion of children eating 5 or more pieces of fruit per day.

c **i** $\mathrm{H}_0 : p = 0.5\, \mathrm{H}_1 : p \neq 0.5; p$ is the probability that the coin shows heads

ii $\mathrm{H}_0 : p = 0.26\, \mathrm{H}_1 : p \neq 0.26; p$ is the proportion of entries in AS level Psychology graded ‘A’

2 **a** **i** p-value $19.3\% > 5\%$, insufficient evidence

ii p-value $91.4\% > 10\%$, insufficient evidence

b **i** p-value $31.8\% > 10\%$, insufficient evidence

ii p-value $18.2\% > 3\%$, insufficient evidence

c **i** p-value $17.9\% > 4\%$, insufficient evidence

ii p-value $41.1\% > 2.5\%$, insufficient evidence

3 **a** $\mathrm{H}_0 :\ p = 0.63\ \mathrm{H}_1 :\ p < 0.63$

b Cannot reject H_0, no evidence of lower car ownership in David’s neighbourhood

4 Insufficient evidence of an increase

5 Insufficient evidence of a probability less than $\frac{1}{6}$

6 Insufficient evidence that a higher proportion of teachers drive to work

7 Insufficient evidence that the new treatment is better

8 Insufficient evidence that the proportion with a younger sibling at the same school is different

9 There is evidence that a six has a probability significantly less than $\frac{1}{6}$

10 Insufficient evidence that the proportion able to run 100 metres in under 12 seconds has changed

11 14

12 It was greater than 4.45%

13 $q<0.0247$

Exercise 18C

1 **a** **i** $X \leqslant 23$

ii $X \leqslant 4$

b **i** $X \leqslant 51$

ii $X \leqslant 51$

c **i** $X \geqslant 37$

ii $X \geqslant 12$

d **i** $X \geqslant 69$

ii $X \geqslant 69$

2 **a** **i** 0.0314

ii 0.0759

b **i** 0.0602

ii $0.009\,60$

c **i** 0.280

ii 0.0875

d **i** 0.0602

ii $0.00\,388$

3 $X \leqslant 10$

4 **a** $\mathrm{H}_0: \ p = 0.85\ \mathrm{H}_1: p > 0.85$ where p is the proportion of patients the drug cures

b $X \leqslant 162$

5 **a** $\mathrm{H}_0: \ p = 0.12\ \mathrm{H}_1: p > 0.12$ where p is the proportion of the population who have purchased the manufacturer's products

b Cannot reject H_0; no significant rise in product purchase history among the population

c 0.0556

6 **a** acceptance region

b 0.06

7 **a** $\mathrm{H}_0: \ p = 0.3\ \mathrm{H}_1: p > 0.3$ where p is the proportion of children who walk to school

b $X \geqslant 62$

c Reject H_0 : significant evidence that the proportion of children who walk to school is greater than 30%

8 **a** $\mathrm{H}_0: \ p = 0.72\ \mathrm{H}_1: p \neq 0.72$ where p is the proportion of students passing the test on their first attempt

b $X \leqslant 29$ *or* $X \geqslant 41$

c $0.0972;\ \mathrm{SL} \geqslant 9.72\%$

Mixed practice 18

1 Quota sampling

2 **a** Cluster sampling

b Students from the same tutor group are likely to go or not go together

c Obtain a list of all students, number sequentially, use a random number generator to generate 50 numbers, select students with those numbers (ignore repeats and numbers larger than the number of students)

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3 **a** $\mathrm{H}_0 : p = \frac{1}{6}\, \mathrm{H}_1 : p > \frac{1}{6}$ where p is the underlying probability of rolling a 6

b There is insufficient evidence that the probability of rolling a 5 is greater than $\frac{1}{6}$

4 $\mathrm{H}_0 : p = \frac{1}{8}\, \mathrm{H}_1 :\ p > \frac{1}{8}$ where p is the underlying proportion of days that Lisa is late. There is insufficient evidence that Lisa's probability of being late has decreased from 1 in 8

5 **a** $\frac{5}{6}$

b Not correct; this sample has the same probability as any other sample

6 **a** Pupils who are interested in politic are more likely to volunteer.

b simple random sample – obtain a list of all pupils, number sequentially, use a random number generator to generate 50 numbers, select pupils with those numbers (ignore repeats and numbers larger than the number of pupils)

7 2

8 $p = 0.00523 < 0.025$, sufficient evidence to reject H_0, there is evidence to conclude that home ownership in Germany is lower than 64%

9 **a** Each possible group of a given size has an equal probability to be included within the sample.

b **i** Stratified sample

ii Attitudes to sport may differ between boys and girls.

c $X \geqslant 34$

d 0.0906

e There is insufficient evidence that the proportion of students who play in a school sports team is greater than 40%.

10 **a** $\mathrm{H}_0 :\ p = 0.19,\ \mathrm{H1} :\ p > 0.4$. Reject H_0 at 10%, $p = 0.0841$

b Letters are not distributed independently.

11 **a** No

b No

c Yes

d No

Focus on ... 3

Focus on ... Proof 3

1 Discussion

2 Proof

3 Proof

Focus on ... Problem solving 3

1 **a** 22.4%

b Still an underestimate

2 **a** 100 million

b assuming no fish have entered or left the North Sea and that the fish have thoroughly mixed up

Focus on ... Modelling 3

1 Proof

2 Proof

3 Proof

Cross-topic review exercise 3

1 $\frac{5}{36}$

2 $\sqrt[4]{\frac{e}{24}}$

3 **a** 0.31

b 0.303

4 **a** **i** Quota sampling

ii In a stratified sample, the 20 men and 20 women would be selected using random sampling.

b $p = 0.274$; insufficient evidence that the proportion is greater

5 **a** **i** 40

ii 48

b **i** Stem-and-leaf diagram

ii Cumulative frequency diagram

6 **a** $6, 10, 16, 7$

b $\text{Mean} = 21.3 \text{ minutes}$, $\text{variance} = 82.8 \text{ minutes}^2$

c Actual data values are not given.

7 $\frac{11}{12}$

8 $\frac{8}{3}$

9 **a** A survey may have only used a sample. The population of the city may have increased.

b $p = 0.044$; insufficient evidence that the proportion is greater

10 **a** $P(A \text{ and } B) \neq P(A) \times P(B)$

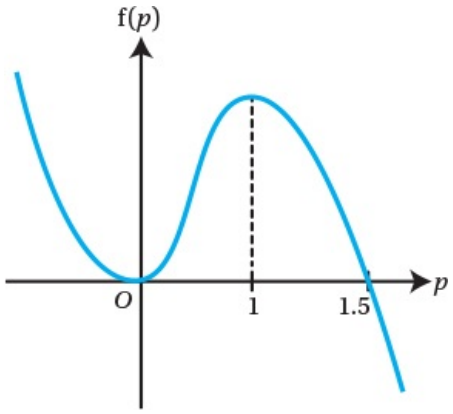
b **i** 0.0199

ii Same probability for each building; completion of different buildings independent of the others

c Evidence of increase $(\text{p} = 0.0294)$

11 **a** Proof

b



c Proof

12 New mean $= 34.9$, new variance $= 6.32$

13 £518

14 a $\ln 5$

b $0.001\,62$

15 a $\text{Binomial},\ n = 12,\ p = 0.1$

b **i** 0.0852

ii 0.718

c 0.317

16 a $\mathrm{p} = \dfrac{4}{5}$

b $p = 0.633$

17 $x = 3, 11$

18 a Selections are not independent. The probabilities with replacement are very close to the probabilities without replacement.

b 0.0240

c The statement is incorrect; all samples are equally likely.

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Answers

Chapter 19

Before you start...

1 **a** $-\frac{7}{5}$

b $\frac{2}{3}$

2 **a** **i** $\frac{dy}{dx} = 6x - 4 - 5x^{-2}$

ii -15

b $(2, 21)$

3 $S = 24,\ T = 42$

4 4

5 $y = 5x - 2x^3 - 1$

6 **a** $2\ \text{m s}^{-2}$

b $40\ \text{m}$

Exercise 19A

1 **a** Yes

b No

c No

d No

2 These are some suggestions for discussion:

a Shape and size of box, which affects air resistance; change in g at different heights

b Size of each ball; where it was hit; friction

c The actual shape of the road; mass of passengers inside the bus; resistance forces (may depend on weather conditions)

d Take-off and landing times; changes in height and related changes in g; weather conditions, turbulence; the path is circular, not straight

Exercise 19B

1 **a** **i** $-410\ \text{m}$

ii $-290\ \text{m}$

b **i** $290\ \text{m}$

ii $210\ \text{m}$

c **i** $-120\ \text{m}$

ii $-210\ \text{m}$

2 **a** **i** $7.32\ \text{m s}^{-1},\ 2.93\ \text{m s}^{-1}$

ii $8.64\ \text{ms}^{-1}, -1.48\ \text{ms}^{-1}$

b **i** $15.3\ \text{ms}^{-1}, 0\ \text{ms}^{-1}$

ii $9.55\ \text{ms}^{-1}, 0\ \text{ms}^{-1}$

3 **a** **i** $1.00\ \text{ms}^{-1}$

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ii $17.2\ \text{ms}^{-1}$

b **i** $18.7\ \text{km h}^{-1}$

ii $0.936\ \text{km h}^{-1}$

c **i** $0.00926\ \text{ms} - 2$

ii $0.0347\ \text{ms} - 2$

d **i** $10\ 600\ \text{km h}^{-2}$

ii $35\ 000\ \text{km h}^{-2}$

4 Any of them

Exercise 19C

1 **a** **i** $v = 9t^2 - 4,\ a = 18t$

ii $v = 4t^3 - 6t + 4,\ a = 12t^2 - 6$

b **i** $v = \frac{2t}{5} - 4.2, a = \frac{2}{5}$

ii $v = -1.5t^2 + \frac{2t}{3}, a = -3t + \frac{2}{3}$

2 **a** **i**

	$x\,(0)$	$x\,(3)$	$v(0)$	$v(3)$	$a(0)$	$a(3)$
a **i**	2	71	-4	77	0	54
ii	0	66	44	94	-6	102
b **i**	0	-10.8	-4.2	-3	$\frac{2}{5}$	$\frac{2}{5}$
ii	0	-10.5	0	-11.5	$\frac{2}{3}$	-8.33

3 **a** **i** $x = \frac{3}{2}t^2 - 4t + 2$

ii $x = t - t^2 + 1$

b **i** $x = 5t - t^3$

ii $x = \frac{1}{3}t^3 - t^2$

4 **a** **i** 19.5

ii -19

b **i** -100

ii 25.7

5 **a** **i** $v = t - t^2 + 3, x = \frac{1}{2}t^2 - \frac{1}{3}t^3 + 3t$

ii $v = t^2 + 2^t + 2t - 2, x = \frac{2}{3}t^3 + t^2 - 2t$

b **i** $v = -5t + 3, x = -\frac{5}{2}t^2 + 3t + 5$

ii $v = 3t - 2, x = \frac{3}{2}t^2 - 2t + 7$

6 **a** $-3.6\ \text{m s}^{-2}$

b $0.96\ \text{m s}^{-1}$

c $x = 1.2t^2 - 0.5t^3$

7 **a** $0.1\ \text{m s}^{-1},\ 1.2\ \text{m s}^{-2}$

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b $1\ \text{m s}^{-1}$

c $1.92\ \text{s},\ 3.00\ \text{m}$

8 **a** $v = 0.04t^3 - 0.72t^2 + 4.32t$,
$x = 0.01t^4 - 0.24t^3 + 2.16t^2$

b $v = 8.64\ \text{m s}^{-1},\ x = 38.9\ \text{m}$

9 **a** $v = 7.5 + t - 0.3t^2$,

b $x = 50\ \text{m}$

c $x = 62.5\ \text{m}$

10 **a** $-116\ \text{m s}^{-1}$

Exercise 19D

1 i. **a.** $0.65\ \text{m s}^{-2}, -0.52\ \text{m s}^{-2}$

b. $1950\ \text{m}$

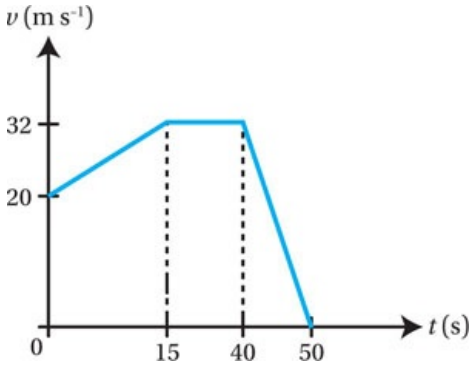
ii. **a.** $1\ \text{m s}^{-2}, -3\ \text{m s}^{-2}$

b. $888\ \text{m}$

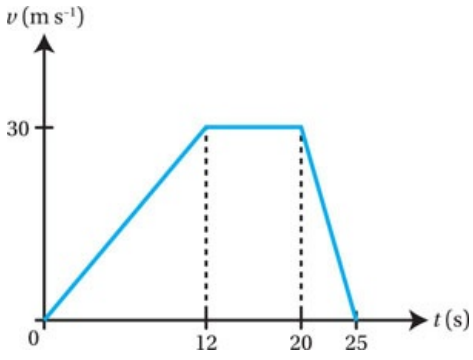
iii. **a.** $-1.39\ \text{m s}^{-2},\ 1.25\ \text{m s}^{-2}$

b. $398\ \text{m}$

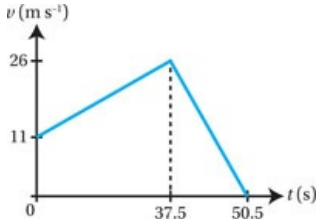
2 **a** $d = 1350\ \text{m}$



b $d = 495\ \text{m}$



c $d = 863\ \text{m}$



3 **a** 50 metres

b awway

c Particle changes direction

d The particle stops moving (it is stationary), 30 m from P

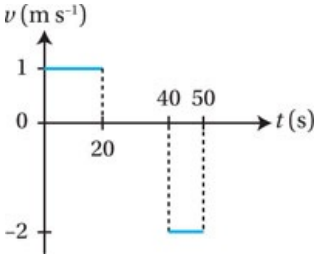
e When $t = 50$

f Decreasing

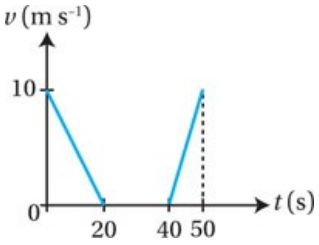
g Speed is decreasing, velocity is increasing

h 190 m

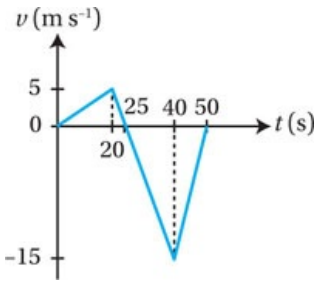
4 **a**



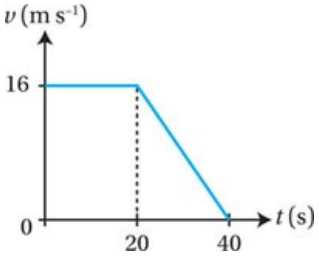
b



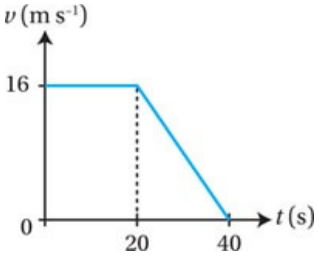
c



d



5 **a**



b 480 m

6 **a** $2\ \text{m s}^{-1}$

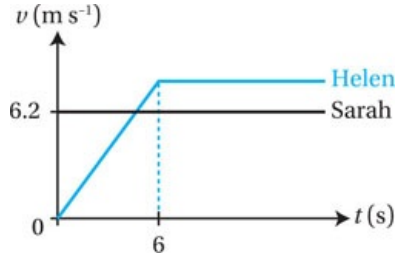
b $25\ s$

c $-7.5\ \text{m s}^{-1}$

d 175 m (to the right)

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7 **a**



b $14.6\ s$

8 **a** 8

b $0.8\ \text{m s}^{-2}$

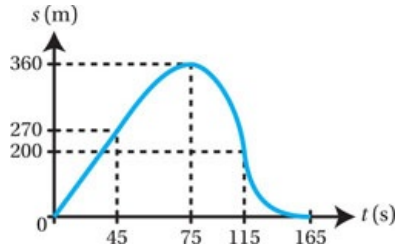
c C

9 $x = 5$

10 **a** 45

b $115\ s$

c



Exercise 19E

1 **i.** **a.** $16.25\ \text{m s}^{-1}$

b. $16.25\ \text{m s}^{-1}$

ii. **a.** $20.2\ \text{m s}^{-1}$

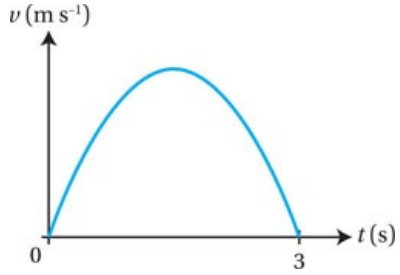
b. $20.2\ \text{m s}^{-1}$

iii. **a.** $6.31\ \text{m s}^{-1}$

b. $3.69\ \text{m s}^{-1}$

2 $2.45\ \text{m s}^{-1}$

3 **a**



b $9\ \text{m s}^{-1}$

4 **a** Stationary

b $200\ \text{m},\ 6.67\ \text{m s}^{-1}$

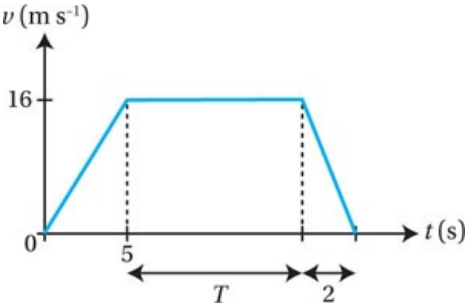
c $13.3\ \text{m s}^{-1}$

5 **a** $t = 4,\ 7$

b $85.75\ \text{m}$

c $12.3\ \text{m s}^{-1},\ 33.5\ \text{m s}^{-1}$

6 **a**



b 7

7 **a** When $t = 0,\ v = 0;\ 8\ s,\ 12\ s$

b $v = 3.6\ \text{m s}^{-1},\ x = 30.6\ \text{m}$

c $3.29\ \text{m s}^{-1}$

8 $28.9\ \text{m s}^{-1}$

9 1.82 s or 9.7 s

10 $10.2\ \text{m s}^{-1}$

Exercise 19F

1 1.5 s

2 **a** when $t = 0,\ x = 0$

b $5.4\ \text{m s}^{-1}$

3 $34.0\ \text{m};\ 5.76\ \text{m s}^{-2}$

4 5.6 s

5 **a** $-2.4\ \text{m s}^{-2}$

b **i** $13.5\ \text{m}$

ii $40.5\ \text{m}$

6 **a** 3.7

b $1.43\ \text{m s}^{-1}$

c $37.8\ \text{m}$

d 7.13 s

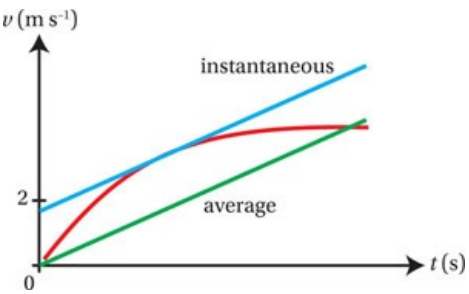
7 **a** $v = 6.5t - 0.65t^2$

b $v = 31.85 - 2.6t$

8 **a** Positive acceleration throughout, starting from rest

b $3.125\ \text{m s}^{-1}$

c



9 **a** $v = t^3 - 7t^2 + 10t$

b **i** $4.06\ \text{m s}^{-1}$

ii $8.21\ \text{m s}^{-1}$

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10 **a** 15 m

b 11.6 s

11 13.3

12 34.7 m

Mixed practice 19

1 **a** 4 m s^{-2}

b $x = t^3 - 4t^2$

2 **a** 11.2 m

b 2.8 m s^{-1}

3 **a** 1.4 m s^{-2}

b $0,\ 12,\ 15\text{ s}$

c 3ms^{-1}

4 **a** 30 m

b 76s

c -0.667 ms^{-2}

5 -15 m s^{-2}

6 **a** $v = \frac{100}{9} + \frac{25}{9}t - \frac{5}{36}t^2$

b $a = \frac{25}{9} - \frac{5}{18}t; t = 10$

c 253 m

d **i** No

ii Yes

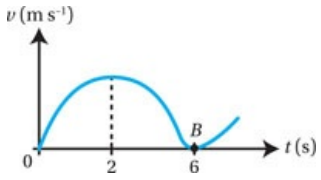
7 4.25

8 **a** $v = 0.04t^3 - 0.48t^2 + 1.44t$

b $a = 0.12t^2 - 0.96t + 1.44$

c 0 m s^{-1}

d Forwards (away from A)



e 4.32 m

9 $9.93\text{s}, 30.1\text{s}$

10 **a** 8.73 s

b Proof

c 0 s

11 17.3 m s^{-1}

Answers

Chapter 20

Before you start...

1 **a** $v = 2t - t^3 + 4.2$

b $x = t^2 - \frac{1}{4}t^4 + 4.2t + 14$

2 **a** $t = 0.249, 2.49$

b $t = 0.563, 1.89$

3 **a** $(0.169, 1.03)$

b $(1.5, 2.28)$

Exercise 20A

1 Proof

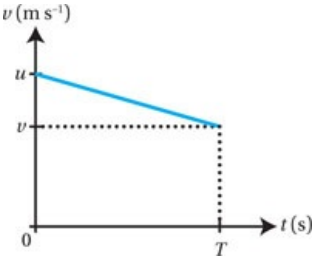
2 **a** Proof

b Proof

3 **a** Proof

b $49.9\ \text{m s}^{-1}$

4 **a**



b Proof

5 **a** $v = u + 0.5kt^2$

b Proof

6 Proof

7 **a** $v = u + aT$

b At the start and after 1 second

c $T = 0$ *or* $1.5\ \text{s}$

8 Proof

Exercise 20B

1 **a** **i** $u = 3.6\ \text{m s}^{-1},\ \text{s} = 64.8\ \text{m}$

ii $u = -23.1\text{m s}^{-1},\ \text{s} = -77.2\ \text{m}$

b **i** $v = 5\ \text{m s}^{-1},\ t = 6.67\ \text{s}$ or $v = -5\ \text{m s}^{-1},\ t = 15\ \text{s}$

ii $v = 12.6\ \text{m s}^{-1},\ t = 2.45\ \text{s}$ or $v = -12.6\ \text{m s}^{-1},\ t = 10.9\ s$

c **i** $a = -0.45\ \text{m s}^{-2},\ t = 13.3\ \text{s}$

ii $a = 1.03\ \text{m s}^{-2},\ t = 18.5\ \text{s}$

d **i** $a = -1.83\ \text{m s}^{-2},\ \text{s} = -3\ m$

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ii $a = -1.3\ \text{m s}^{-2},\ \text{s} = 15\ \text{m}$

2 **a** $1.92\ \text{m s}^{-2}$

b $40.5\ \text{m}$

3 **a** $6.3\ \text{m s}^{-2}$

b $36.8\ \text{m}$

4 **a** $-0.873\ \text{m s}^{-2}$

b $26.6\ \text{m}$

5 $14.1\ \text{s}$

6 $4.43\ \text{s}$

7 $8.05\ \text{s}$

8 $66.7\ \text{m}$

9 **a** $5.29\ \text{m}$

b It will go an unlimited distance in the negative direction.

Work it out 20.1

Solution 2 is correct.

Exercise 20C

1 **a** **i** $21.9\ \text{m s}^{-1},\ 6.14\ \text{m}$

ii $30.8\ \text{m s}^{-1},\ 21.5\ \text{m}$

b **i** $30.4\ \text{m s}^{-1},\ 47.1\ \text{m}$

ii $58.8\ \text{m s}^{-1},\ 176\ \text{m}$

2 **a** **i** $15.5\ \text{m s}^{-1}\ \text{downward},\ -8.13\ \text{m}$

ii $4.16\ \text{m s}^{-1}\ \text{upward},\ 6.46\ \text{m}$

b **i** $4.62\ \text{m s}^{-1}\ \text{upward}, 61.4\ \text{m}$

ii $9.22\ \text{m s}^{-1}\ \text{upward},\ 16.1\ \text{m}$

3 $22.1\ \text{m s}^{-1}$

4 **a** $1.18\ \text{s}$

b $6.95\ \text{m s}^{-1}$

c **i** Increase

ii Decrease

5 $11.0\ \text{m s}^{-1}$

6 **a** $55\ \text{m}$

b $6.21\ \text{s};\ 32.8\ \text{m s}^{-1}$

7 **a** No

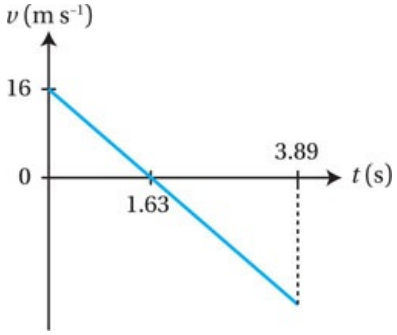
b $12.8\ \text{m s}^{-1}$

c Increase

8 **a** $1.63\ \text{s}$

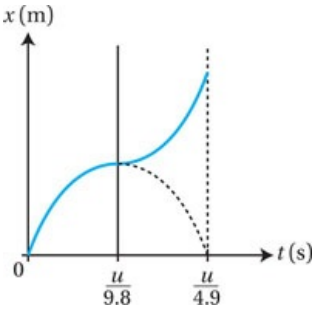
b $3.89\ \text{s}$

c



9 **a** $\frac{u}{4.9}$s

b



10 $13.3\ \text{m}$

Exercise 20D

1 $149\ \text{m}$

2 **a** $10.2\ \text{m s}^{-1}$

b $0.207\ \text{m s}^{-2}$

3 $1.92\ \text{m s}^{-1}$

4 $17.4\ \text{m s}^{-1}$

5 $6.91\ \text{m}$

6 $12.4\ \text{s}$

7 44.2

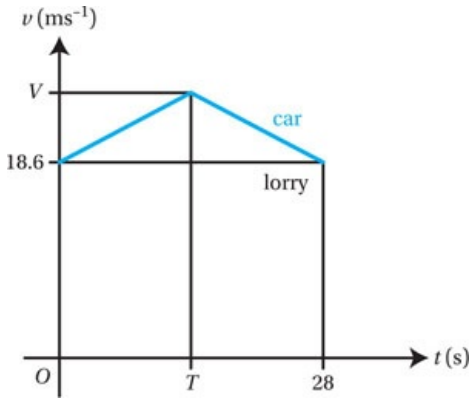
8 Maximum height $38.3\ \text{m}$, speed as it hits the ground $27.4\ \text{m s}^{-1}$

9 $18.1\ \text{m s}^{-1}$

10 **a** $90.2\ \text{m}$

b $25\ \text{m s}^{-1}$

c



Mixed practice 20

1 **a** $8.75\ \text{s}$

b $166.25\ \text{m}$

2 **a** $8.6\ \text{m s}^{-1},$ downwards

b $2.4\ \text{m}$

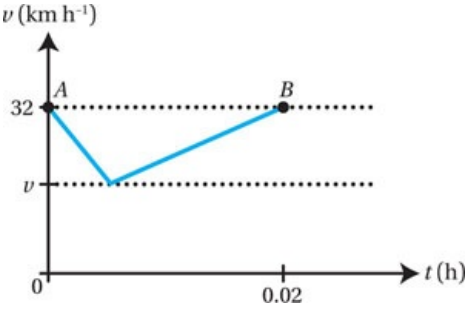
3 $2.2\ \text{m s}^{-1}$

4 **a** $2.8\ \text{m s}^{-1}$

b $2.5\ \text{m}$

c $1.30\ \text{s}$

5 **a**



b $23\ \text{km h}^{-1}$

c $27.5\ \text{km h}^{-1}$

6 **a** Proof

b $614\ \text{m}$

7 **a** $7.93\ \text{m}$

b $9.04\ \text{m s}^{-1},\ 21.8\ \text{m s}^{-1}$

8 Proof

9 **a** Proof

b 5.6

c Proof

10 $120\ \text{m}$

11 $2:1$

12 $u = 150,\ v = 50$

Answers

Chapter 21

Before you start...

1 **a** $0.417\ \text{m s}^{-2}$

b $66\ \text{m}$

2 **a** $\begin{pmatrix} -2 \\ 6 \end{pmatrix}$

b $-4\mathbf{i} + 2\mathbf{j}$

3 **a** Magnitude $2.77,\ 64.4^\circ$ above horizontal

b Magnitude $4.12,\ 14.0^\circ$ below horizontal

Exercise 21A

1 **a** $138\ \text{N}$

b $0.45\ \text{N}$

c $7200\ \text{N}$

d 0.828

e $0\ \text{N}$

f $1.18\ \text{N}$

g $0\ \text{N}$

2 **a** $2.39\ \text{m s}^{-2}$

b $18.3\ \text{m s}^{-2}$

c $0.4\ \text{m s}^{-2}$

d $21.7\ \text{m s}^{-2}$

3 **a** **i** $\begin{pmatrix} 4.8 \\ -7.5 \end{pmatrix}\ \text{N}$

ii $\begin{pmatrix} -3.5 \\ 6.5 \end{pmatrix}\ \text{N}$

b **i** $\begin{pmatrix} 6 \\ 2 \end{pmatrix}\ \text{ms}^{-2}$

ii $\begin{pmatrix} -2 \\ 1.43 \end{pmatrix}\ \text{ms}^{-2}$

c **i** $1.4\ \text{kg}$

ii $0.5\ \text{kg}$

4 **a** Towards the centre

b Deep space

c Depends on the object

5 $1070\ \text{N}$

6 $0.048\ \text{N}$

7 $0.269\ \text{s}$

8 $\begin{pmatrix} 1.08 \\ 1.98 \end{pmatrix}\ \text{N}$

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9 **a** $\left(\frac{-1.31}{1}\right)\text{ms}^{-2}$

b 1.65 m s^{-2}

10 218 N

11 6.47 kg

Exercise 21B

1 **a** **i** 2 N left

ii 5 N right

b **i** 3 N down

ii 6 N up

c **i** 7 N right

ii 4 N right

2 **a** **i** $(13\mathbf{i}+8\mathbf{j})\text{N}$

ii $(-5\mathbf{i}+7\mathbf{j})\text{N}$

b **i** $(7\mathbf{i}+2\mathbf{j})\text{N}$

ii $(2\mathbf{i}+4\mathbf{j})\text{N}$

c **i** $(-4\text{j})\text{N}$

ii $(6\mathbf{i}-4\mathbf{j})\text{N}$

3 **a** **i** $15.3\text{ N},\ 31.6^\circ$

ii $8.60\text{ N},\ 126^\circ$

b **i** $7.28\text{ N},\ 15.9^\circ$

ii $4.47\text{ N},\ 63.4^\circ$

c **i** $4\text{ N}, -90^\circ$

ii $7.21\text{ N}, -33.7^\circ$

4 **a** **i** 3 m s^{-2} right

ii 2 m s^{-2} left

b **i** 3 m s^{-2} down

ii 1.4 m s^{-2} up

c **i** 0 m s^{-2}

ii 0 m s^{-2}

d **i** 6 m s^{-2} left

ii 2 m s^{-2} down

5 **a** **i** $(-4\mathbf{i}+6\mathbf{j})\text{ m s}^{-2}$

ii $(2\mathbf{i}-3\mathbf{j})\text{ m s}^{-2}$

b **i** $(\mathbf{i}+1.5\mathbf{j})\text{ m s}^{-2}$

ii $(0.8\mathbf{i}+0.1\mathbf{j})\text{ m s}^{-2}$

6 **a** **i** 15 N

ii 14 N

b **i** 15 N

ii 8 N

c **i** 12 N

ii 4 N

d **i** 7 N

ii 3 N

7 1200 kg

8 20 kg

9 **a** 8.06 N

b 29.7 degrees

Exercise 21C

1 $2\ \text{m s}^{-2}$

2 269 kg

3 **a** 760 N

b 0.86 s less (8.40 compared to 9.26)

4 $p = 13.2,\ q = 30$

5 $18.4\ \text{N},\ 151^\circ$

6 $0.986\ \text{ms}^{-2}$

7 4.8 N

8 210 N

9 607 N

10 **a** 3.54 seconds

b 52.4 N

11 **a** proof

b 320 N

c 910 m

12 1.69 N against the direction of travel

13 **a** Proof

b 115

c 164

Exercise 21D

1 **a** **i** $0.867\ \text{m s}^{-2}$ upwards

ii $1.7\ \text{m s}^{-2}$ upwards

b **i** $2.3\ \text{m s}^{-2}$ downwards

ii $1.36\ \text{m s}^{-2}$ downwards

2 **a** **i** 110 N

ii 27.5 N

b **i** 89.6 N

ii 35.4 N

3 **a** **i** 3.02 kg

ii 2.4 kg

b **i** $2.36\ \text{kg}$

ii $1.77\ \text{kg}$

4 $6.06\ \text{m s}^{-2}$ downwards

5 **a** $1.22\ \text{kg}$

b $2.45\ \text{m s}^{-2}$ upwards

6 $88.6\ \text{kg}$

7 $1.08\ \text{N}$

8 $0.344\ \text{kg}$

9 **a** $2.26\ \text{s}$

b $1.51\ \text{N}$

10 $5060\ \text{N}$

11 $1440\ \text{N}$

12 **a** $39.3\ \text{N}$

b $1.43\ \text{m s}^{-2}$

13 $1260\ \text{N}$

Exercise 21E

1 **a** No

b Yes

c Yes

d No

2 **a** $14\ \text{N}$

b $7\ \text{N}$

c $16\ \text{N},\ 9\ \text{N}$

d $3\ \text{N},\ 6\ \text{N}$

3 **a** **i** $x = -18,\ y = -7$

ii $x = 6,\ y = -30$

b **i** $x = 2,\ y = -10$

ii $x = -9,\ y = -4$

c **i** $x = -8,\ y = 4$

ii $x = -15,\ y = 9$

d **i** $x = 2,\ y = 3$

ii $x = -1,\ y = 5$

4 $6\ \text{N},\ 5\ \text{N}$

5 $x = 2,\ y = 4$

6 $a = 8,\ b = 10$

7 $7.28\ \text{N}$

8 $12\ \text{N}$

9 $19\ \text{N}$ thrust

10 $a = 6,\ b = -3.5,\ m = 6$

11 $\mathbf{F} = (-57\mathbf{i} + 6\mathbf{j})\ \text{N}$

12 **a** $\mathbf{F} = \begin{pmatrix} 17 \\ -25 \end{pmatrix}$

b $30.2\ \text{N}, \theta = 55.8^\circ$

Mixed practice 21

1 **a** $863\ \text{kg}$

b Lower mass

2 $11.9\ \text{N}$

3 **a** $0.816\ \text{m s}^{-2}$

b $25.2\ \text{s}$

4 **a** $27.3\ \text{kg}$

b $293\ \text{N}$

c $268\ \text{N}$

d **i** Does not contribute to system mass

ii Constant tension throughout, perfectly transmits tension forces

5 **a** $p = -2.6,\ q = -3.8$

b $\begin{pmatrix} 1.04 \\ 1.52 \end{pmatrix}\ \text{m s}^{-2}$

6 **a** $0.327\ \text{m s}^{-2}$

b $6.86\ \text{N}$

7 **a** $481\ \text{N}$

b $8.33\ \text{s}$

8 $98.5\ \text{N},\ 60.8^\circ$

9 $19.2\ \text{N},\ 141^\circ$

10 **a** $19.2\ \text{N},\ 038.7^\circ$

b $19.2\ \text{N},\ 219^\circ$

11 $16.8\ \text{N},\ \text{thrust}$

12 **a** $(7.5\mathbf{i} - 0.75\mathbf{j})\ \text{m s}^{-2}$

b $30.1\ \text{N},\ 174^\circ$

13 $165\ \text{N},\ 13.8^\circ\ \text{above horizontal}$

Answers

Chapter 21

Before you start...

1 $2.92\ \text{m s}^{-2}$

2 $333\ \text{N}$

3 $x = 2,\ y = 6$

4 **a** $0.4\ \text{m s}^{-2}$

b 15 seconds

Exercise 22A

1 $5.49\ \text{m s}^{-2}$

2 $65\ \text{kg}$

3 $5.32\ \text{m}$

4 **a** $0.306\ \text{m s}^{-2}$

b $1.84\ \text{s}$

5 **a** $1.18\ \text{N}$

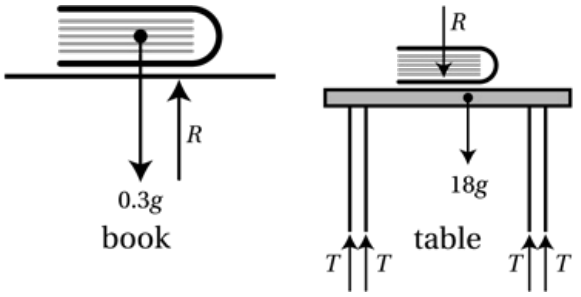
b $1.43\ \text{s}$

c $1.18\ \text{N}$

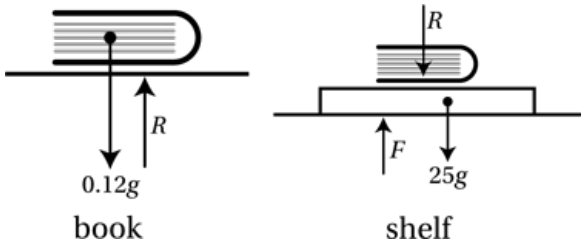
d $2.02 \times 10^{-25}\ \text{m}$

Exercise 22B

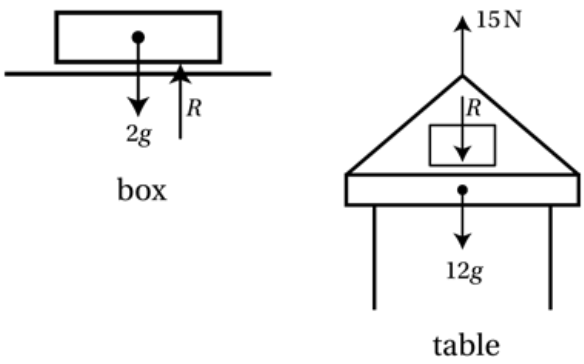
1 **a** **i**



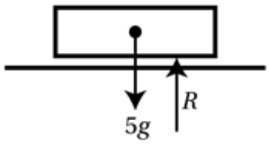
ii



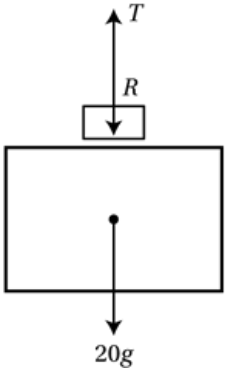
b **i**



ii

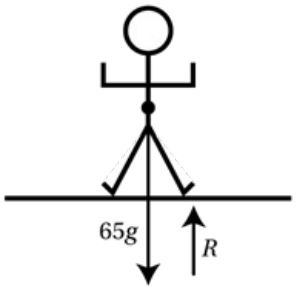


box

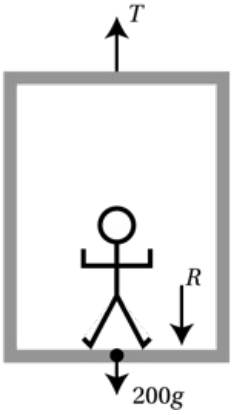


crate

c i

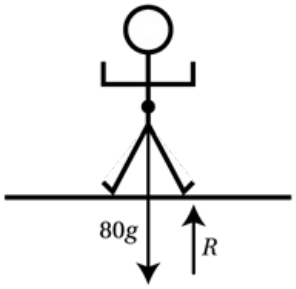


person

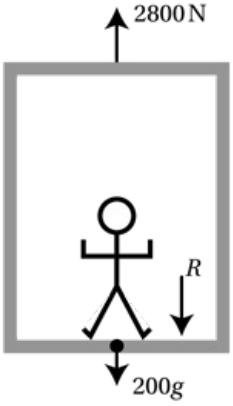


lift

ii

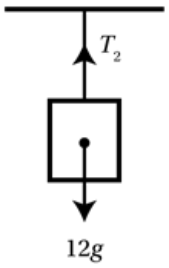


person

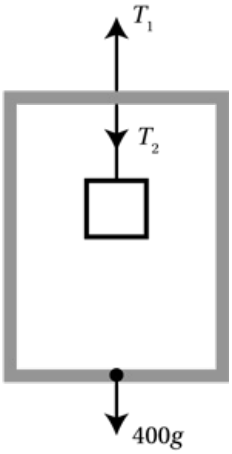


lift

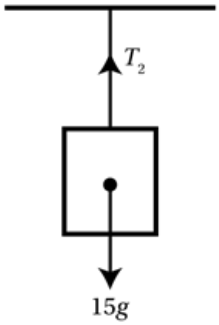
d i



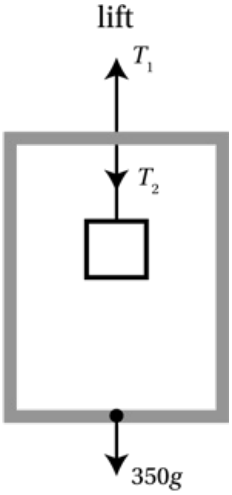
box



ii



box



lift

2 **a** **i** 110 N

ii 80 N

b **i** 95 N

ii 130 N

3 17.9 N

4 79 N

5 1224 N

6 1.18 N

7 11.0 kg

8 **a** 5560 N

b 364 N

9 **a** 1190 N

b 833 N

10 355 kg

11 **a** Up

b 6590 N

Exercise 22C

1 650 N

2 $F = 46$ N, $R = 754$ N

3 250 N

4 **a** 1.54 N

b 11.3 N

5 **a** 25.7 N

b 63.7 N

6 122 N, 54.2 N

7 **a** 5.88 N

b 140 N

8 268 N

Work it out 22.1

Solution 2 is correct.

Exercise 22D

1 **a** $12.7\ \mathrm{m\ s^{-2}}$

b $4850\ \mathrm{N}$

2 $12.6\ \mathrm{N}, 22.1\ \mathrm{N}$

3 $37.5\ \mathrm{N}$

4 $1120\ \mathrm{N}$

5 **a** $312\ \mathrm{N}$

b $0.720\ \mathrm{s}$

6 **a** $901\ \mathrm{N},\ 530\ \mathrm{N}$

b $833\ \mathrm{N},\ 490\ \mathrm{N}$

7 **a** $10.8\ \mathrm{kN}$

b $5.5\ \mathrm{kN},\ 2.75\ \mathrm{kN}$

8 **a** Driving ($\text{driving force } = 2000\ \mathrm{N}$)

b Tension of $1000\ \mathrm{N}$ between the locomotive and the first carriage, thrust of $1000\ \mathrm{N}$ between the two carriages

Exercise 22E

1 $745\ \mathrm{N}$

2 $1.63\ \mathrm{m\ s^{-2}},\ 57.2\ \mathrm{N}$

3 **a** $58.8\ \mathrm{N}$

b $2.8\ \mathrm{m\ s^{-2}},\ 42\ \mathrm{N}$

4 $2.53\ \mathrm{kg}$

5 $k = 5$

6 **a** $19.6\ \mathrm{N}$

b $1.23\ \mathrm{m\ s^{-2}}$

7 $p = 26.8,\ q = 19.6$

8 $18.4\ \mathrm{N},\ 151^\circ$

9 $0.986\ \mathrm{ms^{-2}}$

10 $0.58,\ 1.10$

11 **a** $2x$

b $2a$

c $35.3\ \mathrm{N}$

12 $5 \leqslant M \leqslant 15$

13 Discussion

Mixed practice 22

1 $50.9\ \mathrm{kg}$

2 **a** $0.966\ \mathrm{m\ s^{-2}}$

b $529\ \mathrm{N}$

3 $576\ \text{N}$

4 $1.18\ \text{N}$

5 $29.4\ \text{N},\ 58.8\ \text{N}$

6 $707\ \text{N}$

7 **a** $2790\ \text{N}$

b $1810\ \text{N}$

8 **a** $1130\ \text{N}$

b **i** $2.1\ \text{m s}^{-2}$ downwards

ii $4040\ \text{N}$

9 $42.2\ \text{kg}$

10 **a** 1.5

b $51.0\ \text{N},\ 14.7\ \text{N}$

11 $39.2\ \text{N},\ 46.0\ \text{N}$

12 **a** $33.16\ \text{N}$

b $18\ \text{N}$ or $26\ \text{N}$

c $1.84\ \text{kg}$ or $2.65\ \text{kg}$

13 **a** $400\ \text{N},\ 1300\ \text{N}$

b $700\ \text{N},\ 2350\ \text{N}$

14 **a** $4.85\ \text{N}$

b 0.55

c $0.84\ \text{m s}^{-1}$

d $0.756\ \text{m}$

15 **a** $22.0\ \text{N}$

b No (stops after total distance of $1.30\ \text{m}$)

16 **a** $753\ \text{N}$

b $75\ \text{N}$

c Box B moves on top of box A.

17 **a** $1.43\ \text{s}$

b $3.13\ \text{m}$

c $2.14\ \text{s}$

Focus on … 4

Focus on … Proof 4

1 proof

2 **a** $\dfrac{2s}{a+b}$

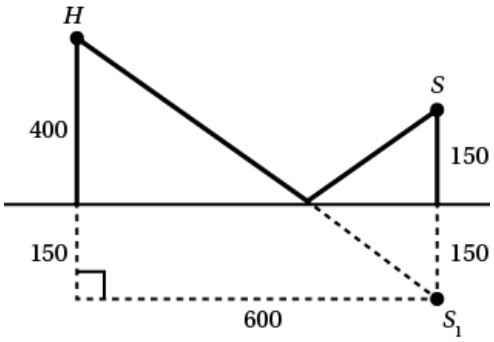
b $\dfrac{s\,(a+b)}{2ab}$

c Proof

d Proof

Focus on ... Problem solving 4

1 814 m



2 $\sqrt{5}\text{ m}$; $\sqrt{3}$ through cube

3 $3\sqrt{2}$; $\sqrt{14}$ through cuboid

4 $\sqrt{h^2+\pi^2r^2}$

Focus on ... Modelling 4

1 12.5 m

2 0.25%

3 0.04%

4 **a** **i** 510 m

ii 0.25%

iii 0.04%

5 $120a = 180 - 40v$; remember that $a = \dfrac{dv}{dt}$

6 $v = 1.5t$

7 Note that when there is air resistance, the cyclist can never exceed the speed of $4.5\,\text{m s}^{-1}$ (this is called **terminal velocity**).

	a $0.5\,\text{m s}^{-1}$	b $2\,\text{m s}^{-1}$	c $5\,\text{m s}^{-1}$
without air resistance	$0.333\,\text{s}$	$1.33\,\text{s}$	$3.33\,\text{s}$
with air resistance	$0.353\,\text{s}$	$1.76\,\text{s}$	–
% difference	5.7%	24%	–

Ignoring air resistance is only reasonable in the initial stages of the motion.

8 **a, b** and **9 a, b**

	a Speed after 1 second (m s^{-1})		b Speed after 10 seconds (m s^{-1})	
	air resistance ignored	with air resistance	air resistance ignored	with air resistance
parachutist	9.8	9.69	98	51.1
coin	9.8	8.79	98	18.8

9 **c** **i** 1 s

ii 10 s

d The model is suitable for short distances (a few metres), although there is already a noticeable difference for coin (but not for the parachutist). It is not suitable over distances as long as $10\,\text{m}$;

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the discrepancy is much larger for the coin.

Cross-topic review exercise 4

1 $(6\mathbf{i} + 4\mathbf{j})\ \text{m s}^{-2}$

2 $2.4\ \text{m s}^{-1}$

3 $6.34\ \text{m s}^{-1}$

4 **a** 1.90

b 71.6°

5 **a** $4.9\ \text{m s}^{-1}$

b $1.23\ \text{m}$

c $5.88\ \text{m s}^{-1}$

6 **a** $4.6\ \text{m s}^{-2}$ downwards

b $2500\ \text{N}$

c mass of the cable was not included in the calculation

7 **a** $u = 4,\ a = 6$

b $783\ \text{kg}$

8 $297\ \text{N}$, thrust

9 **a** **i** $v = 3t^2 - 22t + 35$

ii $\frac{7}{3}$ and $5\ \text{s}$

iii $35\ \text{m s}^{-1}$

b $44\ \text{m}$

10 3

11 **a** $27.4\ \text{m s}^{-1}$

b $107\ \text{m}$

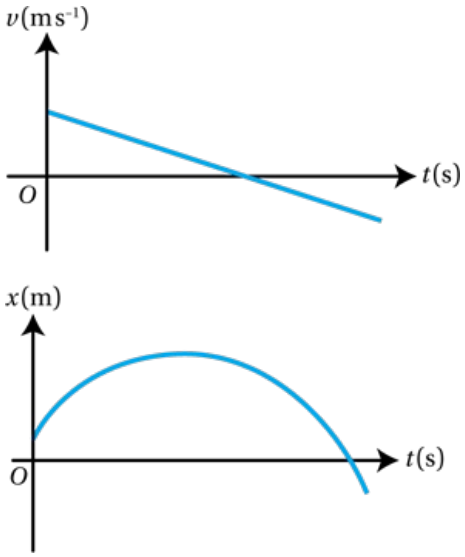
c **i** Figure 2

ii Figure 1 has zero initial velocity; figure has decreasing velocity.

12 **a** $8.62\ \text{m s}^{-1}$

b $1.39\ \text{s}$

c



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13 **a** $a = 0.012t - 0.18$

b Proof

c $30\ \text{m}$

14 **a** $v = 0.3t^2 - 0.6t + 0.2,\ a = 0.6t - 0.6$

b Proof

c $0.423\ \text{s},\ 1.58\ \text{s}$

d $3\ \text{s}$

AS Practice paper 1

1 $\frac{4}{3}$

2 $\frac{19}{7}$

3 **a** $-11\mathbf{i} + 6\mathbf{j}$

b $\sqrt{370}$

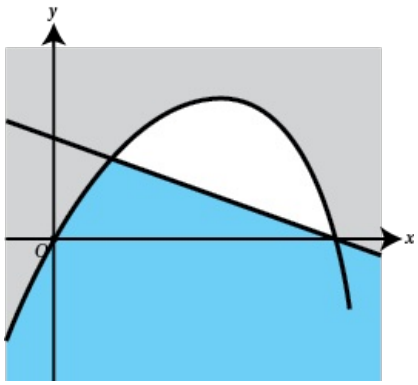
4 $\text{f'}(x) = 6x$

5 **a** $AB = \cos\alpha,\ BC = \sin\alpha$

b Proof

6 **a** $\left(\frac{1}{4}, \frac{23}{8}\right)$ and $(6, 0)$

b



c $p = -2,\ q = -15$

7 $(x - 21)^2 + y^2 = 841$

8 0.0725

9 **a** Obtain a list of all registered voters, numbered sequentially. Use a random number generator to select 200 number from the list, ignoring repeats. The sample is made up of voters with those numbers.

b $p = 0.133 > 0.05$, insufficient evidence that the proportion of undecided voters has decreased

10 **a** No correlation

b 2.24

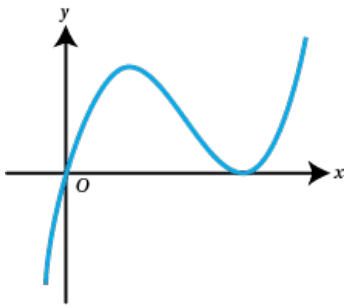
c 6 outliers above 6.23

d Urban

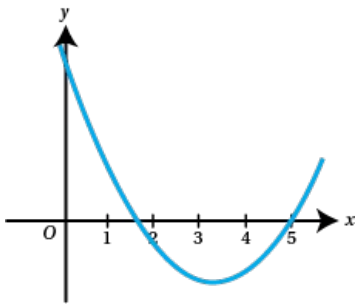
AS Practice paper 2

1 $30°,\ 150°$

2 **a**



b



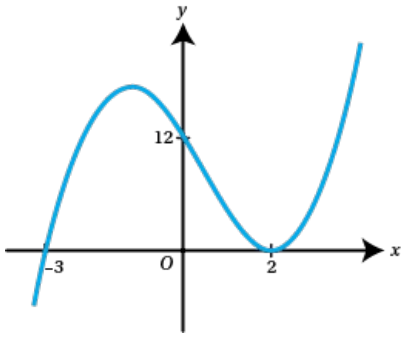
3 $-1,\ 2$

4 **a** $1 - 4x + 7x^2 - \ 7x^2 + ...$

b -21

5 **a** $a = -8,\ \mathrm{f}(x) = (x-2)^2\ (x+3)$

b



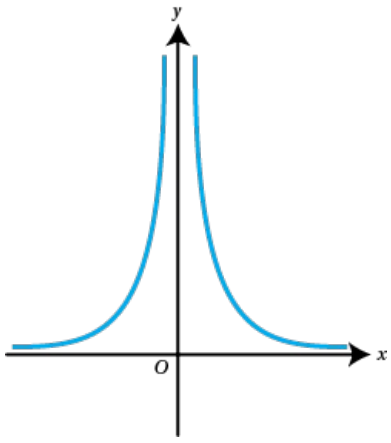
6 **a** e.g. 19

b Proof

7 **a** $A = 30\,000,\ k = 0.4$

b 8.77 days

8 **a**



b Proof

9 $(6.8\mathbf{i} - 0.2\mathbf{j})\ \mathrm{ms}^{-2}$

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10 **a** 7.98ms^{-1} downwards

b 2.47seconds

c t would increase because air resistance would make the ball travel more slowly.

11 $138\,\text{N}$

12 **a** $a = -4.05\text{ms}^{-2}$

b $t = 13$

c $9.96\,\text{s}$

Gateway to A Level – GCSE revision

A. Expanding brackets and factorising

You need to be able to multiply together (or expand) sets of brackets, and simplify the resulting expression by collecting together like terms.

To expand two sets of brackets, multiply each term in the first bracket by each term in the second bracket.

Worked example 1

Expand and simplify $(2+x)(1+3x+y)$

Solution	Comments
$(2+x)(1+3x+y)$	
$=2(1+3x+y)+x(1+3x+y)$ $=2+6x+2y$ $\quad +x+3x^2+xy$	Multiply each term in the second bracket by 2. Multiply each term in the second bracket by $+x$.
$=2+7x+2y+3x^2+xy$	Add the results together, collecting like terms ($6x$ and x).

If there are three brackets, it is a good idea to expand two of them and then multiply the result by the third set of brackets.

Worked example 2

Expand and simplify $(x-3)(x+4)(x-5)$

Solution	Comments
$(x-3)[(x+4)(x-5)]$ $=(x-3)\left[x^2-x-20\right]$	Expand any two brackets, for example the second and third.
$=x\left[x^2-x-20\right]-3\left[x^2-x-20\right]$ $=x^3-x^2-20x$ $\quad -3x^2+3x+60$	Then multiply by the first bracket as in Worked example 1.
$=x^3-4x^2-17x+60$	Add the results together, collecting like terms ($-20x$ and $3x$).

You also need to be able to factorise expressions, which means putting brackets in and writing the sum or difference as a product. You can often do this by finding a factor common to each term in the sum.

Worked example 3

Factorise $12ab^2+9a^2b$

Solution	Comments
$12ab^3 + 9a^2b = 3ab(4b^2 + 3a)$	The largest factor of: • 12 and 9 is 3 • a and a^2 is a • b^3 and b is b So the largest factor in both terms is $3ab$.

Sometimes you can factorise an expression by first splitting it up into two parts and factorising each one separately. This sometimes reveals a common factor in both parts.

Worked example 4

Factorise

Solution	Comments
$ax + bx + ay + by = x(a + b) + y(a + b)$	x is a factor of the first two terms and y of the second two.
$= (x + y)(a + b)$	You can now take out a factor of $(a + b)$.

Note

Gateway to A Level Section B looks at factorising quadratic expressions into two brackets using a method similar to this.

EXERCISE A

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1 Simplify:

a $1 + x + 1 + y$

b $2 + 2a - 3a + a$

c $5x + xy - y^2 + x^2 - 3xy$

d $(3z)^2 + 2z^2$

e $x(x + 2) - 2(x - 1)$

f $x(x + y) + 2y(x + z) + 3z(y + x)$

2 Expand and simplify:

a $(x + 3)(x + 5)$

b $(2a + 1)(3a - 2)$

c $(x - 3)^2$

d $(x + 1)(y + 1)$

e $(2x + y)^2$

f $(a - 1)(a^2 + a + 1)$

g $(x + y - 1)^2$

h $(x + 1)(x + 2)(x + 3)$

i $(3y - 2)(2y + 3)(y - 4)$

3 Factorise:

a $8x + 12y$

b $5y - 10y^2$

c $18x^3y - 4x$

d $14a^2b^2 - 7ab$

e $32x^3y + 4x^2y^2 + 2xy^4$

f $15a^2b^2 - 12ab + 9a^2$

4 Factorise:

a $xy + y + x + 1$

b $x^2 + xy + x + y$

c $ab + 3a + 2b + 6$

d $6ab - 8a + 3b - 4$

e $pq - p - q + 1$

f $6p^2q - 4pq - 15p + 10$

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B. Factorising quadratics

You need to be able to factorise quadratic expressions of the form $\boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c}$.

If the coefficient of x^2 is 1, look for a factorisation of the form $(x - p)(x - q)$. The numbers p and q are such that their product is $\boldsymbol{c}$ and they add up to $\boldsymbol{b}$.

Worked example 1

Factorise $x^2 - 7x + 12$

Solution	Comments
$x^2 - \mathbf{7}x + \mathbf{12}$	Look at factors of $\mathbf{12}$. You need two negative numbers to give $+12$. $12 = (-1) \times (-12)$ $= (-2) \times (-6)$ $= (-3) \times (-4)$
$= (x - 3)(x - 4)$	-3 and -4 add up to $\mathbf{-7}$.

If the coefficient of x^2 is **not** 1 x^2, you need to adapt this procedure slightly. First look for two numbers that multiply to $\boldsymbol{ac}$ and add up to $\boldsymbol{b}$. Then split the middle term and factorise in pairs.

Worked example 2

Factorise $6x^2 + 11x - 10$

Solution	Comments
$\mathbf{6} \times (\mathbf{-10}) = -60$	You need two numbers that multiply to -60 and add up to 11. This means you need one positive and one negative number, with the positive number being larger.
The two numbers are -4 and 15 $(-4 + 15 = 11)$	Look at factors of 60: $-60 = (-1) \times 60$ $= (-3) \times 20$ $= (-4) \times 15$
$6x^2 + \mathbf{11x} - 10 = 6x^2 \mathbf{- 4x + 15x} - 10$	Split the middle term: $11x = -4x + 15x$
$= 2x(3x - 2) + 5(3x - 2)$	Factorise in pairs: The first two terms have a common factor $2x$ and the last two terms have a common factor 5.
$= (3x - 2)(2x + 5)$	Finally, take out the common factor $(3x - 2)$.

An alternative method is to simply look for numbers that work. If the coefficients are small prime

numbers this can be quite quick, but otherwise the method shown in Worked example 2 is more efficient.

Worked example 3

Factorise $5x^2 + 9x - 2$

Solution	Comments
$5x^2 + 9x - 2 = (5x)(x)$	The only way to factorise 5 is 5×1.
$= (5x - 1)(x + 2)$	The missing numbers in brackets are 1 and 2. One is positive one is negative. Try possible combinations until you find one that gives the middle term $+9x$.

Before you use one of these methods, you should check whether there is a common factor that can be taken out of all three terms. For example:

$$\begin{aligned} 7x^2 - 35x + 42 &= 7\left(x^2 - 5x + 6\right) \\ &= 7(x - 2)(x - 3) \end{aligned}$$

A special example of factorising a quadratic is the **difference of two squares**:

$a^2 - b^2 = (a - b)(a + b)$

Worked example 4

Factorise $9x^2 - 25$

Solution	Comments
$9x^2 - 25 = (3x - 5)(3x + 5)$	$9x^2$ is the square of $3x$ and 25 is the square of 5.

EXERCISE B

Factorise these expressions:

1 **a** $x^2 + 2x - 3$

b $x^2 - 2x - 35$

c $a^2 - 8a - 20$

d $b^2 + 13b + 40$

2 **a** $2x^2 - 4x - 30$

b $5x^2 + 5x - 30$

c $2x^2 - 4x - 16$

d $3x^2 - 18x + 27$

3 **a** $b^2 - 81$

b $k^2 - 100$

c $16y^2 - 49$

d $36x^2 - 81$

4 **a** $3x^2 - 14x - 5$

b $2x^2 - x - 3$

c $5x^2 - 14x - 3$

d $2x^2 + x - 10$

e $6x^2 - 5x + 1$

f $15x^2 + 13x + 2$

g $6x^2 - x - 15$

h $10x^2 + x - 21$

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C. Quadratic equations

You have met two ways of solving quadratic equations:

- Factorising
- Using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Worked example 1

Solve the equation:

$x^2 + x = 12$

Solution	Comments
$x^2 + x - 12 = 0$	For both methods, the equation needs to be $= 0$.
$(x + 4)(x - 3) = 0$	You are looking for two numbers that add up to 1 and multiply to give -12. It must be a positive and a negative number.
$x + 4 = 0 \Rightarrow x = -4$ $x - 3 = 0 \Rightarrow x = 3$	Make each bracket $= 0$.

You can also factorise by taking out common factors (rather than using two sets of brackets).

Worked example 2

Solve the equation:

$6x^2 + 4x = 0$

Solution	Comments
$2x(3x + 2) = 0$	Both terms have a common factor $2x$.
$2x = 0 \Rightarrow x = 0$ $3x + 2 = 0 \Rightarrow x = -\dfrac{2}{3}$	This now has two factors: $(2x)$ and $(3x + 2)$. Make each factor $= 0$.

If the equation doesn't seem to factorise, use the formula. (You can also use the formula for equations that *do* factorise: you will then get rational numbers as answers.)

Worked example 3

Solve the equation:

$3x^2 - 5x = 1$

Solution	Comments
$3x^2 - 5x - 1 = 0$	The equation needs to $= 0$.
$x = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2 \times 3}$	The equation doesn't seem to factorise, so use the formula with $a = 3, b = -5, c = -1$.
$= \dfrac{5 \pm \sqrt{25 + 12}}{6}$ $= \dfrac{5 \pm \sqrt{37}}{6}$	Be careful: $(-5)^2$ is $+25$.
$x = 0.180$ or 1.85 (to 3 s. f.)	You usually round the answers to three significant figures.

EXERCISE C

1 Solve by factorising:

a $x^2 + 5x - 6 = 0$

b $x^2 - 8x + 12 = 0$

c $x^2 - 5x - 24 = 0$

d $6x^2 + 15x = 0$

e $x^2 - 81 = 0$

f $9x^2 - 16 = 0$

g $2x^2 - 7x - 4 = 0$

h $3x^2 + 16x - 12 = 0$

i $5x^2 - 19x + 12 = 0$

j $6x^2 + x - 12 = 0$

2 Solve by using the formula, giving your answers to 3 s.f:

a $x^2 + 5x + 2 = 0$

b $x^2 + 3x - 6 = 0$

c $x^2 - 2x - 4 = 0$

d $2x^2 - 6x + 3 = 0$

e $3x^2 - x - 5 = 0$

f $6x^2 - 4x - 1 = 0$

3 Solve these equations. Check whether you can factorise first, and if you can't then use the formula.

a $x^2 - 2x - 15 = 0$

b $x^2 + x - 3 = 0$

c $2x^2 - 4x = 5$

d $2x^2 - 4x - 16 = 0$

e $3x^2 - 18x + 27 = 0$

f $10x^2 - 15x = 0$

g $x^2 - 49 = 0$

h $6x^2 = 24$

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D. Linear equations and inequalities

The most important thing to remember when solving equations is that you have to do the same thing to both sides. A good tactic is to get all the unknowns on one side and everything else on the other side and then divide. Do not expect all the answers to be whole numbers.

Worked example 1

Solve:

$3 + \frac{x}{5} = 2 - x$

Solution	Comments
Multiply by 5 : $15 + x = 10 - 5x$	Getting rid of fractions is a good idea. Remember to multiply the *whole* of each side, not just the terms you are most interested in.
Add $5x$: $15 + 6x = 10$ Subtract 15: $6x = -5$	Get all terms containing x on one side and everything else on the other. Try to get a positive coefficient of x. Make x the subject.
$x = -\frac{5}{6}$	

You can solve linear inequalities in a similar way, with one important exception: when multiplying or dividing by a negative number, you must *reverse* the inequality sign.

Worked example 2

Solve the inequality:

$1 - 4x \leqslant \frac{x}{2} + 1$

Solution	Comments
Multiply by 2 : $2 - 8x \leqslant x + 2$	Getting rid of fractions is a good idea.
Subtract x: $2 - 9x \leqslant 2$ Subtract 2: $-9x \leqslant 0$	Get all terms containing x on one side and everything else on the other.
$x \geqslant 0$	Divide by -9 and remember to reverse the inequality sign.

The issue of changing the inequality sign can be avoided if you always try to get a positive coefficient for x. Worked example 3 demonstrates a different method of solving the same inequality.

Worked example 3

Solve the inequality:

$1 - 4x \leqslant \frac{x}{2} + 1$

Solution	Comments
$2 - 8x \leqslant x + 2$	Multiply through by 2 to get rid of the fraction.
Add $8x$: $2 \leqslant 9x + 2$ Subtract 2 : $0 \leqslant 9x$	Get all terms containing x on one side and everything else on the other. Try to get a positive coefficient of x
$0 \leqslant x$	Divide.
$x \geqslant 0$	Conventionally x is written first, so you need to rewrite this.

EXERCISE D

1 Solve for x:

a $10 + 4x = 7$

b $7 + 5x = 9 - x$

c $1 - (3 - 5x) = 8 + 2x$

d $\frac{4 + 5x}{3} + 1 = x$

e $3 - 2x = \frac{5 + x}{3}$

f $1 + \frac{1}{x} = 3 - \frac{1}{x}$

2 Solve these inequalities:

a $11 + 4x > 29$

b $2 + 5x \leqslant 4 - x$

c $2x + 1 < 4x - 3$

d $3 - \frac{x}{2} > 2x + 1$

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E. Types of numbers

There are many different types of number and it is very important that you know some of the labels applied to them:

$\mathbb{N}$	Natural numbers	$\{\ldots 1,\ 2,\ 3,\ 4\ldots\}$
$\mathbb{Z}$	Integers	$\{\ldots -3, -2, -1,\ 0,\ 1,\ 2,\ 3\ldots\}$
$\mathbb{Q}$	Rational numbers	$\left\{\ldots -\frac{3}{5},\ 0,\ 5.3,\ 6,\ \frac{43\,984}{865}\ldots\right\}$
$\mathbb{Q}$	Irrational numbers	$\{\ldots -\sqrt{2},\ 1-\sqrt{3},\ \pi\ldots\}$
$\mathbb{R}$	Real numbers	$\left\{\ldots -\sqrt{2},\ 0,\ \pi,\ 5.3,\ 6,\ \frac{43\,984}{865}\ldots\right\}$

The rational numbers are any numbers that can be written as a fraction of two integers.

Irrational numbers cannot be written as a fraction of two integers. The main irrational numbers that you know at the moment are surds and anything involving π.

The real numbers comprise all numbers you have met so far. If no other indication is given, assume that you are working with real numbers.

EXERCISE E

1 Tick the boxes to indicate which sets of numbers the numbers are in. The first one has been done for you.

	Number	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
a	0	✓	✓	✓	✓
b	1				
c	$\frac{1}{3}$				
d	-6				
e	-1.5				
f	$0.\dot{3}$				
g	$\sqrt{2}$				
h	$\sqrt{25}$				
i	$\pi + 2$				

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F. Functions

A function is a rule that assigns to each input value a unique output value. To evaluate a function at a particular value, you substitute the given value into the algebraic expression.

Although you will be allowed to use your calculator in examinations, it is important that you are confident about rules of arithmetic so that you can apply them to algebraic expressions.

Worked example 1

If $\mathrm{f}(x) = x^2 - 3x$, find

a $\mathrm{f}(7)$

b $\mathrm{f}(-2)$

c $\mathrm{f}\left(\frac{1}{2}\right)$

Solution	Comments
a $\mathrm{f}(7) = 7^2 - 3 \times 7$ $= 49 - 21$ $= 28$	Substitute $x = 7$ into $x^2 - 3x$.
b $\mathrm{f}(-2) = (-2)^2 - 3 \times (-2)$	Substitute $x = -2$ into $x^2 - 3x$.
$= 4 + 6$ $= 10$	Be careful with negatives: $(-2)^2 = 4$ and $-3 \times (-2) = +6$
c $\mathrm{f}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - 3 \times \frac{1}{2}$	Substitute $x = \frac{1}{2}$ into $x^2 - 3x$.
$= \frac{1}{4} - \frac{3}{2}$ $= \frac{1}{4} - \frac{6}{4}$ $= -\frac{5}{4}$	Make sure you are comfortable with the arithmetic of fractions.

You also need to simplify functions when the input is itself a function of x.

Worked example 2

If $\mathrm{f}(x) = x^2 - 3x$ find

a $\mathrm{f}(2x)$

b $\mathrm{f}(x + 1)$

c $\mathrm{f}(-x)$

Solution	Comments
a $f(2x) = (2x)^2 - 3 \times 2x$	Substitute $2x$ into $x^2 - 3x$.
$= 4x^2 - 6x$	Make sure you square **all** of $2x$: $(2x)^2 = 4x^2$
b $f(x+1) = (x+1)^2 - 3 \times (x+1)$	Substitute $x+1$ into $x^2 - 3x$.
$= x^2 + 2x + 1 - 3x - 3$ $= x^2 - x - 2$	Expand: $(x+1)^2 = (x+1)(x+1) = x^2 + 2x + 1$
c $f(-x) = (-x)^2 - 3 \times (-x)$	Substitute $-x$ into $x^2 - 3x$
$= x^2 + 3x$	Be careful with negatives: $(-x)^2 = x^2$ and $-3 \times (-x) = +3x$

EXERCISE F

1 Without using a calculator, evaluate each function at the given values.

a $f(x) = (2 - 3x)^2$ at

i $f(1)$

ii $f(-4)$

iii $f\left(\frac{1}{6}\right)$

b $g(x) = x^3 - 2x + 1$ at

i $g(3)$

ii $g(-1)$

iii $g(0)$

c $h(x) = \sqrt{2x - 1}$ at

i $h(5)$

ii $h\left(\frac{1}{2}\right)$

iii $h(13)$

d $f(x) = \frac{1}{3 - 4x}$ at

i $f(1)$

ii $f(-3)$

iii $f\left(-\frac{1}{4}\right)$

e $g(x) = \frac{2}{\sqrt{5 - x^2}}$ at

i $g(-1)$

ii $g(2)$

iii $g(-2)$

f $h(x) = \frac{3x + 2}{x - 4}$ at

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i $\mathrm{h}(0)$

ii $\mathrm{h}\left(\frac{2}{3}\right)$

iii $\mathrm{h}(6)$

2 Simplify each function for the given inputs.

a $\mathrm{f}(x) = 2x + 5$ at

i $\mathrm{f}(x - 1)$

ii $\mathrm{f}(x + 2)$

iii $\mathrm{f}(3x)$

b $\mathrm{g}(x) = 4 - 3x$ at

i $\mathrm{g}(-x)$

ii $\mathrm{g}(x - 2)$

iii $\mathrm{g}\left(\frac{1}{3}x\right)$

c $\mathrm{h}(x) = x^2 + 2x - 3$ at

i $\mathrm{h}(2x)$

ii $\mathrm{h}(x + 3)$

iii $\mathrm{h}(x - 1)$

d $\mathrm{f}(x) = 4 - x - 2x^2$ at

i $\mathrm{f}(x - 3)$

ii $\mathrm{f}(-x)$

iii $\mathrm{f}(-2x)$

e $\mathrm{g}(x) = \frac{1}{3 - 2x}$ at

i $\mathrm{g}(x + 1)$

ii $\mathrm{g}(x - 1)$

iii $\mathrm{g}\left(-\frac{1}{2}x\right)$

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G. Rules of indices

You need to be able to evaluate positive, negative and fractional powers.

Worked example 1

Evaluate:

a $(-4)^3$

b $\left(\frac{2}{3}\right)^{-2}$

c $64^{\frac{1}{3}}$

Solution	Comments
a $(-4) \times (-4) \times (-4)$	Write the exponent form as repeated multiplication.
$= 16 \times (-4) = -64$	Evaluate in parts.
b $\left(\frac{2}{3}\right)^2 = \left(\frac{2}{3}\right) \times \left(\frac{2}{3}\right) = \frac{4}{9}$	$x^{-2} = \frac{1}{x^2}$, so evaluate $\left(\frac{2}{3}\right)^2$ first ...
$\therefore \left(\frac{2}{3}\right)^{-2} = \frac{9}{4}$	... and then take the reciprocal.
c $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$	The power $\frac{1}{3}$ means cube root.

You should also know the rules for working with powers:

$$x^a \times x^b = x^{a+b} \qquad \frac{x^a}{x^b} = x^{a-b} \qquad (x^a)^b = x^{ab}$$

Worked example 2

Simplify:

a $\frac{x^3 \times x^5}{x^2}$

b $(x^3)^4 \div (x^5)^2$

Solution	Comments
a $\frac{x^3 \times x^5}{x^2} = \frac{x^8}{x^2}$	Add powers when multiplying.
$= x^6$	Subtract powers when dividing.
b $(x^3)^4 \div (x^5)^2 = x^{12} \div x^{10}$	Multiply powers.
$= x^2$	Subtract powers when dividing.

EXERCISE G

1 Are these statements true or false?

a x^2 is always larger than x

b $3 \times 5^7 = 15^7$

c $6^{14} = 6 \times 6^{13}$

d $1^{99} = 1$

e $0^{40} = 1$

f $9^7 = 7^9$

g $2^4 + 2^4 = 2^5$

h $3^4 + 3^4 = 3^5$

2 Evaluate without a calculator:

a 5×2^3

b $\left(\frac{2}{3}\right)^3$

c $\left(-\frac{1}{4}\right)^2$

d $\left(-\frac{5}{3}\right)^3$

3 Evaluate the following without a calculator, leaving your answers as a fraction where appropriate:

a **i** $4^{\frac{1}{2}}$

ii $8^{\frac{1}{3}}$

b **i** $10000^{0.5}$

ii $81^{0.25}$

c **i** $\left(\frac{1}{25}\right)^{\frac{1}{2}}$

ii $\left(\frac{9}{16}\right)^{\frac{1}{2}}$

d **i** $8^{\frac{2}{3}}$

ii $25^{\frac{3}{2}}$

e **i** $100^{2.5}$

ii $81^{0.75}$

f **i** $\left(\frac{1}{16}\right)^{\frac{5}{4}}$

ii $\left(\frac{8}{27}\right)^{\frac{5}{3}}$

g **i** $8^{-\frac{1}{3}}$

ii $49^{-\frac{1}{2}}$

h **i** $\left(\frac{16}{9}\right)^{-\frac{1}{2}}$

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ii $\left(\frac{9}{16}\right)^{-\frac{3}{2}}$

4 Simplify:

a $x^3 \times x^5 \times x^2$

b $\frac{x^7}{x^2 \times x^3}$

c $\frac{(x^2)^3 \times x^5}{x^4}$

d $\frac{x^7}{(x^5)^2 \times x^3}$

5 **a** Write $64\sqrt{2}$ in the form 2^n.

b Write $\sqrt[3]{81}$ in the form 3^n.

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H. Surds

A surd (also called a root or a radical) is any number of the form $a + b\sqrt{c}$ where a, b and c are fractions or whole numbers.

The most important rule of surds deals with their product:

$\sqrt{ab} = \sqrt{a}\sqrt{b}$

There is no equivalent for the sum. In general:

$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

You can collect square roots of the same number together, so, for example:

$3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$.

One very useful tool for simplifying square roots is to take out any square factors of the number being square rooted. For example:

$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$.

Worked example

Write $\left(\sqrt{6} + \sqrt{2}\right)^2$ in the form $a + b\sqrt{3}$.

Solution	Comments
$\left(\sqrt{6} + \sqrt{2}\right)\left(\sqrt{6} + \sqrt{2}\right)$ $= 6 + \sqrt{6}\sqrt{2} + \sqrt{2}\sqrt{6} + 2$	Treat the expression as two brackets.
$= 8 + 2\sqrt{12}$	Simplify.
$= 8 + 2\sqrt{4}\sqrt{3}$ $= 8 + 4\sqrt{3}$	Then put it into the required form.

EXERCISE H

1 Write in the form $k\sqrt{5}$, where k is a whole number:

a $7\sqrt{5}-2\sqrt{5}$

b $\sqrt{5}+9\sqrt{5}-3\sqrt{5}$

c $\sqrt{20}+\sqrt{45}$

d $\sqrt{5}-2\left(\sqrt{125}-\sqrt{20}\right)$

e $3\sqrt{80}-5\sqrt{20}$

f $\sqrt{80}+7\sqrt{5}$

2 Write in the form $\sqrt{a}$ where a is a whole number:

a $4\sqrt{2}$

b $7\sqrt{3}$

c $\sqrt{7}+2\sqrt{7}$

d $\sqrt{3}+\sqrt{75}$

e $\sqrt{2}+\sqrt{8}$

f $\sqrt{32}+\sqrt{8}$

3 Write in the form $a+b\sqrt{3}$:

a $2\left(3-\sqrt{3}\right)-3\left(1-\sqrt{3}\right)$

b $\left(1+\sqrt{3}\right)-\left(1-\sqrt{3}\right)$

c $\left(1+\sqrt{3}\right)\left(2+\sqrt{3}\right)$

d $\left(1-\sqrt{3}\right)\left(1+2\sqrt{3}\right)$

e $\left(4-2\sqrt{3}\right)^2$

f $\left(\sqrt{15}-\sqrt{5}\right)^2$

Gateway to A Level – GCSE revision

I. Algebra of expressions

You need to be able to use rules of algebra to simplify expressions.

Rules of arithmetic

All the rules of arithmetic apply equally to algebra. In particular the same *order* of operations applies.

Collecting like terms

If you are adding terms together you can group together terms which are alike. This means that, for example, $2x$ and $5x$ can be collected to form $7x$, but x and x^2 *cannot* be collected together.

Dealing with brackets

If you multiply a sum in brackets by x you multiply each term in the sum by x. Most other operations – such as square rooting, or finding the sine – you do *not* do separately to each term.

Multiplying together two sets of brackets

If you are multiplying a sum by another sum you need to multiply each term in the first sum by each term in the second sum.

The difference between expressions and equations

With an equation you can ‘do the same thing’ to both sides and maintain the truth of the equation. With an expression you cannot do anything which will change the expression’s value. You can check whether you have succeeded by substituting values for any unknown elements. The expression should have the same value after simplification as it had beforehand.

Worked example

Expand and simplify $(2+x)(1+3x+y)$.

Solution	Comments
$2+6x+2y$	Multiply everything in the second bracket by 2.
$+x+3x^2+xy$	Multiply everything in the second bracket by $+x$.
$=2+7x+2y+3x^2+xy$	Add the results together, collecting terms which are alike ($6x$ and x).

EXERCISE I

1 Multiply out and simplify:

a $(x+3)(x+5)$

b $(z+1)(z-1)$

c $(2a+1)(3a-2)$

d $(x+1)(y+1)$

e $(x-3)^2$

f $(2x+y)^2$

g $(a-1)(a^2+a+1)$

h $(x+1)(x+2)(x+3)$

i $(x+y-1)^2$

2 Check the following simplifications by putting $x=2$ and $y=3$ into both forms:

a $(x+y)^2$ is the same as x^2+y^2

b $(3x)^2$ is the same as $9x^2$

c $\sqrt{xy}$ is the same as $\sqrt{x}\sqrt{y}$

d $\sqrt{y-x}$ is the same as $\sqrt{y}-\sqrt{x}$

e $\dfrac{1}{\left(\dfrac{x}{y}\right)}$ is the same as $\dfrac{y}{x}$

f $\dfrac{1}{x+y}$ is the same as $\dfrac{1}{x}+\dfrac{1}{y}$

g $\dfrac{x+y}{1+y}$ is the same as $\dfrac{x}{1}$

h $(x^y)^2$ is the same as x^{2y}

i 2^{x+y} is the same as 2^x+2^y

j $(5x)^2$ is the same as $5x^2$

Tip

Just because a simplification works for some values you try does not guarantee that it is true. However, if it fails to work for *any* values then you can be sure that it is wrong.

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J. Simultaneous equations

If you have two equations and two unknowns then you can often solve them to find just one set of values that works in both equations. There are two algebraic methods for doing this: elimination and substitution.

1 Elimination

You can add or subtract entire equations (and multiples of them) to entirely eliminate one of the variables.

Worked example 1

Solve the simultaneous equations:

$3x + 5y = 0$

$6x - 2y = 10$

Solution	Comments
$3x + 5y = 0 \quad (1)$ $6x - 2y = 10 \quad (2)$	Labelling the equations with numbers in brackets to show clearly how you are manipulating the equations.
$(1) \times 2: \quad 6x + 10y = 0 \quad (3)$ $(3) - (2): \quad 12y = -10$ $\Leftrightarrow y = -\frac{5}{6}$ Substituting into (1): $3x - \frac{25}{6} = 0$ $\Leftrightarrow x = \frac{25}{18}$	Look for a way to multiply one equation to match the coefficient of a variable in another.

2 Substitution

You can rearrange one of the equations to make one of the variables the subject and then substitute this into the other equation. Again, you will have formed an equation with just one variable. The advantage of this method is that you can use it with some non-linear simultaneous equations.

Worked example 2

Solve the simultaneous equations:

$x + y = 7$

$x^2 + y^2 = 169$

Solution	Comments

$y = 7 - x$	Rearrange the simpler equation to make one variable the subject.
$x^2 + (7 - x)^2 = 169$ $\Leftrightarrow x^2 + 49 - 14x + x^2 = 169$ $\Leftrightarrow 2x^2 - 14x = 120$ $\Leftrightarrow x^2 - 7x = 60$ $\Leftrightarrow x^2 - 7x - 60 = 0$ $\Leftrightarrow (x - 12)(x + 5) = 0$ $\Leftrightarrow x = 12 \text{ or } x = -5$	Substitute into the second equation.
When $x = 12, y = -5$ When $x = -5, y = 12$	Use $y = 7 - x$ to find y for each solution for x.

EXERCISE J

1 Find the solution of these simultaneous equations. Use whichever method you prefer:

a $y = 3x + 1$ and $y = 5x - 3$

b $x + 3y = 1$ and $y + x = 2$

c $2x + 2y = 1$ and $x - y = 2$

d $3y + 2x = 5$ and $x - y = 0$

e $2x - y - 1 = 0$ and $x + y = 5$

f $y = 2x - 1$ and $y + 2x = 7$

2 Solve these simultaneous equations:

a $x + y = 7$ and $x^2 + y^2 = 25$

b $y - x = 10$ and $x^2 + 2y = 44$

c $y^2 = 2x + 1$ and $2x + 3y = 17$

d $x + 2y = 13$ and $xy = 20$

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K. Direct and inverse proportion

Proportion is another word for fraction – it is a way of comparing one group to the whole quantity. So the proportion of vowels in the alphabet is $\frac{5}{26}$.

If two variables are 'in proportion' then if you divide one by the other the result is always the same. i.e. if y is proportional to x then $\frac{y}{x} = k$, a constant. You usually just rewrite this as $y = kx$. (Sometimes you use the phrase 'direct proportion' for this type of relationship.)

Worked example 1

If a is proportional to b, and when $a = 6, b = 10$, what is the value of a when $b = 34$?

Solution	Comments
If a is proportional to b then $a = kb$.	
When $a = 6, b = 10$ so: $6 = 10k$ $k = 0.6$	Use the given values to find k.
So $a = 0.6 \times 34 = 20.4$	

There are other types of proportional relationships. For example:

- y is proportional to x^2 means that $y = kx^2$
- y is inversely proportional to x means that $y = \frac{k}{x}$.

Worked example 2

Given that y is inversely proportional to x^2, and that $y = 40$ when $x = 2$, find an expression for y in terms of x.

Solution	Comments
$y = \frac{k}{x^2}$	
$40 = \frac{k}{2^2} \Rightarrow k = 40 \times 4 = 160$	Use the given information to find k.
$\therefore y = \frac{160}{x^2}$	

EXERCISE K

1 **a** If a is proportional to b and when $\mathrm{a} = 3, \mathrm{b} = 6$, find a formula for a in terms of b.

b If r is proportional to s and when $r = 20$, $s = 15$, find a formula for r in terms of s.

c If x is proportional to y and when $x = 4, y = 10$, find x when $\mathrm{y} = 12$.

d If x is proportional to y and when $x = 10, y = 7$, find y when $x = 25$.

e If p is proportional to q^2 and when $p = 5, q = 10$, find a formula for p in terms of q.

f If r is proportional to $\sqrt{s}$ and when $r = 1, s = 4$, find r when $s = 16$.

2 **a** Given that y is inversely proportional to x, and that $y = 4$ when $x = 3$, find the value of y when $x = 2$.

b F is inversely proportional to r^2 and $F = 35$ when $r = 2$. Find an expression for F in terms of r.

c Given that a is inversely proportional to $\sqrt{S}$, and that $a = 3$ when $S = 16$, find the value of S when $a = 6$.

d If c is inversely proportional to d, and $c = 12$ when $d = 3$, find an expression for c in terms of d.

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L. Using coordinates

You can describe any point in a plane relative to an origin using two numbers, which you call *coordinates*. The x-coordinate describes how far to the right the point is from the origin and the y-coordinate describes how far up a point is from the origin. A negative coordinate means the point is to the left or down respectively.

You can calculate a gradient between two points, giving a measure of the steepness of the line connecting the two points. If the gradient is positive it *increases* from left to right. If it is negative it *decreases* from left to right. If the two points have coordinates (x_1, y_1) and (x_2, y_2) then:

Key point

Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

There are also simple formulae for the point exactly half-way between two coordinates and the distance between two points.

Key point

Midpoint $= \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Tip

Notice that this is just the average of the coordinates of the two points.

Key point

Distance between two points $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Tip

This formula follows from Pythagoras' theorem.

Worked example

Point A has coordinates $(1, 5)$ Point B has coordinates $(-3, 3)$

a What is the gradient of the line connecting A and B?

b What are the coordinates of the midpoint of A and B?

c What is the distance between A and B?

Solution	Comments
	Use the formula for gradient.

a	$\text{Gradient} = \dfrac{y_2 - y_1}{x_2 - x_1}$ $= \dfrac{3-5}{-3-1}$ $= -\dfrac{2}{-4} = \dfrac{1}{2}$	
b	$\text{Midpoint} = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$ $= \left(\dfrac{1+-3}{2}, \dfrac{5+3}{2}\right)$ $= \left(\dfrac{-2}{2}, \dfrac{8}{2}\right)$ $= (-1, 4)$	Use the formula for midpoint.
c	$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $= \sqrt{(-3-1)^2 + (3-5)^2}$ $= \sqrt{(-4)^2 + (-2)^2}$ $= \sqrt{16+4} = \sqrt{20}$	Use the formula for distance.

EXERCISE L

1 Find the gradient of the line connecting:

a $(1, 4)$ and $(3, 7)$

b $(4, -2)$ and $(9, 2)$

c $(-3, -3)$ and $(5, -3)$

2 Find the midpoint of these points:

a $(1, 4)$ and $(3, 7)$

b $(4, -2)$ and $(9, 2)$

c $(-3, -3)$ and $(5, -3)$

3 Find the distance between the two points:

a $(1, 4)$ and $(3, 7)$

b $(4, -2)$ and $(9, 2)$

c $(-3, -3)$ and $(5, -3)$

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M. Straight line graphs

If you are given a rule connecting the x-coordinate and the y-coordinate, only some points on the plane will satisfy them. If the rule is of the form $y = mx + c$ then those points will lie in a straight line with gradient m and the line will meet the y-axis at $(0, c)$.

If you are given the gradient and one point the line passes through, you can use that information to find the value of c.

If two lines have the same gradient then they are parallel.

Worked example

A line has equation $3x + 2y = 2$. Find the equation of a line parallel to this line through the point $(1, 2)$.

Solution	Comments
$2y = 2 - 3x \Rightarrow y = 1 - \frac{3}{2}x$ So the gradient is $-\frac{3}{2}$	Rearrange into the form $y = mx + c$ to find the gradient.
The required line also has gradient $-\frac{3}{2}$ and passes through the point $(1, 2)$	Parallel lines have the same gradient.
$y = mx + c$: $2 = -\frac{3}{2} \times 1 + c$ $\Rightarrow c = \frac{7}{2}$	You know m and one pair of (x, y) values.
$\therefore y = -\frac{3}{2}x + \frac{7}{2}$	

EXERCISE M

1 Find the gradient and y-intercept of these straight lines:

a $y = 2x - 7$

b $y = 5 - x$

c $3x + y = 9$

d $y + 5 = 3(x - 1)$

e $5y = 10x + 1$

f $3y + 2x = 0$

2 Find the equation of a line:

a through $(1, 3)$ with gradient 7

b through $(2, 2)$ with gradient $-\frac{1}{2}$

c through $(0, 0)$ with gradient -5.

3 Find in the form $y = mx + c$ the equation of:

a a line parallel to $y = 3x - 1$ through $(1, 1)$

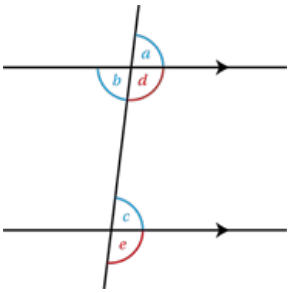
b a line parallel to $x + y = 5$ through $(2, 4)$

c a line parallel to $3x - 2y + 4 = 0$ through $(0, 0)$.

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N. Angles and bearings

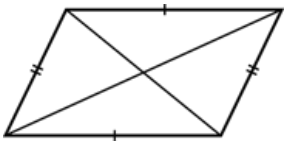
Parallel lines



- $a = b = c$
- $d = e = 180^\circ - a$

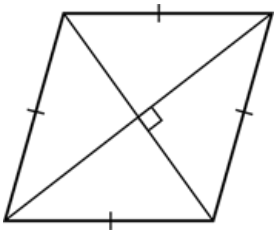
Parallelograms and rhombuses

Parallelogram:



- opposite sides are equal
- opposite angles are equal
- adjacent angles add up to 180°
- diagonals bisect each other

Rhombus:

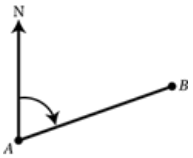


A rhombus is a type of parallelogram in which all four sides are equal. There are two further useful facts about a rhombus:

- the diagonals are perpendicular to each other
- the area is equal to $\frac{d_1 d_2}{2}$, where d_1 and d_2 are the lengths of the diagonals.

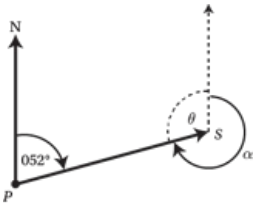
Bearings

Bearings are a way of describing directions by specifying the angle measured clockwise from the north.



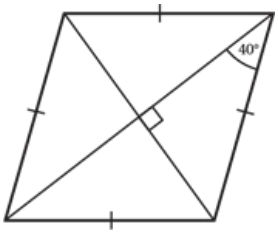
Worked example

The bearing of S from P is 052° Find the bearing of P from S.

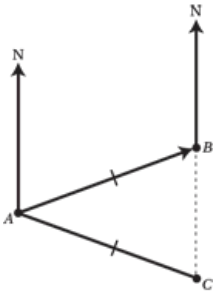
Solution	Comments
	Draw a diagram and label required angles.
$\theta = 180 - 52 = 128^\circ$ $\therefore\ \alpha = 360 - \theta = 232^\circ$	Angles 52° and θ add up to 180° (see 'parallel lines').

EXERCISE N

1 Find the angles at the four vertices of this rhombus:



2 The bearing of B from A is 040°. The distance from A to C is the same as the distance from A to B, and C is directly south of B.



Find the bearing of:

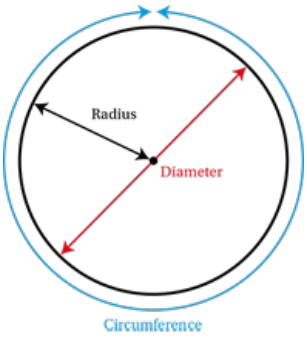
a A from B

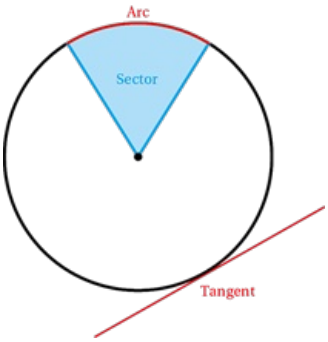
b A from C.

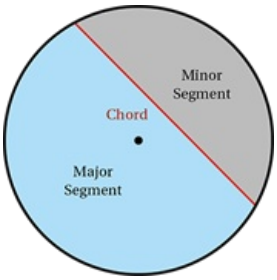
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O. Circle theorems

You need to know this terminology:

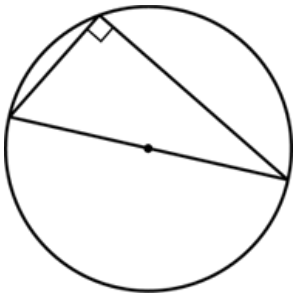




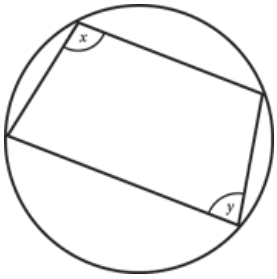


These theorems about angles in circles can be useful in solving problems about diameters and tangents:

1. The angle in a semicircle is 90°.

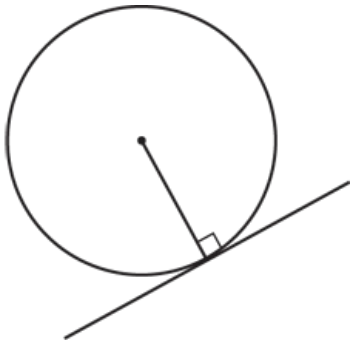


2. Opposite angles in a cyclic quadrilateral (a quadrilateral with all four corners on the circumference of a circle) add up to 180°.



$x + y = 180^\circ$

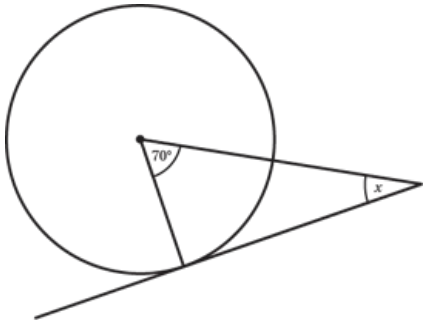
3 A tangent meets its radius at right angles.



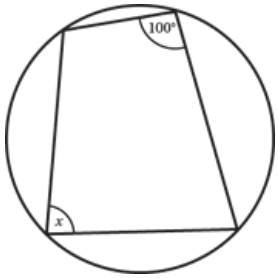
EXERCISE O

1 Find the value of the angle marked x in these diagrams, giving reasons for your answers:

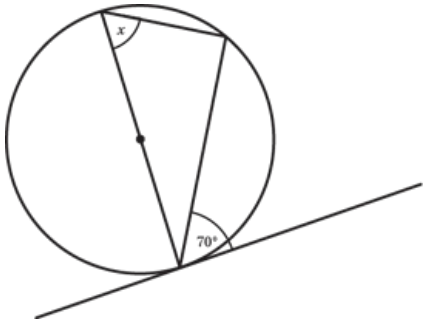
a



b



c



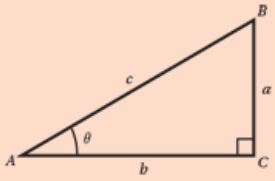
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P. Pythagoras and trigonometry

There is a convention for labelling triangles with the angles in capital letters opposite the side with the same letter but in lower case.

In a right-angled triangle the longest side is always opposite the right angle and it is called the hypotenuse. These rules *only* apply to right-angled triangles:

Key point



Pythagoras' theorem:

$$c^2 = a^2 + b^2$$

Trigonometric ratios:

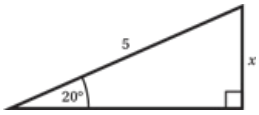
$$\sin\theta = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \cos\theta = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \tan\theta = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}$$

Pythagoras' theorem also works in reverse – if you have a triangle with $c^2 = a^2 + b^2$ then it is right-angled with the right angle at C.

If you know the lengths in a right-angled triangle you can then use these to find the angles. To do this you have to 'undo' one of the trigonometric ratios to get just θ using the $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ buttons on your calculator.

Worked example 1

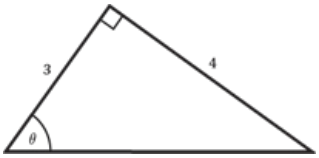
Find the length x in this diagram:



Solution	Comments
$\sin 20^\circ = \frac{x}{5}$ $x = 5\sin 20^\circ = 1.71 (3\text{s.f.})$	First decide which trig ratio to use. The relevant sides are the side opposite the angle and the hypotenuse – this means sin.

Worked example 2

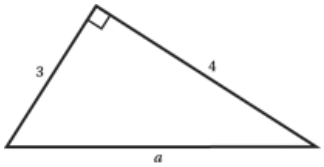
Find the angle θ in this diagram:

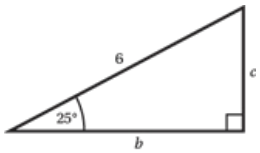


Solution	Comments
$\tan\theta = \frac{4}{3}$	First decide which trig ratio to use. The relevant sides are the side opposite the angle and the side adjacent to the angle – this means tan.
$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ \left(3\text{ s.f.}\right)$	To find θ you need to undo the tangent operation.

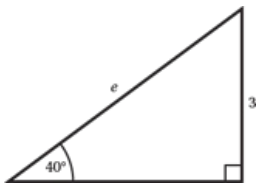
EXERCISE P

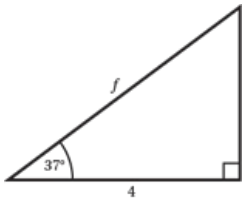
1 Find the unknown lengths a to f :



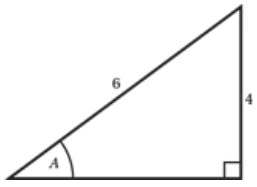


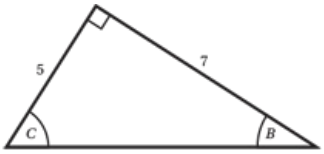


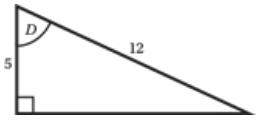


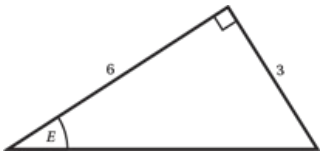


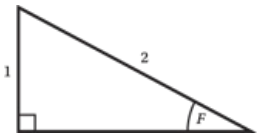
2 Find the unknown angles A to F:











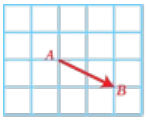
3 A ship is $10\,\mathrm{km}$ east and $8\,\mathrm{km}$ north of a lighthouse. Find the bearing of the ship from the lighthouse.

Gateway to A Level – GCSE revision

Q. Vectors

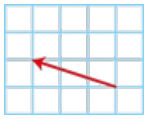
You can represent vectors either by drawing arrows or by using the column vector notation.

Worked example 1



a Write the vector $\overrightarrow{AB}$ as a column vector.

b Represent the vector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ on the grid.

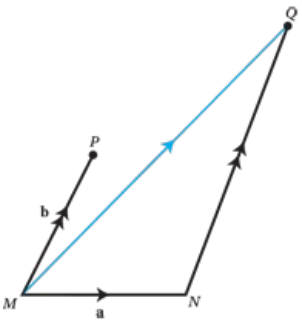
Solution	Comments
$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$	The top number shows the number of units to the right. The bottom number shows the number of units up. The vector goes 1 unit down, so it's a *negative* number.
	The top number is *negative*, so the vector goes to the left and up.

You can add vectors by joining the endpoint of one vector to the start of another. You can use this to express vectors in terms of other vectors.

Worked example 2

The line NQ is parallel to MP and $NQ = 2MP$.

Express the vector $\overrightarrow{MQ}$ in terms of vectors $\mathbf{a}$ and $\mathbf{b}$.

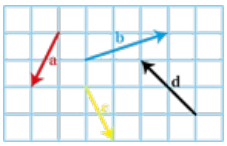


Solution	Comments
$\overrightarrow{NQ} = 2\mathbf{b}$	Going from N to Q is in the same direction but twice the distance as from M to P.

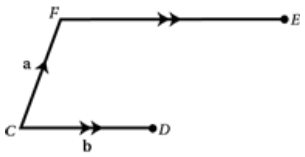
$\overrightarrow{MQ} = \overrightarrow{MN} + \overrightarrow{NQ}$ $= \mathbf{a} + 2\mathbf{b}$	You can get from M to Q by going from M to N and then from N to Q.

EXERCISE Q

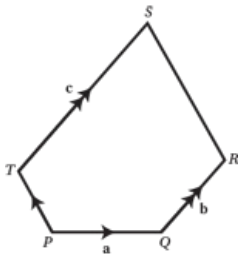
1 Write these as column vectors:



2 **a** FE is parallel to CD and $FE = 2CD$. Express $\overrightarrow{CE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.



b RS is parallel to PT and $RS = 2PT$. TS is parallel to QR and $TS = 2QR$.

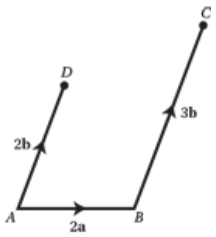


Express in terms of $\mathbf{a}$ and $\mathbf{b}$:

i $\overrightarrow{RP}$

ii **c**

c Express $\overrightarrow{DC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.



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R. Area and volume

You need to know how to find the areas of common shapes:

- Area of a parallelogram $A = (b \times h)$, where b is the base, h is the height
- Area of a triangle $A = \frac{1}{2}(b \times h)$, where b is the base, h is the height
- Area of a trapezium $A = \frac{1}{2}(a + b)h$, where a and b are the parallel sides, h is the height
- Area of a circle $A = \pi r^2$, where r is the radius.

You also need to be able to find volumes of these objects:

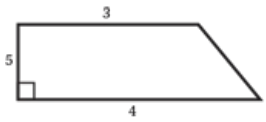
- Volume of a pyramid $V = \frac{1}{3}(\text{area of base } \times \text{ vertical height})$
- Volume of a cuboid $V = l \times w \times h$, where l is the length, w is the width, h is the height
- Volume of a cylinder $V = \pi r^2 h$, where r is the radius, h is the height
- Volume of a cone $V = \frac{1}{3}\pi r^2 h$, where r is the radius, h is the height
- Volume of a sphere $V = \frac{4}{3}\pi r^3$, where r is the radius.

You sometimes need to calculate curved surface areas:

- Area of the curved surface of a cylinder $S = 2\pi rh$, where r is the radius, h is the height
- Area of the curved surface of a cone $S = \pi rl$, where r is the radius and l is the length of the sloping side.

Worked example 1

Find the area of this shape:



Solution	Comments
Quadrilateral with two parallel sides: trapezium	Identify the shape.
$a = 3$, $b = 4$, $h = 5$	Find what is needed for the formula.
$A = \frac{1}{2}(a + b)h$ $= \frac{1}{2}(3 + 4) \times 5$ $= 17.5$	Quote formula, then substitute the values and calculate.

Worked example 2

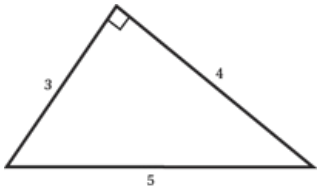
A cylinder has height 12 cm and curved surface area $168\pi\text{ cm}^2$. Find its volume.

Solution	Comments
$S = 2\pi rh$: $168\pi = 2\pi r(12)$ $\Rightarrow r = \frac{168}{24} = 7\text{ cm}$	Use the formula for the curved surface area to find the radius.
$V = \pi r^2 h$ $= \pi \times 7^2 \times 12 = 588\pi\text{ c m}^3$	Now use radius and height to find the volume.

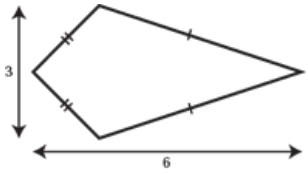
EXERCISE R

1 Find the area of each of these shapes:

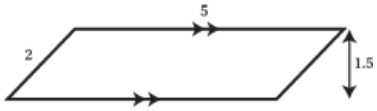
a



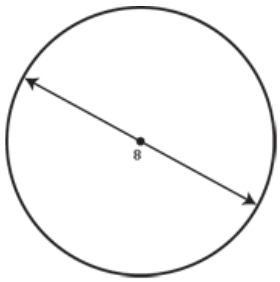
b



c

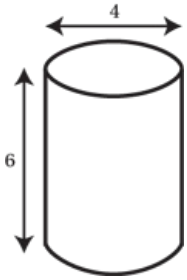


d

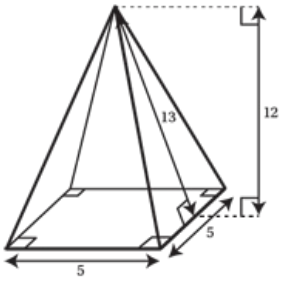


2 Find the volume of each of these symmetrical shapes:

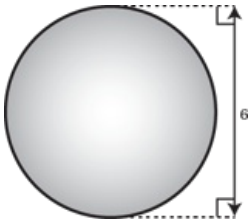
a



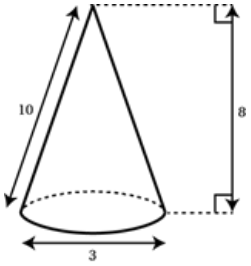
b



c



d



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S. Working with formulae

A formula is a set of instructions for finding one variable if you know some others. If you treat a formula as an equation you can change the subject of the formula from one variable to another. There are a few useful ideas you need to bear in mind when you are trying to do this:

- If the new subject occurs in any squares or square roots, isolate these before undoing them.
- Get all the terms involving the new subject on one side and everything else on the other side. Then factorise.

Worked example 1

Make x the subject of the formula $y = \dfrac{3+x}{5-x}$.

Solution	Comments
$\times(5-x) : y(5-x) = 3+x$ $\Rightarrow 5y - xy = 3+x$	Get rid of fractions and expand brackets.
$\Rightarrow 5y - 3 = xy + x$ $\Rightarrow 5y - 3 = x(y+1)$ $\Rightarrow \dfrac{5y-3}{y+1} = x$	Get everything involving x on one side, everything else on the other side, then factorise.
$x = \dfrac{5y-3}{y+1}$	It is good practice to have the subject on the left-hand side.

Another important tool is substituting one formula into another.

Worked example 2

If $A = \pi r^2 + 2\pi rl$ and $r = 3l$, find A in terms of l.

Solution	Comments
$A = \pi(3l)^2 + 2\pi(3l)l$ $= 9\pi l^2 + 6\pi l^2$ $= 15\pi l^2$	Simplify the expression.

EXERCISE S

1 **a** If $A = xy + y$, find A when $x = 6, y = \frac{1}{2}$.

b If $B = (3p)^2$, find B when $p = 2$.

c If $C = \sqrt{x(1 + y^2)}$, find C when $x = 9$ and $y = 2$.

2 Make x the subject of each of these formulae:

a $A = 3x + 7$

b $B = 3 - x^2$

c $C = 1 + \frac{1}{x}$

d $D = 7 + \sqrt{1 - 3x}$

e $E = \frac{3 + x}{x - 2}$

f $F = \frac{10 + x}{3 + 2x}$

g $G = \frac{1 + x^2}{2 + x^2}$

h $H = \frac{a + x}{8 + x}$

i $I = \frac{a + bx}{c + dx}$

3 Write z in terms of x, giving your answer in a form without brackets:

a $z = 5y + 5, y = 3x$

b $z = 6y - 1, y = 2x + 1$

c $z = 2y^2, y = 3x$

d $z = y^2 + 2y + 5, y = x + 1$

e $z = 1 + \frac{y}{2}, y = 6x + 4$

f $z = \frac{1 + y}{1 - y}, y = x - 1$

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T. Statistical diagrams

You should be able to interpret different types of statistical diagrams.

Pie charts

Pie charts represent proportions by sizes of sectors.

Bar charts

Bar charts represent frequencies by the heights of bars.

Pictograms

Frequencies are proportional to the number of icons displayed in a pictogram.

Stem-and-leaf diagrams

In a stem-and-leaf diagram, the last one (or occasionally two) significant figures of each data item is listed from a ‘stem’ of all previous significant figures.

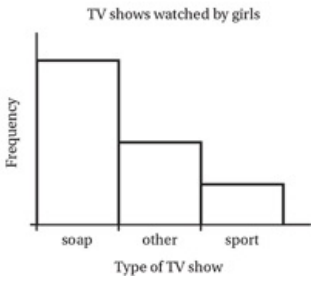
Line graphs

A line graph is normally used with time along the x-axis and the values recorded along the y-axis. Consecutive data points are joined by a straight line, representing an assumption that the data changes at a constant rate in the time interval.

Worked example

This pie chart and bar chart show the types of TV shows watched by girls and boys in a class.





a In which group did a greater proportion of students watch soap operas?

b Did more boys than girls watch sport?

Solution	Comments
a Soap is the least popular category for boys but the most popular for girls so a greater proportion of girls watch soaps.	
b You cannot answer this because the pie chart shows *proportions* not numbers.	

EXERCISE T

1 This pie chart represents the favourite colours of 100 16-year-old students.



a How many students have each favourite colour?

i yellow

ii blue

iii red

b In a survey of the favourite colour of 12-year-olds, the results were:

Colour	Number
Blue	18
Yellow	12
Red	42

In a pie chart, what angle would the sector representing each of these colours take up?

i blue

ii yellow

iii red

c Draw bar charts showing the results for 16-year-olds and 12-year-olds.

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U. Statistical measures

Mean, median and mode

Mean, median and mode are all measures of the centre of a group of data.

- The mean is the total of all the data divided by the number of data items.
- The median is the value of the middle item when all the data is arranged in order. If there are two middle items, the mean of these two is taken.
- The mode is the value of the most common data item. There may be more than one mode, or no mode at all.

Range

Range is a measure of the spread of the data.

The range is the difference between the smallest and the largest data value.

Worked example

For the data 7, 5, 6, 8, 5 find its:

a **mean**
b **median**
c **mode**
d **range**

Solution	Comments
a $7+5+6+8+5=31$	Add up all the data.
5 data items	Count how many data items there are.
Mean $=\frac{31}{5}=6.2$	Find the ratio of these.
b 5, 5, 6, 7, 8	Arrange the data in order.
With 5 data items, the 3 rd item is in the middle.	Decide which is the middle data item.
Median is 6	
c Mode is 5	Decide which data item has occurred most often.
d The range is $8-5=3$.	Identify the smallest and the largest data values.

EXERCISE U

1 Find:

i the mean

ii the median

iii the mode

iv the range

of these data:

a $3, 6, 4, 4, 4$

b $8, 12, 6, 9, 9, 10$

c $2, 2, 4, 4, 6, 12$

d $-2, 0, 7, 9, 11, 15, 20$

e $98, 100, 107, 111, 115, 120, 109$

f $0.2, 0.2, 0.4, 0.4, 0.6, 1.2$

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V. Probability

One way to find probabilities is to list all possible outcomes. This can be in the form of a list, table or grid. The list of all possible outcomes is called a sample space.

This method only works if all the outcomes are equally likely.

Worked example 1

a List the sample space for the sexes of a two-child family, assuming no twins.

b Hence find the theoretical probability that the two children are a boy and a girl.

Solution	Comments
a First child: boy, boy, girl, girl Second child: boy, girl, boy, girl	A table is a systematic way to list the possibilities.
b $\mathrm{P}(\text{a boy and a girl}) = \frac{2}{4} = \frac{1}{2}$	There are four outcomes, and two of them are a boy and a girl.

Tip

Don’t forget that since you can distinguish between the boy first–girl second and girl first–boy second cases; you must count them separately in the sample space.

Worked example 2

What is the probability of getting a sum of 7 when two fair dice are rolled?

Solution	Comments
	Draw a sample space grid diagram showing all possible totals when two dice are rolled.

		Dice A					
		1	**2**	**3**	**4**	**5**	**6**
	1	2	3	4	5	6	**7**
	2	3	4	5	6	**7**	8

Dice B	**3**	4	5	6	**7**	8	9
	4	5	6	**7**	8	9	10
	5	6	**7**	8	9	10	11
	6	**7**	8	9	10	11	12

There are 36 items in the sample space, each of equal probability.	Count how many items are in the sample space.
There are 6 sums of 7, therefore the probability of a sum of 7 is $\frac{6}{36} = \frac{1}{6}$	Count how many sums of 7 there are.

Tip

A '3' on dice **A** and a '1' on dice **B** has to be counted as a separate event from a '1' on dice **a** and a '3' on dice **B**. However, a '3' on dice **A** and a '3' on dice **B** is only one event. This often causes confusion.

EXERCISE V

1 List the sample space for:

a a fair six-sided dice

b rearrangements of the word 'RED'

c the sexes of a three-child family

d a six-sided dice with three '1 's and the numbers from 2 to 4.

2 Two fair six-sided dice numbered 1 to 6 are rolled. By drawing probability grid diagrams find the probability that:

a the sum is 8

b the product is greater than or equal to 8

c the product is 24 or 12

d the largest value is 4.

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W. Tree diagrams

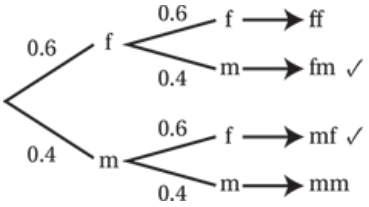
Listing all possible outcomes only helps calculate probabilities when the outcomes are equally likely. When this is not the case, tree diagrams can help find the probabilities instead.

The rule for calculating probabilities from a tree diagram is:

- multiply along branches
- add between branches.

Worked example 1

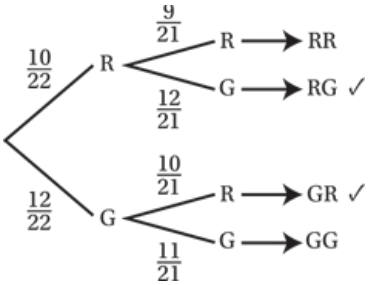
In a particular species of animal, the probability of a female offspring is 0.6. A female has two offspring. Find the probability that there is a male and a female.

Solution	Comments
	There are two possible outcomes, but they are not equally likely. So listing all possible combinations in a table wouldn't give the correct answer.
	In a tree diagram, the first set of branches represents possible outcomes for the first offspring, and the second set the possible outcomes for the second offspring.
$(0.6 \times 0.4) + (0.4 \times 0.6) = 0.48$	There are two ways to get a male and a female offspring. Multiply along branches and add between branches.

The probabilities on the two sets of branches don't have to be the same; the probabilities can change depending on the outcome of the first event.

Worked example 2

A bag contains 10 red balls and 12 green balls. Two balls are chosen at random and not replaced. Find the probability that the two balls are different colours.

Solution	Comments
	The first ball is chosen out of 22. The second ball is chosen out of 21; but the probability of it being red depends on the colour of the first ball.

$\left(\frac{10}{22} \times \frac{12}{21}\right) + \left(\frac{12}{22} \times \frac{10}{21}\right) = 0.519$	There are two ways to get one red and one green ball.

EXERCISE W

1 The probability that Daniel is late for school on any given day is 0.02. Find the probability that he is late on exactly one out of two consecutive days.

2 A box contains 6 triangular tokens and 10 square tokens.

a Two tokens are picked at random and not replaced. Find the probability that they are both square tokens.

b How does the answer change if the first token is replaced in the box?

3 The probability that it is windy on any given day is 0.3. If it is windy, Rosie walks to work with probability 0.8 and cycles with probability 0.2. If it is not windy, Rosie cycles with probability 0.9 and walks with probability 0.1. What is the probability that Rosie cycles to work on a randomly selected day?

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X. Travel graphs

For an object moving in a straight line, you can represent the displacement, velocity and acceleration on a travel graph. These all have time on the horizontal axis.

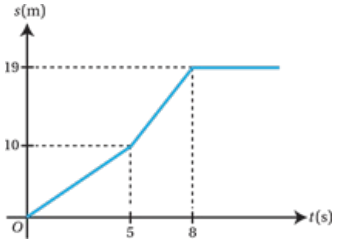
For a displacement–time graph:

- the vertical axis shows the displacement from the starting point
- the gradient gives the velocity.

For a velocity–time graph:

- the vertical axis shows velocity
- the gradient gives acceleration
- the area under the graph gives distance travelled.

Worked example 1



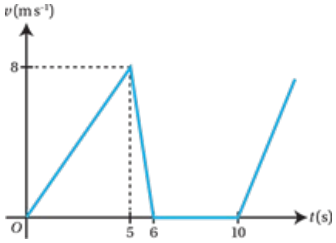
Use this displacement–time graph to find:

a the distance travelled by the object between $t = 5$ and $t = 8$

b the speed of the object when $t = 6$.

Solution	Comments
a The displacement changes from $10\ \mathrm{m}$ to $19\mathrm{m}$, so the distance travelled is $9\mathrm{m}$.	The object is always moving in the same direction, so the distance is the difference between the initial and the final displacements.
b $v = \dfrac{19-10}{8-5} = 3\ \mathrm{m\ s^{-1}}$	The speed is the gradient of the graph. You need to look at the part of the graph between $t = 5$ and $t = 8$.

Worked example 2



Use this velocity–time graph to find:

a the acceleration of the object between $t = 0$ and $t = 5$.

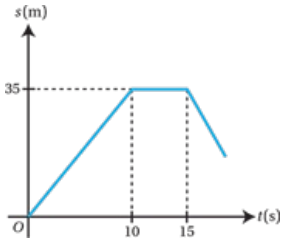
b the length of time during which the object is at rest

c the distance travelled during the first 6 seconds.

Solution	Comments
a $a = \dfrac{8-0}{5} = 1.6 \text{ m s}^{-2}$	The acceleration is the gradient of the graph.
b The object is at rest between $t = 6$ and 10, so for 4 seconds.	The object is at rest when $v = 0$.
c $\text{distance} = \dfrac{1}{2}(6)(8) = 24 \text{ m}$	The distance is the area under the graph. This is a triangle with base 6 and height 8.

EXERCISE X

1 **a** Use this distance–time graph to find:

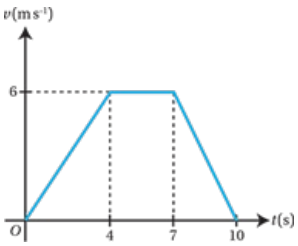


i the velocity of the object during the first 10 seconds

ii between which times the object is at rest.

b Is the object moving towards or away from the starting point when $t = 20$?

2 Use this speed–time graph to find:



a the acceleration of the object when $t = 3$.

b the distance travelled in the first 10 seconds.

Gateway to A Level – GCSE revision

Y. Calculations with powers and roots

Although you will be allowed to use your calculator in the exams, it is important that you are confident using the rules of arithmetic so that you can apply them to algebraic expressions.

You need to know the mathematical convention for the order in which you carry out operations:

1. **B**rackets
2. **I**ndices (this includes both powers and roots)
3. **D**ivision and **M**ultiplication
4. **A**ddition and **S**ubtraction

If several instances of operations in the same category occur in the same calculation you work through them from left to right.

Worked example 1

Evaluate:

a $4 + 2^3 - (-1)^6$

b $\sqrt{144 + 25} - 2 \times 3^2$

Solution	Comments
a $4 + 2^3 - (-1)^6 = 4 + 8 - 1$	Evaluate the powers first.
$= 12 - 1$ $= 11$	Do the addition and subtraction in order from left to right.
b $\sqrt{144 + 25} - 2 \times 3^2 = \sqrt{169} - 2 \times 3^2$	The square root acts as brackets: do $144 + 25$ first.
$= 13 - 2 \times 9$	Next do the root and the power.
$= 13 - 18 = -5$	Multiplication before subtraction.

Worked example 2

Expand and simplify $(\sqrt{6} - 1)(\sqrt{6} + 2)$.

Solution	Comments
$\sqrt{36} + 2\sqrt{6} - \sqrt{6} - 2$	Expand the brackets.
$= 6 + \sqrt{6} - 2$ $= 4 + \sqrt{6}$	Simplify.

EXERCISE Y

1 Evaluate without a calculator:

a $7 - 3^2$

b $5 - 3\,(8 - 1)$

c -5^2

d $(-5)^2$

e $\dfrac{65}{5} + 98$

f $12 \div (6 - 2)$

2 Evaluate without a calculator:

a $\sqrt{25 - 16} - 2^3$

b $\sqrt[3]{64} - \sqrt[3]{-27}$

c $\left(\sqrt{5} - 2\right)\left(\sqrt{5} - 3\right)$

d $\dfrac{6\sqrt{10}}{2} - 2\left(\sqrt{10} - 1\right)$

Gateway to A Level – GCSE revision

Z. Factorising

Factorisation means taking a sum and re-expressing it as a product. This is often done by finding a factor *common* to each term in the sum.

Worked example 1

Factorise:

$12ab^2 + 9a^2b$

Solution	Comments
$= 3ab(4b + 3a)$	The largest factor in both terms is $3ab$.

You must also be able to factorise quadratic equations like $x^2 + bx + c$. If the coefficient of x^2 is 1 you look for a factorisation of the form $(x - p)(x - q)$. You try to find p and q such that their product is c and they add up to b. If there is a common term throughout the whole expression, do not forget to take that out first.

Worked example 2

Factorise:

$x^2 - 7x + 12$

Solution	Comments
$= (x - 3)(x - 4)$	$-3, -4$ are two numbers which add to give -7 and multiply to give 12.

A special example of factorising is the *difference of two squares*:

$a^2 - b^2 = (a - b)(a + b)$

Worked example 3

Factorise:

$9x^2 - 25$

Solution	Comments
$9x^2 - 25 = (3x)^2 - 5^2 = (3x - 5)(3x + 5)$	$9x^2$ is the square of $3x$ and 25 is the square of 5.

Sometimes it is possible to factorise an expression by first splitting it up into two parts and factorising each one separately. This sometimes reveals a common factor in both parts.

Worked example 4

Factorise:

$ax + bx + ay + by$

Solution	Comments
$= x(a + b) + y(a + b)$	
$= (a + b)(x + y)$	A common factor of $(a + b)$ can be taken out.

EXERCISE Z

1 Factorise the following:

a $8x + 12y$

b $5y - 10y^2$

c $18x^3y - 4x$

d $14a^2b^2 - 7ab$

e $32x^3y + 4x^2y^2 + 2xy^4$

f $15a^2b^2 - 12ab + 9a^2$

2 Factorise the following:

a $x^2 + 2x - 3$

b $x^2 - 2x - 35$

c $a^2 - 8a - 20$

d $b^2 - 81$

e $16y^2 - 49$

f $2x^2 + 10x + 12$

g $5x^2 + 20x + 15$

3 Factorise the following:

a $xy + y + x + 1$

b $x^2 + xy + x + y$

c $ab + 3a + 2b + 6$

d $6ab - 8a + 3b - 4$

e $pq - p - q + 1$

f $6p^2q - 4pq - 15p + 10$

Gateway to A Level – GCSE revision answers

A. Expanding brackets and factorising

1 **a** $2 + x + y$

b 2

c $5x - 2xy + x^2 - y^2$

d $11z^2$

e $x^2 + 2$

f $x^2 + 3xy + 3xz + 5yz$

2 **a** $x^2 + 8x + 15$

b $6a^2 - a - 2$

c $x^2 - 6x + 9$

d $xy + x + y + 1$

e $4x^2 + 4xy + y^2$

f $a^3 - 1$

g $x^2 + 2xy + y^2 - 2x - 2y + 1$

h $x^3 + 6x^2 + 11x + 6$

i $6y^3 - 19y^2 - 26y + 24$

3 **a** $4(2x + 3y)$

b $5y(1 - 2y)$

c $2x(9x^2y - 2)$

d $7ab(2ab - 1)$

e $2xy(16x^2 + 2xy + y^3)$

f $3a(5ab^2 - 4b + 3a)$

4 **a** $(x + 1)(y + 1)$

b $(x + 1)(x + y)$

c $(a + 2)(b + 3)$

d $(2a + 1)(3b - 4)$

e $(p - 1)(q - 1)$

f $(2pq - 5)(3p - 2)$

Gateway to A Level – GCSE revision answers

B. Factorising quadratics

1 **a** $(x+3)(x-1)$

b $(x+5)(x-7)$

c $(a-10)(a+2)$

d $(b+5)(b+8)$

2 **a** $2(x+3)(x-5)$

b $5(x+3)(x-2)$

c $2(x-4)(x+2)$

d $3(x-3)(x-3)$

3 **a** $(b-9)(b+9)$

b $(k-10)(k+10)$

c $(4y-7)(4y+7)$

d $9(2x+3)(2x-3)$

4 **a** $(3x+1)(x-5)$

b $(2x-3)(x+1)$

c $(5x+1)(x-3)$

d $(2x+5)(x-2)$

e $(3x-1)(2x-1)$

f $(5x+1)(3x+2)$

g $(3x-5)(2x+3)$

h $(5x-7)(2x+3)$

Gateway to A Level – GCSE revision answers

C. Quadratic equations

1 **a** $1, -6$

b $2, 6$

c $8, -3$

d $0, -\frac{5}{2}$

e ± 9

f $\pm \frac{4}{3}$

g $4, -\frac{1}{2}$

h $\frac{2}{3}, -6$

i $\frac{4}{5}, 3$

j $\frac{4}{3}, -\frac{3}{2}$

2 **a** $-4.56, -0.438$

b $1.37, -4.37$

c $3.24, -1.24$

d $0.634, 2.37$

e $1.47, -1.14$

f $0.860, -0.194$

3 **a** $5, -3$

b $1.30, -2.30$

c $2.87, -0.871$

d $4, -2$

e 3

f $0, 1.5$

g $7, -7$

h $2, -2$

Gateway to A Level – GCSE revision answers

D. Linear equations and inequalities

1 **a** $x = -\frac{3}{4}$

b $x = \frac{1}{3}$

c $x = \frac{10}{3}$

d $x = -\frac{7}{2}$

e $x = \frac{4}{7}$

f $x = 1$

2 **a** $x > \frac{9}{2}$

b $x \leqslant \frac{1}{3}$

c $x > 2$

d $x < \frac{4}{5}$

Gateway to A Level – GCSE revision answers

E. Types of numbers

1 **b** $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

c $\mathbb{Q}, \mathbb{R}$

d $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$

e $\mathbb{Q}, \mathbb{R}$

f $\mathbb{Q}, \mathbb{R}$

g $\mathbb{R}$

h $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

i $\mathbb{R}$

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Gateway to A Level – GCSE revision answers

F. Functions

1 **a** **i** 1

ii 196

iii $\frac{9}{4}$

b **i** 22

ii 2

iii 1

c **i** 3

ii 0

iii 5

d **i** -1

ii $\frac{1}{15}$

iii $\frac{1}{4}$

e **i** 1

ii 2

iii 2

f **i** $-\frac{1}{2}$

ii $-\frac{6}{5}$

iii 10

2 **a** **i** $2x + 3$

ii $2x + 9$

iii $6x + 5$

b **i** $4 + 3x$

ii $10 - 3x$

iii $4 - x$

c **i** $4x^2 + 4x - 3$

ii $x^2 + 8x + 12$

iii $x^2 - 4$

d **i** $-2x^2 + 11x - 11$

ii $4 + x - 2x^2$

iii $4 + 2x - 8x^2$

e **i** $\dfrac{1}{1-2x}$

ii $\dfrac{1}{5-2x}$

iii $\dfrac{1}{3+x}$

Gateway to A Level – GCSE revision answers

G. Rules of indices

1 **a** False

b False

c True

d True

e False

f False

g True

h False

2 **a** 40

b $\frac{8}{27}$

c $\frac{1}{16}$

d $-\frac{125}{27}$

3 **a** **i** 2

ii 2

b **i** 100

ii 3

c **i** $\frac{1}{5}$

ii $\frac{3}{4}$

d **i** 4

ii 125

e **i** $100\,000$

ii 27

f **i** $\frac{1}{32}$

ii $\frac{32}{243}$

g **i** $\frac{1}{2}$

ii $\frac{1}{7}$

h **i** $\frac{3}{4}$

ii $\frac{64}{27}$

4 **a** x^{10}

b x^{2}

c x^{7}

d x^{-6}

5 **a** $2^{6.5}$

b $3^{\frac{4}{3}}$

Gateway to A Level – GCSE revision answers

H. Surds

1 **a** $5\sqrt{5}$

b $7\sqrt{5}$

c $5\sqrt{5}$

d $-5\sqrt{5}$

e $2\sqrt{5}$

f $11\sqrt{5}$

2 **a** $\sqrt{32}$

b $\sqrt{147}$

c $\sqrt{63}$

d $\sqrt{108}$

e $\sqrt{18}$

f $\sqrt{72}$

3 **a** $3+\sqrt{3}$

b $2\sqrt{3}$

c $5+3\sqrt{3}$

d $-5+\sqrt{3}$

e $28-16\sqrt{3}$

f $20-10\sqrt{3}$

Gateway to A Level – GCSE revision answers

I. Algebra of expressions

1 **a** $x^2 + 8x + 15$

b $z^2 - 1$

c $6a^2 - a - 2$

d $xy + x + y + 1$

e $x^2 - 6x + 9$

f $4x^2 + 4xy + y^2$

g $a^3 - 1$

h $x^3 + 6x^2 + 11x + 6$

i $x^2 + 2xy + y^2 - 2x - 2y + 1$

2 **a** False

b True

c True

d False

e True

f False

g False

h True

i False

j False

Gateway to A Level – GCSE revision answers

J. Simultaneous equations

1 **a** $(2, 7)$

b $\left(2\frac{1}{2}, -\frac{1}{2}\right)$

c $\left(\frac{5}{4}, -\frac{3}{4}\right)$

d $(1, 1)$

e $(2, 3)$

f $(2, 3)$

2 **a** $(x, y) = (3, 4)$ or $(4, 3)$

b $(x, y) = (4, 14)$ or $(-6, 4)$

c $(x, y) = (4, 3)$ or $\left(17\frac{1}{2}, -6\right)$

d $(x, y) = (5, 4)$ or $\left(8, 2\frac{1}{2}\right)$

Gateway to A Level – GCSE revision answers

K. Direct and inverse proportion

1 **a** $a = \frac{b}{2}$

b $r = \frac{4s}{3}$

c $x = 4.8$

d $y = 17.5$

e $p = \frac{q^2}{20}$

f $r = 2$

2 **a** $y = 6$

b $F = \frac{140}{r^2}$

c $S = 4$

d $c = \frac{36}{d}$

Gateway to A Level – GCSE revision answers

L. Using coordinates

1 **a** $\frac{3}{2}$

b $\frac{4}{5}$

c 0

2 **a** $\left(2, 5\frac{1}{2}\right)$

b $\left(6\frac{1}{2}, 0\right)$

c $(1, -3)$

3 **a** $\sqrt{13}$

b $\sqrt{41}$

c 8

Gateway to A Level – GCSE revision answers

M. Straight line graphs

1 **a** $2, (0, -7)$

b $-1, (0, 5)$

c $-3, (0, 9)$

d $3, (0, -8)$

e $2, \left(0, \frac{1}{5}\right)$

f $-\frac{2}{3}, (0, 0)$

2 **a** $y = 7x - 4$

b $y = -\frac{1}{2}x + 3$

c $y = -5x$

3 **a** $y = 3x - 2$

b $y = -x + 6$

c $y = \frac{3}{2}x$

Gateway to A Level – GCSE revision answers

N. Angles and bearings

1 $80^\circ, 100^\circ$

2 **a** 220°

b 320°

Gateway to A Level – GCSE revision answers

O. Circle theorems

1 **a** 20° (right-angled triangle)

b 80° (opposite angles in cyclic quadrilateral are complementary)

c 70° (radius meets tangent at right-angle, angle in a semicircle is 90°)

Gateway to A Level – GCSE revision answers

P. Pythagoras and trigonometry

1 **a** 5

b 5.44

c 2.54

d 2.91

e 4.67

f 5.01

2 **a** 41.8°

b 35.5°

c 54.5°

d 65.4°

e 26.6°

f 30°

3 051°

Gateway to A Level – GCSE revision answers

Q. Vectors

1 **a** $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$

b $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

c $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

d $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$

2 **a** $a + 2b$

b **i** $-a - b$

ii $2b$

c $2a + b$

Gateway to A Level – GCSE revision answers

R. Area and volume

1 **a** 6

b 9

c 7.5

d $16\pi = 50.3$

2 **a** $24\pi = 75.4$

b 100

c $36\pi = 113$

d $6\pi = 18.8$

Gateway to A Level – GCSE revision answers

S. Working with formulae

1 **a** $A = 3\frac{1}{2}$

b $B = 36$

c $C = 3\sqrt{5}$

2 **a** $x = \frac{A - 7}{3}$

b $x = \sqrt{3 - B}$

c $x = \frac{1}{C - 1}$

d $x = \frac{1 - (D - 7)^2}{3}$

e $x = \frac{3 + 2E}{E - 1}$

f $x = \frac{10 - 3F}{2F - 1}$

g $x = \sqrt{\frac{1 - 2G}{G - 1}}$

h $x = \frac{a - 8H}{H - 1}$

i $x = \frac{a - cI}{dI - b}$

3 **a** $z = 15x + 5$

b $z = 12x + 5$

c $z = 18x^2$

d $z = x^2 + 4x + 8$

e $z = 3x + 3$

f $z = \frac{x}{2 - x}$

Gateway to A Level – GCSE revision answers

T. Statistical diagrams

1 a i 20

ii 25

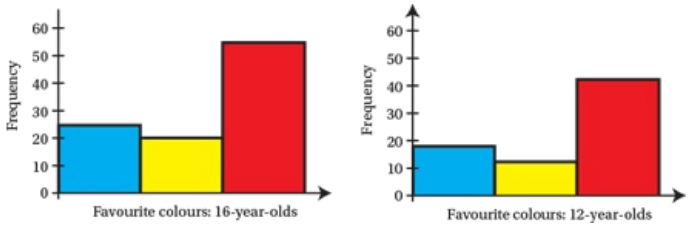
iii 55

b i 90°

ii 60°

iii 210°

c



Gateway to A Level – GCSE revision answers

U. Statistical measures

1 **a** **i** 4.2

ii 4

iii 4

iv 3

b **i** 9

ii 9

iii 9

iv 6

c **i** 5

ii 4

iii $2, 4$

iv 10

d **i** 8.6

ii 9

iii None

iv 22

e **i** 109

ii 109

iii None

iv 22

f **i** 0.5

ii 0.4

iii $0.2, 0.4$

iv 1

Gateway to A Level – GCSE revision answers

V. Probability

1 **a** $1, 2, 3, 4, 5, 6$

b RED, RDE, ERD, EDR, DRE, DER

c BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG

d $1, 1, 1, 2, 3, 4$

2 **a** $\frac{5}{36}$

b $\frac{22}{36} = \frac{11}{18}$

c $\frac{6}{36} = \frac{1}{6}$

d $\frac{7}{36}$

Gateway to A Level – GCSE revision answers

W. Tree diagrams

1 0.039

2 **a** 0.375

b 0.391

3 0.69

Gateway to A Level – GCSE revision answers

X. Travel graphs

1 **a** **i** $3.5\ \mathrm{m\,s^{-1}}$

ii between $t = 10$ and $t = 15$

b Towards the starting point

2 **a** $1.5\ \mathrm{m\,s^{-2}}$

b $39\ \mathrm{m}$

Gateway to A Level – GCSE revision answers

Y. Calculations with powers and roots

1 **a** -2

b -16

c -25

d 25

e 111

f 3

2 **a** -5

b 7

c $11-5\sqrt{5}$

d $\sqrt{10}+2$

Gateway to A Level – GCSE revision answers

Z. Factorising

1 **a** $4(2x + 3y)$

b $5y(1 - 2y)$

c $2x(9x^2y - 2)$

d $7ab(2ab - 1)$

e $2xy(16x^2 + 2xy + y^3)$

f $3a(5ab^2 - 4b + 3a)$

2 **a** $(x - 1)(x + 3)$

b $(x - 7)(x + 5)$

c $(a - 10)(a + 2)$

d $(b - 9)(b + 9)$

e $(4y - 7)(4y + 7)$

f $2(x + 2)(x + 3)$

g $5(x + 1)(x + 3)$

3 **a** $(x + 1)(y + 1)$

b $(x + 1)(x + y)$

c $(a + 2)(b + 3)$

d $(2a + 1)(3b - 4)$

e $(p - 1)(q - 1)$

f $(2pq - 5)(3p - 2)$

Chapter 1 worked solutions

1 Proof and mathematical communication

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 1A

3 Expanding brackets:

$$\begin{aligned}(x+a)(x-b)^2 &= (x+a)(x^2-2bx+b^2)\\ &= x^3+x^2(a-2b)+x(b^2-2ab)+ab^2\\ &= x^3-4x^2-3x+18\end{aligned}$$

Comparing coefficients:

$x^3: 1=1$
$x^2: \ -4=a-2b \ (1)$
$x^1: b^2-2ab=-3 \ (2)$
$x^0: ab^2=18 \quad (3)$
$(1) \Rightarrow a=2b-4$

Substituting into (2) :

$b^2-(4b^2-8b)=-3$ therefore $3b^2-8b-3=0$

$(3b+1)(b-3)=0$ so $b=-\frac{1}{3}$ or $b=3$

$(1): \ b=-\frac{1}{3} \Rightarrow a=-\frac{14}{3}, b=3 \Rightarrow a=2$

Testing in (3): Only $a=2, b=3$ works as a solution.

4 Expanding brackets:

$$\begin{aligned}(x^3+a)(x+b) &= x^4+bx^3+ax+ab\\ &= x^4+8x^3+2x+16\end{aligned}$$

Comparing coefficients:

$x^4: 1 \ =1$
$x^3: b \ =8 \ (1)$
$x^1: a \ =2 \ (2)$
$x^0: ab=16 \ (3)$

Testing in (3): $a=2, b=8$ works as a solution.

Tip

The check may seem unnecessary, but it is an easy way to spot an arithmetic error. In questions where you may not be given the factorised form but have worked it out in an earlier part, the check also allows you to validate your previous working.

5 This is an example of 'stream of consciousness' working.

When calculating $2x+2$ for $x=4$, you might think:

'Two fours are eight, then add two and you get ten', which is perfectly acceptable, as prose.

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When writing mathematical equations however, any time you use an equals sign it **must** mean that the value on the left is the same as the value on the right.

$2 \times 4 = 8$
$8 + 2 = 10$

would be accurate, if not compelling, working. Better would be something like this:

$2 \times 4 + 2$
$= 8 + 2$
$= 10$

where the whole problem is laid out in every line, and gradually simplified to get the answer.

6 Written with implication symbols, you get:

$$\begin{aligned} & 1 = 3 \\ \Leftrightarrow\ & -1 = 1 \\ \Rightarrow\ & 1 = 1 \end{aligned}$$

The direction of the arrows means that if the start line is true then the end line is true, but you cannot read the argument backwards because not all the implication signs are reversible.

7 **a**
$$\begin{aligned} & \sqrt{x^2 + 9} = 3x - 7 \\ \Rightarrow\ & x^2 + 9 = 9x^2 - 42x + 49 \\ \Leftrightarrow\ & 0 = 8x^2 - 42x + 40 \\ \Leftrightarrow\ & 0 = 4x^2 - 21x + 20 \\ \Leftrightarrow\ & 0 = (4x - 5)(x - 4) \\ \Leftrightarrow\ & x = \frac{5}{4} \text{ or } x = 4 \end{aligned}$$

b When squaring, unless both sides are already known to have the same sign, you introduce additional solutions.

$x = 4$ is the solution to the initial problem.

$x = \frac{5}{4}$ is the solution to the problem $\sqrt{x^2 + 9} = -(3x - 7)$, which gives precisely the same second line when you square both sides.

8 **a**
$$\begin{aligned} & \frac{1}{x^2} = \frac{2}{x} \\ \Rightarrow\ & x = 2x^2 \\ \Leftrightarrow\ & 0 = 2x^2 - x \\ \Leftrightarrow\ & 0 = x(2x - 1) \\ \Leftrightarrow\ & x = 0 \text{ or } x = \frac{1}{2} \end{aligned}$$

b Multiplying through by x^3 introduced the false solution $x = 0$.

9 **a**
$$\begin{aligned} & x^2 + 3x = 4x + 12 \\ \Leftrightarrow\ & x(x + 3) = 4(x + 3) \\ \Leftarrow\ & x = 4 \end{aligned}$$

b The penultimate line does not imply the final line, which results in losing the solution $x = -3$.

10 It depends on exactly how you define the word or. In formal logical proof the terms OR and XOR have two different meanings. (XOR means that either one or other but not both conditions are true.)

11 Cancelling is a process which needs care. When you say 'cancel $(a - b)$', what you are actually doing is dividing both sides by $(a - b)$. But $a = b$ so this is division by zero, which causes the remaining logic to break down.

Exactly equivalently, you can truthfully assert $0 \times 1 = 0 \times 2$.

But you cannot 'cancel' the multiplication by zero to get $1 = 2$.

EXERCISE 1C

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1 Any value other than $x = 0$.

2 Any value for which $\cos x \neq 1$.

3 Any negative value for x.

4 2 is the only even prime number, so any counter example must have the form $2p$ for some prime p.

5 Any square number will have its square root as a single factor while all others factors come in pairs, so all square numbers have an odd number of factors.

6 If x is irrational and y is rational then $y - x$ is irrational, and anything of the form $(y - x) + x$ will be the sum of two irrationals and have a rational value.

7 The correct solution to this (as will be seen later in the course) is $-3 < x < 3$, so any values less than or equal to -3 will provide a counter example.

8 If the quadratic expression is to be non-prime, it must have a factor other than 1.

Since it has no linear factors with integer coefficients (cannot be written as $(x + a)(x + b)$ for integer values a and b), the only way the expression can have a factor is if there is a common factor to all terms.

41 is a prime, so 41 is the only possible such factor other than 1.

So you need $n^2 + n$ to be a multiple of 41.

$n^2 + n = n(n + 1)$ so any value of $n > 0$ which is either a multiple of 41 or one below a multiple of 41 will provide a counter example.

9 In two dimensions, non-parallel lines must intersect. However, in three (or more) dimensions, non-parallel lines can be 'skew', and not intersect.

Imagine two lines running horizontally, one running north–south and the other running east–west and set a distance above the first. They are not parallel but neither do they intersect.

EXERCISE 1D

1 n is odd so can be expressed as $2m + 1$ for some integer m : $n^2 = (2m + 1)(2m + 1)$

$= 4m^2 + 4m + 1$

$= 2q + 1$ where $q = 2m^2 + 2m$ is an integer.

Since n^2 can be expressed as $2q + 1$ for an integer q, it must be odd.

2 An even number a can be expressed as $a = 2n$ for some integer n.

An odd number b can be expressed as $b = 2m + 1$ for some integer m.

Their sum $a + b = 2n + 2m + 1 = 2q + 1$ for some integer $q = m + n$.

So the sum equals one more than an even number and is therefore odd.

3 Call the lowest of the three consecutive integers x.

Then the integers are x, $x + 1$ and $x + 2$.

The sum of the integers is $3x + 3 = 3(x + 1)$ which is a multiple of 3.

4 **a** Let the first multiple of 5 be $5x$, then the next is $5(x + 1)$.

The sum of these values is $5x + 5(x + 1) = 10x + 5$.

Since $10x$ must be even, it follows that $10x + 5$ must be odd.

b Let the first multiple of 5 by $5x$, then the next is $5(x + 1)$.

The product of these values is $25x(x + 1)$.

IRANINOVIN

Since either x or $x+1$ must be even, it follows that the total product must be even.

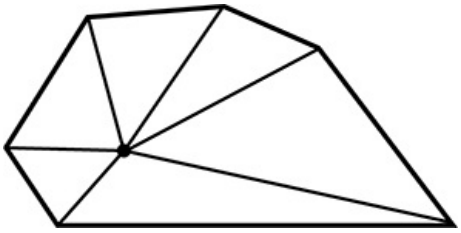
5 The area of the right-angled triangle with sides a and b is $\text{area} = \frac{1}{2}ab$.

The hypotenuse, by Pythagoras' theorem, is $\sqrt{a^2+b^2}$.

If the altitude is h then $\text{area} = \frac{1}{2}h\sqrt{a^2+b^2}$ therefore $\frac{1}{2}h\sqrt{a^2+b^2} = \frac{1}{2}ab$.

So $h = \dfrac{ab}{\sqrt{a^2+b^2}}$

6 Pick any point in the interior of the hexagon and connect each of the external vertices to it. This divides the hexagon into six triangles.



The total interior angles of these triangles must equal the total interior angles of the hexagon (H) plus the 360° at the interior point.

$6 \times 180^\circ = H + 360^\circ$ therefore $H = 720^\circ$.

Tip

There are many ways to approach this. Alternatives include arguments using exterior angles or division of the hexagon using chords from one vertex to the five others.

7 If the number n leaves a remainder 2 when divided by 3 then $n = 3m+2$ for some integer m.

$n^2 = (3m+2)(3m+2) = 9m^2+12m+4.$

$= 3q+1$ where $q = 3m^2+4m+1$ is an integer.

So when n^2 is divided by 3, it leaves remainder 1.

8 **a** $(x+2)^2 = x^2+4x+4$

b $y = x^2+4x+10 = (x+2)^2+6$

Since $(x+2)^2 \geqslant 0$, $y \geqslant 0+6 > 0$.

9 If the exterior angle is x then the interior angle adjacent to it must equal $180^\circ - x$.

The interior angles of a triangle sum to 180°.

The other two interior angles therefore sum to $180^\circ - (180^\circ - x) = x$.

10 $n^2+3n+2 = (n+1)(n+2)$

For integer $n \geqslant 1$, the value of n^2+3n+2 has two integer factors greater than 1: $n+1$ and $n+2$.

Therefore, for integer $n \geqslant 1$, n^2+3n+2 cannot be prime.

11 **a** By convention, you use base 10 as our standard notation, and use place value to allow compact statement of large values (and small ones).

The right-hand entry (in the absence of a decimal point) indicates units (10^0), the next left is tens (10^1), then hundreds (10^2) and thousands (10^3), etc.

So in $abcd$. the d entry indicates $d \times 10^0$, the c entry indicates $c \times 10^1$, etc. and the total number represents the sum of these elements, as given in the question.

b $n = 1000a+100b+10c+d = 999a+99b+9c+(a+b+c+d)$

$= 9(111a+11b+c)+(a+b+c+d)$ so $n-(a+b+c+d) = 9(111a+11b+c)$

IRANINOVIN

So the difference between n and $a+b+c+d$ is a multiple of 9; either both are multiples of 9 or neither is.

c $n = 1000a + 100b + 10c + d$
$= 1001a + 99b + 11c + (b - a + d - c)$
$= 11(91a + 9b + c) + (b - a + d - c)$ so $n(a - b + c - d)$
$= 11(91a + 9b + c)$

So the sum of n and $a - b + c - d$ is a multiple of 11; either both are multiples of 11 or neither is.

12 $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational, $\sqrt{2}$ is known to be irrational.

Consider $x = \sqrt{2}^{\sqrt{2}}$:

If x is irrational, you know that $x^{\sqrt{2}} = 2$, an example of an irrational value to an irrational power being rational.

If x is rational, you know that $\sqrt{2}^{\sqrt{2}} = x$, an example of an irrational value to an irrational power being rational.

So, without actually knowing whether x is rational or irrational, in either case you have an example of an irrational value to an irrational power being rational.

EXERCISE 1E

1 Check all primes $< \sqrt{11}$, i.e. all primes < 4:

2 is not a factor of 11, 3 is not a factor of 11.

Therefore 11 is prime.

2 Check all primes $< \sqrt{83}$, i.e. all primes < 10:

2 is not a factor of 83, 3 is not a factor of 83, 5 is not a factor of 83, 7 is not a factor of 83.

Therefore 83 is prime.

3 When $n = 3$, each angle is $\dfrac{180}{3} = 60^\circ$

When $n = 4$, each angle is $\dfrac{360}{4} = 90^\circ$

When $n = 5$, each angle is $\dfrac{540}{5} = 108^\circ$

When $n = 6$, each angle is $\dfrac{720}{6} = 120^\circ$

In each case the value of the angle is an integer.

4 $1^2 = 1,\ 2^2 = 4,\ 3^2 = 9, 4^2 = 16,\ 5^2 = 25,$
$6^2 = 36, 7^2 = 49, 8^2 = 64,\ 9^2 = 81$

In each case the final digit is never a 7.

5 It seems unrealistic to look for a pattern or rule here; just approach this proof by considering each case in turn. You can be reasonably quick, noting for example that all primes p will have $\mathrm{f}(p) = 2$.

n	1	2	3	4	5	6	7	8	9
$\mathrm{f}(n)$	1	2	2	3	2	4	2	4	3

So, clearly, $\mathrm{f}(n) \leqslant n$ for all the single-digit positive values.

6 $1^2 + 2 = 3$ not divisible by 4

$2^2 + 2 = 6$ not divisible by 4

$3^2 + 2 = 11$ not divisible by 4

IRANINOVIN

$4^2 + 2 = 18$ not divisible by 4

$5^2 + 2 = 27$ not divisible by 4

Tip

You could prove this for all n by considering three cases (n odd, n even and a multiple of 4, n even and not a multiple of 4) – left to the reader as an exercise. However, setting this up will take more time than is needed if you only need to prove it for $1 \leqslant n \leqslant 5$; you can just prove it by exhaustion.

7 Break into two cases:

If n is even: n can be written as $n = 2m$.

$n^2 + n = (2m)^2 + 2m = 4m^2 + 2m$

$= 2(m^2 + m) = 2q$ for some integer $q = m^2 + m$.

If n is odd: n can be written as $n = 2m + 1$.

$n^2 + n = (2m + 1)^2 + 2m + 1 = 4m^2 + 6m + 2$

$= 2(2m^2 + 3m + 1) = 2q$ for some integer $q = 2m^2 + 2m + 1$.

Either way, $n^2 + n$ is a multiple of 2 (even).

8 Every number is either a multiple of 5 or within 2 of a multiple of 5.

Examine the three cases $n = 5m, n = 5m \pm 1$ or $n = 5m \pm 2$.

If $n = 5m$ then $n^2 = 25m^2 = 5q$ where $q = 5m^2$ is an integer.

$$\begin{aligned}\text{If } n = 5m \pm 1 \text{ then } n^2 &= (5m \pm 1)(5m \pm 1)\\ &= 25m^2 \pm 10m + 1\\ &= 5q + 1 \text{ where } q = 5m^2 \pm 2m \text{ is an integer.}\\ \text{If } n = 5m \pm 2 \text{ then } n^2 &= (5m \pm 2)(5m \pm 2) = 25m^2 \pm 20m + 4\\ &= 5q \text{ where } q = 5m^2 \pm 4m \text{ is an integer.}\end{aligned}$$

So n^2 can be expressed as $5q$ or $5q + 1$ or $5q + 4$ for some integer q.

Therefore the remainder when n^2 is divide by 5 is always $0, 1$ or 4.

9 $2x^3 + 3x^2 + x = x(2x + 1)(x + 1)$

For integer x, it must be the case that either x or $x + 1$ is even.

Therefore the expression has at least one even factor and so is a multiple of 2.

Now consider the three cases $x = 3m, x = 3m + 1$ or $x = 3m + 2$.

If $x = 3m$ then the expression has a factor $3m$ which is a multiple of 3.

If $x = 3m + 1$ then $2x + 1 = 6m + 3 = 3(2m + 1)$ so the expression has a factor which is a multiple of 3.

If $x = 3m + 2$ then $x + 1 = 3m + 3 = 3(m + 1)$ so the expression has a factor which is a multiple of 3.

So, irrespective of the value of x the expression has at least one factor of 2 and a factor of 3 and is therefore a multiple of 6.

10 Break into two cases: a, b same sign.

$a > -b \geqslant 0$

$-b > a \geqslant 0$

(Since the statement to be proved is symmetrical in a and b, the cases $b > -a > 0$ and $-a > b > 0$ can be proved by the final two cases just swapping the letters).

If a, b have the same sign then $|a+b| = |a| + |b|$.

If $a > -b > 0$ then $a + b > 0$ and $|a+b| = |a| - |b| \leqslant |a| + |b|$.

If $-b > a > 0$ then $a + b < 0$ and $|a+b| = |b| - |a| \leqslant |a| + |b|$.

MIXED PRACTICE 1

1 Two odd numbers can be expressed as $2m+1$ and $2n+1$ for some integers m and n.

Their product is:

$$\begin{aligned}(2m+1)(2n+1) &= 4mn + 2m + 2n + 1 \\ &= 2q + 1\end{aligned}$$

where $q = 2mn + m + n$ is an integer.

The product is therefore one greater than an even number and so is odd.

2 If n is even then it can be expressed as $n = 2m$ for some integer m.

Then $n^2 = 4m^2 = 4q$ where $q = m^2$ is an integer.

Therefore n^2 is an integer multiple of 4, so is divisible by 4.

3 Any integer value of $q \neq 0, 1$ will give rise to a counter example with $a = cq$ and $b = qd$; non-reduced fractions are equivalent in value to their reduced counterparts.

Counter example: $\frac{2}{6} \equiv \frac{1}{3}$

4 If the two values are both positive, the statement is true, but if either or both is negative then the statement will be false.

Counter example: $a = 3, b = -3$ so $a + b = 0 < a - b = 6$

5 A rational number is any value which can be expressed as the ratio of two integer values, the denominator not taking the value zero.

Let the two rational values be $\frac{a}{b}$ and $\frac{c}{d}$, with a, b, c, d integers and $b, d \neq 0$

Then the product is $\frac{ac}{bd}$.

Since $b, d \neq 0$, it follows that $bd \neq 0$.

Since a, c are integers, it follows that ac is an integer.

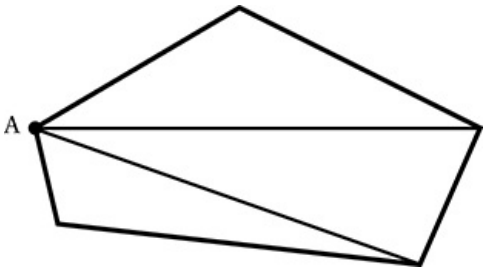
Therefore, by the definition of rational numbers, $\frac{ac}{bd}$ is also rational.

6 Pick one of the n vertices of the n-gon and label it A.

Draw a chord from A to each of the $n-3$ non-adjacent vertices (the adjacent ones are already connected to A by the two edges). This divides the n-gon into $n-2$ triangles.

The interior angle at A has been divided among the $n-2$ triangles, the two adjacent angles are unaffected and each other angle has been split between two of the triangles. The total of the interior angles of all these triangles must equal the total interior angles of the n-gon.

$n-2$ triangles have total interior angle $(n-2) \times 180^\circ = (180n - 360)^\circ$



Tip

This is an alternative approach to that you saw in question 6 in Exercise 1D.

7 $x^3 + y^3 \equiv (x + y)(ax^2 + bxy + cy^2)$

Expanding RHS: $ax^3 + x^2y(a + b) + xy^2(b + c) + cy^3$

For equivalence, all coefficients must match.

$x^3 : 1 = a$

$x^2y : 0 = a + b \Rightarrow b = -1$

$xy^2 : 0 = b + c \Rightarrow c = 1$

$y^3 : 1 = c$ consistent with $c = 1$

$a = 1, b = -1, c = 1$

8 If all you needed was to show the expression is even, then you could argue that n^2 is odd, so $n^2 + 3$ is even and therefore the whole expression is also even.

To prove it is a multiple of 4 requires a more sophisticated approach:

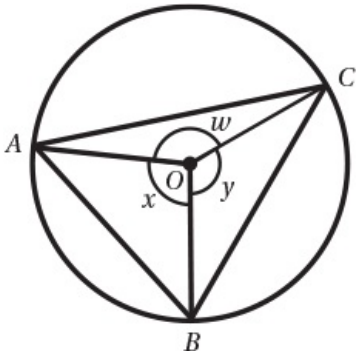
$n^2 + 4n + 3 = (n + 1)(n + 3)$

If n is odd then $n + 1$ and $n + 3$ are both even: $n + 1 = 2m$, $n + 3 = 2(m + 1)$

Then $n^2 + 4n + 3 = 2m \times 2(m + 1) = 4m(m + 1)$ which is clearly a multiple of 4.

9 Connect the centre of the circle to each of the three points A, B, C on the circumference

Let the angle at the circle centre be x for AOB, w for AOC and v for BOC.



Each of the three triangles is isosceles, since $OA = OB = OC = r$, the circle radius

Therefore $ACO = \frac{1}{2}(180^\circ - w)$ and $BCO = \frac{1}{2}(180^\circ - v)$

$ACB = ACO + OCB = 180^\circ - \frac{1}{2}(v + w)$

But $x + v + w = 360^\circ$, since the three angles complete the centre of the circle.

So $x = 360^\circ - (v + w) = 2ACB$

The angle (x) subtended by chord AB to the centre of the circle is twice the angle subtended to any point on the circumference of the major sector.

10 Break into two cases: $n = 3m$ or $n = 3m \pm 1$ (every integer is either a multiple of 3 or one distant from a multiple of 3).

If $n = 3m$ for some integer m then $n^3 = 27m^3 = 9(3m^3)$ is a multiple of 9.

If $n = 3m \pm 1$ for some integer m then

$n^3 = 27m^3 \pm 27m^2 + 9m \pm 1 = 9\left(3m^3 \pm 3m^2 + m\right) \pm 1$

which is one distant from a multiple of 9.

IRANINOVIN

11 a False: two integers have an integer product but it is not true that an integer product requires two integer components. For example, $ab = 3$ is an integer but $a = 5, b = 0.6$ are not both integers.

b False: two irrational number can have a rational product (e.g. $\sqrt{2} \times \sqrt{2} = 2$) and also an irrational product does not mean both components have to be irrational (e.g. $2\sqrt{3} = 2 \times \sqrt{3}$).

12 Let the middle value of the consecutive integers be n.

Then the integers are $n - 1$, n and $n + 1$.

Exactly one of these numbers must be a multiple of 3:

If $n = 3m$ then n is a multiple of 3.

If $n = 3m + 1$ then $n - 1$ is a multiple of 3.

If $n = 3m + 2$ then $n + 1$ is a multiple of 3.

Therefore the product must also be a multiple of 3.

At least one of these numbers must be a multiple of 2:

If n is even then n is a multiple of 2.

If n is odd then $n + 1$ is even and is a multiple of 2.

Therefore the product must be a multiple of 2.

Since the product is a multiple of 2 and of 3, it must be a multiple of 6.

13 The two odd numbers can be expressed as $2m + 1$ and $2n + 1$.

The difference between the squares of these values is therefore given as:

$$\begin{aligned}(2m+1)^2 - (2n+1)^2 &= 4m^2 + 4m + 1 - (4n^2 + 4n + 1) \\ &= 4m^2 + 4m + 4n^2 + 4n \\ &= 4(m(m+1) + n(n+1))\end{aligned}$$

If m is odd then $m + 1$ is a multiple of 2 and if m is even then m is a multiple of 2.

Therefore $m\,(m + 1)$ is always a multiple of 2: $m(m + 1) = 2q$ for some integer q

By similar reasoning, $n(n + 1) = 2r$ for some integer r

$$\begin{aligned}4(m(n+1) + n(n+1)) &= 4(2q + 2r) \\ &= 8(q + r)\end{aligned}$$

So the difference between the squares of any two odd integers must be a multiple of 8.

14 a As you did in question 8 in Exercise 1C, select a value for n which is a multiple of the constant term of the expression. Any positive multiple of 1601 will clearly lead to 1601 being a factor of the result.

b $n^2 - 1 = (n + 1)(n - 1)$

For $n > 2$, both $n - 1$ and $n + 1$, factors of the expression value, are greater than 1, so $n^2 - 1$ is the product of two integers greater than 1 and is not prime.

15 $x = a^2 - b^2 = (a + b)(a - b)$

Consider the two cases where $a + b$ is even or $a + b$ is odd:

If $a + b$ is even then either both are even or both are odd; either way, $a - b$ is also even.

So $a + b = 2m$ for some integer m and $a - b = 2q$ for some integer q.

Then $x = 2m \times 2q = 4r$ where $r = mq$ is an integer.

Therefore x is a multiple of 4.

If $a + b$ is odd then one of a, b is odd and the other is even; either way, $a - b$ is also odd

So $a+b=2m+1$ for some integer m and $a-b=2q+1$ for some integer q.

$$\begin{aligned}\text{Then } x &= (2m+1)(2q+1) = 4mq+2m+2q+1 \\ &= 2r+1 \text{ where } r = 2mq+m+q \text{ is an integer.}\end{aligned}$$

Therefore x is odd.

Chapter 2 worked solutions

2 Indices and surds

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 2A

7 $(100x^4)^{-\frac{1}{2}} = \dfrac{1}{\sqrt{100x^4}} = \dfrac{1}{10x^2}$

8 $\dfrac{x^{\frac{2}{3}} \times x^{\frac{1}{2}}}{x^{\frac{1}{6}}} = x^{\frac{2}{3}+\frac{1}{2}-\frac{1}{6}} = x^1 = x$

9 $\dfrac{\sqrt[3]{64p^6q^4}}{2q} = \dfrac{4p^2q^{\frac{4}{3}}}{2q} = 2p^2q^{\frac{1}{3}}$

10 $\dfrac{1-x^3}{\sqrt[3]{x}} = \dfrac{1-x^3}{x^{\frac{1}{3}}} = x^{-\frac{1}{3}} - x^{\frac{8}{3}}$

$a = -\dfrac{1}{3}, b = \dfrac{8}{3}$

11 $n = 10^6$. Half a second is half a million (5×10^5) microseconds.

$k \times n^{1.5} = k \times 10^9 = 5 \times 10^5$ so $k = 5 \times 10^{-4}$

12 a $243 = k \times 81^{1.5}$ therefore $3^5 = k \times \left(3^4\right)^{1.5} = k \times 3^6$

$k = 3^{-1} = \dfrac{1}{3}$

b $\dfrac{64}{3} = \dfrac{1}{3} \times A^{1.5}$ therefore $2^6 = A^{1.5}$

$\left(2^4\right)^{1.5} = A^{1.5}$ so $A = 2^4 = 16\ \text{cm}^2$

13 $2y \times y^2 = 128 = 2y^3$ therefore $y^3 = 64$ so $y = 4$

$x = 2y = 8\ \text{cm}$

14 $\left(3x^9 + \dfrac{3}{8}x^9\right)^{-\frac{1}{3}} = \left(\dfrac{27x^9}{8}\right)^{-\frac{1}{3}} = \left(\dfrac{2^3}{3^3x^9}\right)^{\frac{1}{3}} = \dfrac{2}{3x^3}$

15 $\sqrt{3^{2a+b}} = \dfrac{27^a}{3^b}$

$\left(3^{2a+b}\right)^{\frac{1}{2}} = \dfrac{\left(3^3\right)^a}{3^b}$

$3^{a+\frac{b}{2}} = 3^{3a-b}$

$a + \dfrac{b}{2} = 3a - b$

$\dfrac{3b}{2} = 2a$ so $a = \dfrac{3b}{4}$

16 $\left(3x^{\frac{3}{4}} - x^{-\frac{3}{4}}\right)^2 = 9\left(x^{\frac{3}{4}}\right)^2 - 6\left(x^{\frac{3}{4}}\right)\left(x^{-\frac{3}{4}}\right) + \left(x^{-\frac{3}{4}}\right)^2$

$= 9x^{\frac{3}{2}} - 6 + x^{-\frac{3}{2}}$

$= 9x^{\frac{3}{2}} + \frac{1}{x^{\frac{3}{2}}} - 6$

$a = 9, b = 1, c = -6, n = 1.5$

17 $4^{ax} = b \times 8^x$ so $b = \left(\frac{4^a}{8}\right)^x$

If multiple values of x satisfy this for a single set of values (a, b) then the bracket must be unvarying to different indices. The only values for which this is true are $0, 1$ and -1:

$\frac{4^a}{8} = 0$ has no finite solution for a.

$\frac{4^a}{8} = 1$ gives $a = \frac{3}{2}, b = 1$, and the problem simplifies to $8^x = 8^x$, true for all values of x.

$\frac{4^a}{8} = -1$ has no real solution for a. a.

Hence the only real, finite solution is $a = \frac{3}{2}, b = 1$.

Tip

You may reassess this question if you go on to study complex numbers in Student Book 2, and you will then realise that if a and b can be complex, the restriction $|b| = 1, \mathrm{Re}(a) = \frac{3}{2}$ allows an infinite number of solutions for x.

18 The accepted answer is that $0^0 = 1$, in the same way that the factorial form $0! = 1$; it is an empty product, and so returns the multiplicative identity 1.

When considered as a limit, it is undefined if there are no specific details of which limit is intended.

$a^0 = 1$ for all non-zero values of a.

So if you take sequentially decreasing values of $|a|$, the pattern is clear – the result value is stable at 1.

You say that the limit as $a \to 0$ of a^0 is 1: $\lim_{a \to 0}\left\{a^0\right\} = 1$

$0^b = 0$ for all positive values of b.

So if you take sequentially decreasing values of $b > 0$, the pattern is clear – the result value is stable at 0. Importantly, you cannot extend into negative values of b, and this suggests that the border value, $b = 0$, is potentially problematic.

You say that the limit as $b \to 0^+$ of 0^b is 0: $\lim_{b \to 0^+}\left\{0^b\right\} = 0$

You could also consider other patterns, such as c^c as $c \to 0$ for positive values of c which gives a pattern tending to 1 (use a calculator to try some values to see this).

To justify the definition $0^0 = 1$, consider how you write out powers long hand:

$1 \times 0^2 = 1 \times 0 \times 0$

$1 \times 0^1 = 1 \times 0$

$1 \times 0^0 = 1$

This last line clearly states that $0^0 = 1$. As with $0! = 1$, this evaluation allows a whole slew of general mathematical results (ranging from polynomial definitions through differentiation, binomial theorem and more well beyond the scope of A level) to be stated without restrictions on zero, and hence is arguably a mathematical convenience more than a determinable result.

19 Evaluating most distant exponents first would be the standard, but in this specific case it doesn't actually matter:

$2^{2^2} = 2^{(2^2)} = 2^4 = 16$

> Note
>
> That if you incorrectly interpret the expression as $\left(2^2\right)^2$ you also reach the same value: 16.

Try the same with 3^{3^3}: If you get the answer $19\,683$, think again – you are in error by a factor of nearly a billion!

EXERCISE 2B

5 $\dfrac{1}{1+\sqrt{n}} + \dfrac{1}{1-\sqrt{n}} = \dfrac{1-\sqrt{n}}{1-n} + \dfrac{1+\sqrt{n}}{1-n} = \dfrac{2}{1-n}$

6 $\dfrac{4}{\sqrt{20}-\sqrt{12}} = \dfrac{4}{2\sqrt{5}-2\sqrt{3}} = \dfrac{2}{\sqrt{5}-\sqrt{3}} \times \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

$= \dfrac{2\left(\sqrt{5}+\sqrt{3}\right)}{5-3} = \sqrt{5}+\sqrt{3}$

7 $\dfrac{5+\sqrt{2}}{3-2\sqrt{2}} = \dfrac{5+\sqrt{2}}{3-2\sqrt{2}} \times \dfrac{3+2\sqrt{2}}{3+2\sqrt{2}} = \dfrac{15+4+10\sqrt{2}+3\sqrt{2}}{9-8}$

$= 19+13\sqrt{2}$

8 $3\sqrt{2} = \sqrt{18} > \sqrt{12} = 2\sqrt{3}$

9 $x\sqrt{27} = 5x\sqrt{3} + 2\sqrt{48}$

$x\left(3\sqrt{3}-5\sqrt{3}\right) = 2 \times 4\sqrt{3}$

$x = \dfrac{8\sqrt{3}}{-2\sqrt{3}} = -4$

10 $\dfrac{1}{2\sqrt{n}-3} = \dfrac{1}{2\sqrt{n}-3} \times \dfrac{2\sqrt{n}+3}{2\sqrt{n}+3} = \dfrac{2\sqrt{n}+3}{4n-9}$

11 $\left(n\sqrt{15}-\sqrt{5}\right)^2 = 15n^2 - 2n\sqrt{75} + 5 = 15n^2 + 5 - 10n\sqrt{3}$

12 a $\text{Area} = \left(a+b\sqrt{2}\right)\left(b-a\sqrt{2}\right) = ab + b^2\sqrt{2} - a^2\sqrt{2} - 2ab$

$= -ab + \left(b^2-a^2\right)\sqrt{2}$

b By Pythagoras, the diagonal length l satisfies:

$l^2 = \left(a+b\sqrt{2}\right)^2 + \left(b-a\sqrt{2}\right)^2$

$= a^2 + 2b^2 + 2ab\sqrt{2} + b^2 + 2a^2 - 2ab\sqrt{2}$

$= 3\left(a^2+b^2\right)$

$l = \sqrt{3\left(a^2+b^2\right)}$

13 a $\sqrt{27}+\sqrt{3} = 3\sqrt{3}+\sqrt{3} = 4\sqrt{3} = \sqrt{48}$

b $\sqrt{20}+\sqrt{5} = 2\sqrt{5}+\sqrt{5} = 3\sqrt{5} = \sqrt{45}$

Since $\sqrt{48} > \sqrt{45}$, it follows that $\sqrt{27}+\sqrt{3} > \sqrt{20}+\sqrt{5}$ so $\sqrt{27}-\sqrt{20} > \sqrt{5}-\sqrt{3}$

14 a $\left(a+b\sqrt{2}\right)^2 = a^2 + 2b^2 + 2ab\sqrt{2}$

b $\left(1-\sqrt{2}\right)^4 = \left(1+2-2\sqrt{2}\right)^2$ from question 14a.

$= \left(3-2\sqrt{2}\right)^2 = 9+8-12\sqrt{2} = 17-12\sqrt{2}$

Since $\left(1-\sqrt{2}\right)^4 > 0$, it follows that $17-12\sqrt{2} > 0$:

$17 > 12\sqrt{2}$ so $\sqrt{2} < \dfrac{17}{12}$

15 a $\begin{aligned}(a-b)(a^2+ab+b^2) &= a(a^2+ab+b^2) - b(a^2+ab+b^2)\\ &= (a^3+a^2b+ab^2) - (a^2b+ab^2+b^3)\\ &= a^3 - b^3\end{aligned}$

b From question 15a, using $a = \sqrt[3]{3}$ and $b = \sqrt[3]{2}$:

$$\begin{aligned}\frac{1}{\sqrt[3]{3}-\sqrt[3]{2}} &= \frac{1}{\sqrt[3]{3}-\sqrt[3]{2}} \times \frac{\sqrt[3]{3^2}+\sqrt[3]{6}+\sqrt[3]{4}}{\sqrt[3]{3^2}+\sqrt[3]{6}+\sqrt[3]{4}}\\ &= \frac{\sqrt[3]{3^2}+\sqrt[3]{6}+\sqrt[3]{4}}{3-2} = \sqrt[3]{3^2}+\sqrt[3]{6}+\sqrt[3]{4}\end{aligned}$$

16 Not true; for real values of x, $\sqrt{x^2} = \begin{cases} x & \text{for } x \geqslant 0 \\ -x & \text{for } x \leqslant 0 \end{cases}$, also called the *modulus* of x, and written more simply as $|x|$.

For any negative value of x,$\sqrt{x^2} = -x$.

MIXED PRACTICE 2

1 $\left(n+\sqrt{5}\right)^2 = n^2 + 2n\sqrt{5} + 5 = (n^2+5) + (2n)\sqrt{5}$

2 $z = x(3x)^2 = 9x^3$

3 $\dfrac{10}{\sqrt{28}-\sqrt{8}} = \dfrac{10}{\sqrt{28}-\sqrt{8}} \times \dfrac{\sqrt{28}+\sqrt{8}}{\sqrt{28}+\sqrt{8}} = \dfrac{10\left(2\sqrt{7}+2\sqrt{2}\right)}{28-8} = \sqrt{7}+\sqrt{2}$

4 $y^{-4} = \left(\dfrac{\sqrt{x}}{2}\right)^4 = \dfrac{x^2}{16}$

5 $3x\sqrt{8} = x\sqrt{2} + \sqrt{32}$

$x\left(6\sqrt{2}-\sqrt{2}\right) = 4\sqrt{2}$

$5x = 4$

$x = \dfrac{4}{5}$

6 **a** $\left(x^{\frac{1}{3}}\right)^6 = x^{\frac{1}{3}\times 6} = x^2$

b $\dfrac{3y^4 \times 1000y^3}{2y^5} = \dfrac{3000y^7}{2y^5} = 1500y^2$

7 **a** $\dfrac{1}{9} = \dfrac{1}{3^2} = 3^{-2}$

b $\sqrt[3]{3} = 3^{\frac{1}{3}}$

c $3^{10} \times 9^{135} = 3^{10} \times \left(3^2\right)^{135} = 3^{10} \times 3^{270} = 3^{280}$

8 $\left(x^4 + 7x^3 \times \dfrac{x}{9}\right)^{-\frac{1}{2}} = \left(\dfrac{16x^4}{9}\right)^{-\frac{1}{2}} = \left(\dfrac{9}{16x^4}\right)^{\frac{1}{2}} = \dfrac{3}{4x^2}$

9 $\dfrac{\sqrt{n}+1}{\sqrt{n}-1} = \dfrac{\sqrt{n}+1}{\sqrt{n}-1} \times \dfrac{\sqrt{n}+1}{\sqrt{n}+1} = \dfrac{n+1+2\sqrt{n}}{n-1}$

10 **a** $\left(a+b\sqrt{5}\right)^2 = a^2 + 5b^2 + 2ab\sqrt{5}$

b $\left(2-\sqrt{5}\right)^4 = \left(4+5-4\sqrt{5}\right)^2 = \left(9-4\sqrt{5}\ \right)^2 = 81 + 80 - 72\sqrt{5}$

$\left(2-\sqrt{5}\right)^4 > 0 \Rightarrow 161 - 72\sqrt{5} > 0$ therefore $\sqrt{5} < \dfrac{161}{72}$

c $\left(2-\sqrt{5}\right)^3 = \left(2-\sqrt{5}\right)\left(9-4\sqrt{5}\right) = 18 + 20 - 17\sqrt{5}$

$\left(2-\sqrt{5}\right)^3 < 0 \Rightarrow 38 - 17\sqrt{5} < 0$ therefore $\sqrt{5} > \dfrac{38}{17}$

d **i** $3 > \sqrt{5}$ so $\left(3-\sqrt{5}\right)^3 > 0$

Expanding gives an inequality of the form $a - b\sqrt{5} > 0$.

This rearranges to $\sqrt{5} < \dfrac{a}{b}$, giving an upper bound on $\sqrt{5}$.

However, since $\sqrt{5}$ is nearer to 2 than to 3 and the expression uses a higher power, the error will be much less, so the bound in question 10b is better.

ii $2 < \sqrt{5} < 3$ so it follows that while increasing powers of $\left(2-\sqrt{5}\right)$ or $\left(3-\sqrt{5}\right)$ get progressively closer to zero (and hence any bound achieved from rearranging gets tighter to the actual value of $\sqrt{5}$), increasing powers of $\left(4-\sqrt{5}\right) > 1$ get progressively further from zero or one, and hence give bounds that are not as good.

Chapter 3 worked solutions

3 Quadratic functions

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 3A

3 $8x - 9 = (3x - 1)(x + 3)$

$8x - 9 = 3x^2 + 8x - 3$ therefore $3x^2 + 6 = 0$ and so $x^2 = -2$

No real solutions.

4 $6x = 5 + 4x^{-1}$

$6x^2 - 5x - 4 = 0$ therefore $(3x - 4)(2x + 1) = 0$ and so $x = \frac{4}{3}$ or $-\frac{1}{2}$

5 $x + x^{-1} = 3$

$x^2 - 3x + 1 = 0$

$x = \frac{3 \pm \sqrt{(-3)^2 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}$

6 $x^2 + 8k^2 = 6kx$

$x^2 - 6kx + 8k^2 = 0$

$x = \frac{6k \pm \sqrt{(-6k)^2 - 32k^2}}{2} = \frac{6k \pm \sqrt{4k^2}}{2} = 3k \pm k = 2k$ or $4k$

7 Solve the equation $2\sqrt{5}x^2 + 3x - \sqrt{5} = 0$, giving your answers in simplified surd form.

$x = \frac{-3 \pm \sqrt{3^2 + 4 \times 2\sqrt{5} \times \sqrt{5}}}{4\sqrt{5}} = \frac{-3 \pm \sqrt{49}}{4\sqrt{5}}$

$= -\frac{10}{4\sqrt{5}}$ or $\frac{4}{4\sqrt{5}} = -\frac{\sqrt{5}}{2}$ or $\frac{\sqrt{5}}{5}$

8 $x = \frac{-2\sqrt{5} \pm \sqrt{(2\sqrt{5})^2 + 4 \times \sqrt{2} \times 3\sqrt{2}}}{2\sqrt{2}} = \frac{-2\sqrt{5} \pm \sqrt{44}}{2\sqrt{2}}$

$= -\frac{\sqrt{10}}{2} \pm \frac{\sqrt{22}}{2}$

9 $ax^2 + bx + c - y = 0$ therefore $x = \frac{-b \pm \sqrt{b^2 - 4a(c - y)}}{2a}$

10 $x^2 + kx + 3 = 0$ therefore $x = \frac{-k \pm \sqrt{k^2 - 12}}{2}$

Difference between the roots is $2 \times \frac{\sqrt{k^2 - 12}}{2} = \sqrt{69}$

$k^2 - 12 = 69$ simplifying $k^2 = 81$ therefore $k = \pm 9$

EXERCISE 3C

4 **a** $y = x^2 - 6x + 11 = (x - 3)^2 - 9 + 11 = (x - 3)^2 + 2$

b Vertex at $(3, 2)$: minimum value is 2.

5 **a** $y = a(x + b)^2 + c$

Minimum at $(3, 6) = (-b, c)$: $b = -3, c = 6$

b Require that $y(1) = 14$

$14 = a(1 - 3)^2 + 6 = 6 + 4a$ therefore $a = 2$

6 **a** $y = 8x - x^2 - 21 = -\left[x^2 - 8x\right] - 21 = -\left[(x - 4)^2 - 16\right]$

$-21 = -(x - 4)^2 - 5$

$a = -5, b = -4$

b Turning point at $(4, -5)$.

c The curve is a negative quadratic with a vertex below the x-axis, so never intersects the axis, and has no roots.

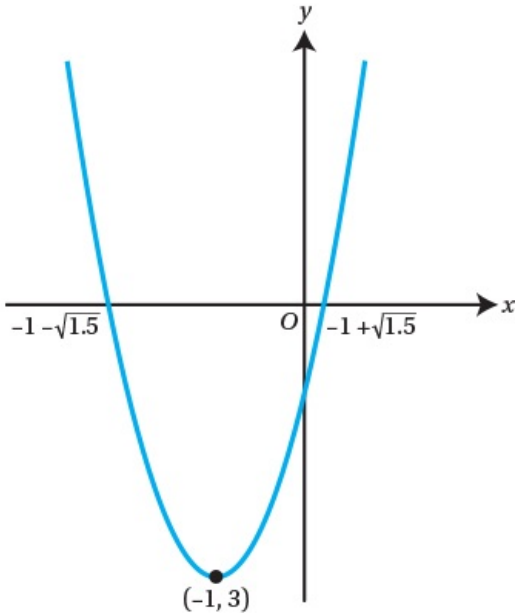
7 **a** $2x^2 + 4x - 1$

$= 2\left(x^2 + 2x\right) - 1$

$= 2\left((x + 1)^2 - 1\right) - 1$

$= 2(x + 1)^2 - 3$

b



$2(x + 1)^2 - 3 = 0$ therefore $(x + 1)^2 = \frac{3}{2}$ so $x = -1 \pm \sqrt{\frac{3}{2}}$

c The vertex is at $(-1, -3)$.

d Symmetry line is $x = -1$.

8 **a** $x^2 + 6x = (x + 3)^2 - 9$

b $(x + 3)^2 - 9 \geqslant -9$

Solutions to $(x + 3)^2 - 9 = p$ have solutions for $p \geqslant -9$

9 **a** $\mathrm{f}(x) = 3x^2 + 2x + 1 = 3\left[x^2 + \frac{2}{3}x\right] + 1$

$= 3\left[\left(x + \frac{1}{3}\right)^2 - \frac{1}{9}\right] + 1$

$= 3\left(x + \frac{1}{3}\right)^2 + \frac{2}{3}$

IRANINOVIN

b From the completed square form, the minimum value of $f(x)$ is $\frac{2}{3}$ and so $f(x) = 0$ has no real solutions.

c $x = -\frac{1}{3}$

10 $-2x^2 + 8x - 13 = -2\left[x^2 - 4x\right] - 13$

$= -2\left[(x-2)^2 - 4\right] - 13$

$= -2(x-2)^2 - 5 \leqslant -5$ for all x

There is therefore no real solution to the equation $-2x^2 + 8x - 13 = 0$.

EXERCISE 3D

3 $2x^2 + x - 6 > 0$

Roots at $x = \frac{-1 \pm \sqrt{1+48}}{4} = -2$ or 1.5

Positive quadratic is greater than zero outside the roots

4 $6x = 5 + 4x^{-1}$

$6x^2 - 5x - 4 = 0$ so $(3x-4)(2x+1) = 0$

$x = \frac{4}{3}$ or $-\frac{1}{2}$

5 $7t - 4.9t^2 > 1.5$ so $4.9t^2 - 7t + 1.5 < 0$

Roots at $t = \frac{7 \pm \sqrt{49 - 29.4}}{9.8}$

Length of time between the roots (i.e. time from when the ball passes 1.5 m going up to when it passes 1.5 m going down again) $2 \times \frac{\sqrt{49-29.4}}{9.8} = 0.904$ seconds.

6 **a** **i** $7x - 5 < 3x + 5$

$4x < 10$

$x < 2.5$

ii $2x^2 - 7x < 4x - 5$

$2x^2 - 11x + 5 < 0$

Roots at $x = \frac{11 \pm \sqrt{121-40}}{4} = 0.5$ or 5

Positive quadratic is less than zero between the roots

$0.5 < x < 5$

b Require $x < 2.5$ and $0.5 < x < 5$: $0.5 < x < 2.5$

7 $x^2 - 5x + 6 > 0 \quad (1)$

$(x-2)(x-3) > 0$

Positive quadratic is greater than zero outside the roots $x < 2$ or $x > 3$.

$x^2 - 1 \geqslant 0 \quad (2)$

Positive quadratic is greater than zero outside the roots $x \leqslant -1$ or $x \geqslant 1$.

For both conditions to be true, $x \leqslant -1$ or $1 \leqslant x < 2$ or $x > 3$.

8 $2x^2 - 4x \geqslant 0$

$2x(x-2) \geqslant 0$

Positive quadratic is greater than zero outside the roots

IRANINOVIN

$x \leqslant 0$ or $x \geqslant 2$

$5x^2 - 13x - 6 \leqslant 0$

$(5x + 2)(x - 3) \leqslant 0$

Positive quadratic is less than zero between the roots

$-\frac{2}{5} \leqslant x \leqslant 3$

Require both conditions to be true:

$-\frac{2}{5} \leqslant x \leqslant 0$ or $2 \leqslant x \leqslant 3$

9 Total profit (£) equals $n(280 - 5n) - (950 + 63n) = 217n - 950 - 5n^2$

Require $217n - 950 - 5n^2 > 0$

Roots at $n = \frac{-217 \pm \sqrt{217^2 - 20 \times 950}}{-10} = 38.5$ or 4.9

Negative quadratic is greater than zero between the roots.

Require integer values of n, so $5 \leqslant n \leqslant 38$.

EXERCISE 3E

4 Equal roots: discriminant $\Delta = 0$.

$\Delta = (-4)^2 - 4m(2m) = 16 - 8m^2$ so $8m^2 = 16$ therefore $m = \pm\sqrt{2}$.

5 Repeated root: discriminant $\Delta = 0$.

$\Delta = (2k + 1)^2 - 4(-3)(-4k) = 4k^2 + 4k + 1 - 48k$

$4k^2 - 44k + 1 = 0$

$k = \frac{44 \pm \sqrt{44^2 - 16}}{8} = \frac{11 \pm \sqrt{120}}{2} = \frac{11 \pm 2\sqrt{30}}{2}$

6 Positive quadratic $\geqslant 0$ for all x: require 1 or 0 roots

Discriminant $\Delta \leqslant 0$:

$\Delta = (-3)^2 - 4(2)(2c + 1)$ so $9 - 16c - 8 \leqslant 0$ therefore $c \geqslant \frac{1}{16}$.

7 No real solutions: discriminant $\Delta < 0$.

$\Delta = (-2k)^2 - 4(6k)$ so $4k^2 - 24k < 0$ therefore $(k - 6) < 0$.

Positive quadratic is negative between the roots: $0 < k < 6$.

8 No real roots: discriminant $\Delta < 0$.

$\Delta = (k + 3)^2 + 4k$ so $k^2 + 10k + 9 < 0$ therefore $(k + 1)(k + 9) < 0$.

Positive quadratic is less than zero between the roots: $-9 < k < -1$

9 Real root(s): discriminant $\Delta \geqslant 0$.

$\Delta = m^2 + 8m$ so $m(m + 8) \geqslant 0$

Positive quadratic is greater than zero outside the roots: $m \leqslant -8$ or $m \geqslant 0$.

10 Negative quadratic < 0 for all x: require 0 roots.

Discriminant $\Delta < 0$.

$\Delta = 3^2 - 4m(-4)$ so $9 + 16m < 0$ therefore $m < -\frac{9}{16}$.

EXERCISE 3F

2 $y^2 - y - 6 = 0$ so $(y-3)(y+2) = 0$

$y = 3$ or $y = -2$ (Reject negative solution since $y = \sqrt{x} \geqslant 0$.)

$\sqrt{x} = 3$ so $x = 9$

3 Let $y = x^2$.

$y - 10 + \dfrac{9}{y} = 0$ therefore $y^2 - 10y + 9 = 0$

$(y-1)(y-9) = 0$ so $y = 1$ or 9

$x = \pm 1$ or ± 3

4 Let $u = \dfrac{1}{3x+1}$.

Then $u^2 - 6u + 5 = 0$ so $(u-1)(u-5) = 0$.

Solutions are $u = 1$ or 5, so $x = \dfrac{1-u}{3u} = 0$ or $-\dfrac{4}{15}$.

5 Let $y = x^{1.5}$.

$y^2 - 9y + 8 = 0$ therefore $(y-1)(y-8) = 0$ so $y = 1$ or 8

$x = 1$ or 4

6 $9\left(1 + 9^{x-1}\right) = 10 \times 3^x$

Expanding:

$9^x - 10 \times 3^x + 9 = 0$ therefore $(3^x)^2 - 10 \times 3^x + 9 = 0$

$3^{2x} - 10 \times 3^x + 9 = 0$

$(3^x - 1)(3^x - 9) = 0$ therefore $3^x = 1$ or 9

$x = 0$ or 2

7 Let $y = 5^x$.

$y = 6 - \dfrac{5}{y}$ therefore $y^2 - 6y + 5 = 0$

$(y-1)(y-5) = 0$ so $y = 1$ or 5

$x = 0$ or 1

8 Let $u = \dfrac{1}{x-1}$.

Then $3u^2 - 5u - 2 = 0$ so $(3u+1)(u-2) = 0$.

Solutions are $u = 2$ or $-\dfrac{1}{3}$, so $x = \dfrac{u+1}{u} = \dfrac{3}{2}$ or -2.

9 **a** $\begin{aligned} x^4 - 4x^3 - 7x^2 &= x^2\left[x^2 - 4x - 7\right] \\ &= x^2\left[(x-2)^2 - 11\right] \\ &= \left(x^2 - 2x\right)^2 - 11x^2 \end{aligned}$

b Let $y = x^2 - 2x$.

$\begin{aligned} \text{Then } x^4 - 4x^3 - 7x^2 + 22x + 24 &= \left(x^2 - 2x\right)^2 - 11x^2 + 22x + 24 \\ &= \left(x^2 - 2x\right)^2 - 11\left(x^2 - 2x\right) + 24 \\ &= y^2 - 11y + 24 \\ &= (y-3)(y-8) = 0 \end{aligned}$

Solutions are $y = 3$ or 8, so $x^2 - 2x - 3 = 0$ or $x^2 - 2x - 8 = 0$.

$x^2 - 2x - 3 = 0$: $(x-3)(x+1) = 0$ so $x = 3$ or -1

IRANINOVIN

$x^2 - 2x - 8 = 0 : (x + 2)(x - 4) = 0$ so $x = -2$ or 4

Solutions are $x = -1, -2, 3$ or 4.

10 $4^{x+0.5} - 17 \times 2^x + 8 = 0$

Re-write $4^{x+0.5} = \left(2^2\right)^{x+0.5} = 2^{2x+1} = 2 \times (2^x)^2$

Put $A = 2^x$

$2A^2 - 17A + 8 = 0$

$(2A - 1)(A - 8) = 0$

$A = 8$ or $A = \dfrac{1}{2}$

So $2^x = 8 = 2^3$; $x = 3$

or $2^x = \dfrac{1}{2} = 2^{-1}$; $x = -1$

11 Let $y = \sqrt{x}$

$y^2 = y + 12$ therefore $y^2 - y - 12 = 0$

$(y - 4)(y + 3) = 0$ so $y = 4$ or -3 (reject negative solution since $y = \sqrt{x} > 0$)

So $x = 16$

MIXED PRACTICE 3

1 Quadratic is symmetrical about turning point, so it lies halfway between the roots.

Roots at k and $k + 4$ so the x-coordinate of the vertex is $k + 2$.

2 $(2x - 3)(x - 5) = (x - 3)^2$

Expanding and rearranging:

$2x^2 - 13x + 15 = x^2 - 6x + 9$ therefore $x^2 - 7x + 6 = 0$

$(x - 1)(x - 6) = 0$ so $x = 1$ or 6.

3 Substituting $y = x^2$:

$y^2 - 5y + 4 = 0$ therefore $(y - 1)(y - 4) = 0$ so $y = 1$ or 4

$x = \pm 1$ or ± 2.

4 **a** Positive quadratic must have a minimum turning point.

b $y = (x - a)^2 + b$ has turning point $(a, b) = (3, 7)$ so $a = 3, b = 7$.

5

Tip

Two methods are given. Method 1 uses the given points and values and, using simultaneous equations, gradually evaluates a, b and c. Method 2 creates a quadratic with the required details and compares. See which you prefer.

Method 1

$y = a(x - b)^2 + c$

$y(-2) = 0 = a(-2 - b)^2 + c = ab^2 + 4ab + 4a + c \qquad (1)$

$y(6) = 0 = a(6 - b)^2 + c = ab^2 - 12ab + 36a + c \qquad (2)$

$(1) - (2) : 16ab - 32a = 0 = 16a(b - 2)$

Since $a \neq 0$ (or the function is constant, and cannot have a maximum at 48) it follows that $b = 2$.

The maximum value in the completed square form is given by c so $c = 48$.

Then $(1) : 4a + 8a + 4a + 48 = 0$

$16a = -48$

$a = -3$

$a = -3, b = 2, c = 48$

Method 2

Require a quadratic with roots at -2 and 6:

$y = k(x + 2)(x - 6)$

Require vertex at $y = 48$. Vertex occurs midway between roots, so at $x = 2$.

$y(2) = 48 = -16k$

$k = -3$

Expanding and completing the square:

$y = -3\left(x^2 - 4x - 12\right) = -3\left((x - 2)^2 - 4 - 12\right) = -3(x - 2)^2 + 48$

Comparing with the given equation:

$a = -3, b = 2, c = 48.$

6 **a** Roots are at -1 and 4 so $p = 1, q = 4$.

b Vertex is midway between roots, so at $x = 1.5$.

7 **a** Discriminant $\Delta = (-4)^2 - 4k^2 = 16 - 4k^2$

b Equal roots: $\Delta = 0$.

$k^2 = 4$ so $k = \pm 2$.

8 $x^2 - 2x = x(x - 2) > 0 \qquad (1)$

Positive quadratic is greater than zero outside the roots: $x < 0$ or $x > 2$

$x^2 - 4x + 3 = (x - 1)(x - 3) \geqslant 0 \qquad (2)$

Positive quadratic is greater than zero outside the roots: $x \leqslant 1$ or $x \geqslant 3$.

Taking both inequalities simultaneously: $x < 0$ or $x \geqslant 3$.

9 Negative quadratic with a single positive root and a negative axis intercept.

a is negative.

Axis intercept c is negative.

Discriminant $b^2 - 4ac$ is zero for a repeated root.

The root occurs at $x = -\dfrac{b}{2a} > 0$ so since $a < 0$ it follows that b is positive.

10 **a** $(x - 5)^2 - 25 + 35 = (x - 5)^2 + 10$

b Maximum value occurs for minimum positive denominator.

Maximum value is $\dfrac{1}{10^3} = \dfrac{1}{1000}$

11 No real roots: discriminant $\Delta < 0$.

$\Delta = (k + 1)^2 - 4(2k) = k^2 - 6k + 1 < 0$

Roots at $k = \dfrac{6 \pm \sqrt{6^2 - 4}}{2}$

$k = 3 \pm \sqrt{8}$ therefore $k = 3 \pm 2\sqrt{2}$.

Positive quadratic is less than zero between the roots:

$3-2\sqrt{2}<k<3+2\sqrt{2}$

12 Let $y=x^{1/4}$

$y+\dfrac{2}{y}=3$ therefore $y^2-3y+2=0$

$(y-1)(y-2)=0$ so $y=1$ or 2

$x=1$ or 16

13 Let $y=\dfrac{1}{5x+2}$.

Substituting: $49y^2-14y+1=(7y-1)^2=0$

Solution is $y=\dfrac{1}{7}=\dfrac{1}{5x+2}$ so $x=1$.

14 a $2x^2-6x+9$

$=2\left[x^2-3x\right]+9$

$=2\left[(x-1.5)^2-2.25\right]+9$

$=2(x-1.5)^2+4.5$

b Vertex is at $(1.5, 4.5)$.

c Positive quadratic with a vertex above the x-axis has no real roots.

15 a By Pythagoras' theorem, the oblique side has length $\sqrt{(3x)^2+(4x)^2}=5x$

Total perimeter $P=3x+2+x+5x+2+5x=14x+4$

b Lawn consists of a rectangle $3x\times(2+x)$ and a triangle with base $4x$ and altitude $3x$.

$A=3x(2+x)+\dfrac{1}{2}(4x)(3x)=6x+3x^2+6x^2=9x^2+6x$

c Require $P\geqslant 39$ and $A<99$.

For the area:

$9x^2+6x-99<0$ therefore $3x^2+2x-33<0$ so $(3x+11)(x-3)<0$

Positive quadratic is less than zero between the roots: $-\dfrac{11}{3}<x<3$

For the perimeter:

$14x+4\geqslant 39$ therefore $14x\geqslant 35$ so $x\geqslant 2.5$

For both restrictions to be fulfilled, $2.5\leqslant x<3$.

16 Let the actual roots be x_1 and x_2.

Alexia correctly interpreted c, which is the product of the roots: $c=x_1x_2=6\times 1$

Michaela correctly interpreted b, which is the negative sum of the roots:

$b=-x_1-x_2=-4-1$

So the actual quadratic is $x^2-5x+6=0$

$(x-3)(x-2)=0$ therefore $x=2$ or $x=3$

17 The tangent meets the curve (a circle) at a single point.

The intersection of the two must therefore have a single repeated solution

$x^2+(2x-k)^2=5$ so $x^2+4x^2-4kx+k^2-5=0$

$5x^2-4kx+k^2-5=0$

Require discriminant $\Delta=0$: $(-4k)^2-4(5)\left(k^2-5\right)=0$

$16k^2 - 20k^2 + 100 = 0$ therefore $4k^2 = 100$ so $k = \pm 5$

18 a

Tip

As is often the case, you can approach this methodically by citing a formula and comparing to the given equation, but there is also a quicker solution if you can spot a factorisation right at the start. Both approaches are given here.

Method 1: Formula and simultaneous equations

For a quadratic $x^2 + bx + c$, the product of the roots equals c and the sum of the roots equals $-b$.

$$\begin{cases} \alpha\beta = k - 1 & (1) \\ \alpha + \beta = k & (2) \end{cases}$$

$(2) : \alpha = k - \beta$

Substituting into (1):

$\beta(k - \beta) = k - 1$ so $\beta^2 - k\beta + k - 1 = 0$

$(\beta - 1)(\beta - k + 1) = 0$

The roots are 1 and $k - 1$.

Method 2: Factorising

$x^2 - kx + (k - 1) = 0$

Factorising: $(x - k + 1)(x - 1) = 0$

The roots are $k - 1$ and 1.

b $1^2 + (k - 1)^2 = 17$ therefore $(k - 1)^2 = 16$

$k - 1 = \pm 4$ so $k = -3$ or 5

19 $q(x) = kx^2 + (k - 2)x - 2$

Discriminant $\Delta = (k - 2)^2 - 4 \times k \times (-2)$

$\Delta = k^2 - 4k + 4 + 8k = k^2 + 4k + 4 = (k + 2)^2$

Therefore $\Delta \geqslant 0$ for all values of k, and so there must always be at least one real root for $q(x)$.

20 Car 1 has position $(-50 + 20t, 0)$ and Car 2 has position $(0, -30 + 15t)$ at t hours.

a By Pythagoras' theorem, distance between the cars d is given by

$$\begin{aligned} d^2 &= (-50 + 20t)^2 + (-30 + 15t)^2 \\ &= 2500 - 2000t + 400t^2 + 900 - 900t + 225t^2 \\ &= 625t^2 - 2900t + 3400 \end{aligned}$$

b Completing the square:

$$\begin{aligned} d^2 &= 625\left(t^2 - 4.64t\right) + 3400 \\ &= 625\left((t - 2.32)^2 - 5.3824\right) + 3400 \end{aligned}$$

So d^2 has minimum value 36.

Since d is never negative, the minimum value of d is $6\ \text{km}$.

Chapter 4 worked solutions

4 Polynomials

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 4A

3 The fully expanded form is preferable when assessing coefficients (such as the constant term which equals the y-intercept) or when comparing, adding or subtracting polynomials.

The factorised form is more useful for determining roots and cancelling common factors in equations.

4 **a** Yes: the term in x^n is unaffected by adding an order $(n-1)$ polynomial.

b No: if the lead coefficients have a zero sum then the sum of the polynomials will not have a term in x^n, so the resultant will be of lower order. For example:

$\mathrm{f}(x) = 3x^5 - x^2 + 2$ and $\mathrm{g}(x) = -3x^5 + x + 9$

$\mathrm{f}(x) + \mathrm{g}(x) = -x^2 + x + 11$

$\mathrm{f}(x)$ and $\mathrm{g}(x)$ are both order 5 but their sum is an order 2 polynomial.

EXERCISE 4C

Tip

It can be helpful in writing solutions to label a function as $\mathrm{f}(x)$ or $\mathrm{g}(x)$, so that evaluating at particular values of x can be clearly described.

5 **a** $\mathrm{p}(2) = 2^3 - 3 \times 2^2 - 10 \times 2 + 24 = 8 - 12 - 20 + 24 = 0$

By the factor theorem, $x - 2$ is a factor of $\mathrm{p}(x)$.

b Factorising, the cubic $\mathrm{p}(x)$ must be a product of $(x-2)$ and a quadratic.

$$\mathrm{p}(x) = (x-2)(ax^2 + bx + c) = x^3 - 3x^2 - 10x + 24$$
$$= ax^3 + x^2(-2a + b) + x(-2b + c) - 2c$$

Comparing coefficients:

$x^3 : a = 1$

$x^2 : b - 2a = -3$ so $b = 2a - 3 = -1$

$x^1 : c - 2b = -10$ so $c = 2b - 10 = -12$

$x^0 : -2c = 24$ is consistent with $c = -12$.

$\mathrm{p}(x) = (x-2)(x^2 - x - 12) = (x-2)(x-4)(x+3)$

$\mathrm{p}(x)$ has roots $x = 2, x = 4$ and $x = -3$.

6 **a** $\mathrm{p}(x) = x^3 - x^2 - 2x - 12$ and $\mathrm{p}(3) = 3^3 - 3^2 - 2 \times 3 - 12 = 0$

By the factor theorem, if $\mathrm{p}(3) = 0$ then $(x-3)$ is a factor of $\mathrm{p}(x)$.

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b Factorising, the cubic $\mathrm{p}(x)$ must be a product of $(x-3)$ and a quadratic.

$\mathrm{p}(x) = (x-3)(ax^2+bx+c) = x^3 - x^2 - 2x - 12$

$= ax^3 + x^2(-3a+b) + x(-3b+c) - 3c$

Comparing coefficients:

$x^3 : a = 1$

$x^2 : b - 3a = -1 \text{ so } b = 3a - 1 = 2$

$x^1 : c - 3b = -2 \text{ so } c = 3b - 2 = 4$

$x^0 : -3c = -12$ is consistent with $c = 4$.

$\mathrm{p}(x) = (x-3)(x^2+2x+4)$

The discriminant of the quadratic factor is $\Delta = 2^2 - 4 \times 4 < 0$.

Negative discriminant means there are no real roots to the quadratic.

Hence the cubic has only one real root, $x = 3$.

7 $\mathrm{f}(x) = x^3 + 7x^2 + cx + d$

By the factor theorem: $\mathrm{f}(-1) = 0 \Rightarrow 6 - c + d = 0$ and $\mathrm{f}(-2) = 0 \Rightarrow 20 - 2c + d = 0$

Solving simultaneous equations gives $c = 14$, $d = 8$.

8 $\mathrm{f}(x) = x^3 - ax^2 - bx + 168$

a By the factor theorem: $\mathrm{f}(7) = 0 = 343 - 49a - 7b + 168 \text{ so } b = -7a + 73$ (*)

By the factor theorem: $\mathrm{f}(3) = 0 = 27 - 9a - 3b + 168 \text{ so } 9a + 3b = 195$

Substituting (*) : $-12a = -24$ therefore $a = 2, b = 59$.

b $\mathrm{f}(x)$ is a cubic with lead coefficient 1.

The third factor must have the form $(x-k)$ for some value k.

$\mathrm{f}(x) = (x-3)(x-7)(x-k)$

Comparing the constant term: $(-3)(-7)(-k) = 168, \text{so } k = -8$

The remaining factor is $(x+8)$.

9 $\mathrm{f}(x) = x^2 + kx - 8k$

By the factor theorem: $\mathrm{f}(k) = 0 = k^2 + k^2 - 8k$

$2k^2 - 8k = 0$ therefore $k(k-4) = 0$

$\text{so } k = 0$ or $k = 4$.

10 $\mathrm{f}(x) = x^2 - (k+1)x - 3$

By the factor theorem: $\mathrm{f}(k-1) = 0 = (k-1)^2 - (k+1)(k-1) - 3$

$-2k - 1 = 0 \text{ so } k = -\frac{1}{2}$.

11 $\mathrm{f}(x) = 2x^3 - 15x^2 + ax + b$

Tip

There are two sensible approaches here.

You may recognise that the quadratic given factorises readily into $(x-2)(x-3)$ so it would be possible to apply the factor theorem to the cubic and solve $\mathrm{f}(2) = \mathrm{f}(3) = 0$ to find a and b.

Alternatively, propose a final factor $(2x-k)$, chosen to ensure the lead coefficient will be correct, and then expand and compare coefficients. Both methods are given below.

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Method 1

$x^2 - 5x + 6 = (x - 3)(x - 2)$ so both $(x - 2)$ and $(x - 3)$ are factors of $\mathrm{f}(x)$.

By the factor theorem: $\mathrm{f}(2) = 0 = 16 - 60 + 2a + b$

So $b = 44 - 2a$ $(*)$

By the factor theorem: $\mathrm{f}(3) = 0 = 54 - 135 + 3a + b$ so $3a + b = 81$

Substituting $(*)$: $a + 44 = 81$ therefore $a = 37, b = -30$.

Method 2

$\mathrm{f}(x) = (2x - k)(x^2 - 5x + 6) = 2x^3 + (-k - 10)x^2 + (5k + 12)x - 6k$

Comparing coefficients:

$x^3 : 2 = 2$

$x^2 : -15 = -10 - k \Rightarrow k = 5$

$x^1 : \ a = 5k + 12 = 37$

$x^0 : \ b = -6k = -30$

Tip

Although the two methods are of similar difficulty, Method 1 requires that you spot the factors of the quadratic, not needed for Method 2. Method 2 also produces the final factor, which may be useful in a multi-part question.

EXERCISE 4D

4 **a** $\mathrm{f}(2) = 2 \times 2^3 - 5 \times 2^2 + 2 + 2 = 16 - 20 + 2 + 2 = 0$

By the factor theorem, $(x - 2)$ is a factor of $\mathrm{f}(x)$.

b $\mathrm{f}(x)$ is a cubic with lead coefficient 2.

The remaining quadratic factor must have the form $(2x^2 + ax + b)$.

$$\begin{aligned}\mathrm{f}(x) &= (x - 2)(2x^2 + ax + b) \\ &= 2x^3 + x^2(-4 + a) + x(-2a + b) - 2b\end{aligned}$$

Comparing coefficients (or use long division):

$x^3 : 2 = 2$

$x^2 : -5 = -4 + a \Rightarrow a = -1$

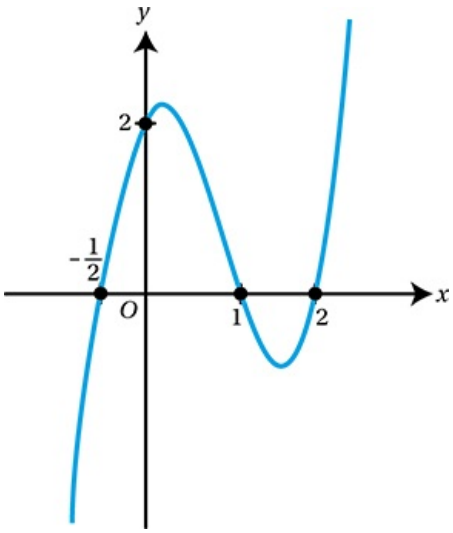
$x^1 : 1 = -2a + b \Rightarrow b = -1$

$x^0 : 2 = -2b$ is consistent with $b = -1$

$\mathrm{f}(x) = (x - 2)(2x^2 - x - 1)$ therefore $\mathrm{f}(x) = (x - 2)(2x + 1)(x - 1)$.

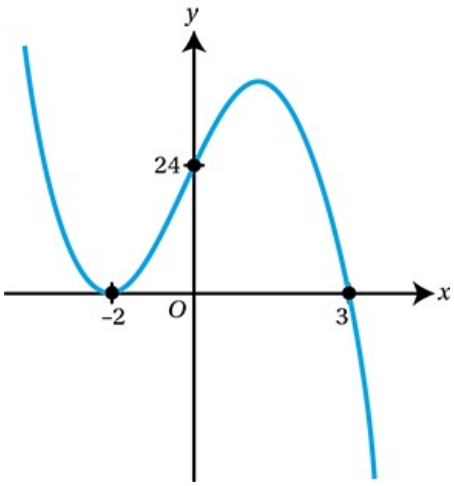
c Roots at $1, 2$ and $-\frac{1}{2}$ and y-intercept at 2.

Positive cubic shape.



5 Repeated root at $x = -2$, single root at $x = 3$, y-intercept at 24.

Negative cubic shape.



6 **a** Repeated root at $x = 3 \Rightarrow$ factor of $(x-3)^2$

Single root at $x = -2 \Rightarrow$ factor of $(x+2)$

$$y = k(x-3)^2(x+2) = k(x^3 - 4x^2 - 3x + 18)$$
$$y(0) = 36 = 18k \Rightarrow k = 2$$
$$y = 2x^3 - 8x^2 - 6x + 36$$
$$p = 2,\ q = -8,\ r = -6,\ s = 36$$

b Repeated root at $x = 0 \Rightarrow$ factor of x^2.

Single root at $x = 3 \Rightarrow$ factor of $(x-3)$.

$$y = kx^2(x-3)$$
$$y(2) = 4 = -4k \Rightarrow k = -1$$
$$y = -x^3 + 3x^2$$
$$p - 1,\ q = 3,\ r = s = 0$$

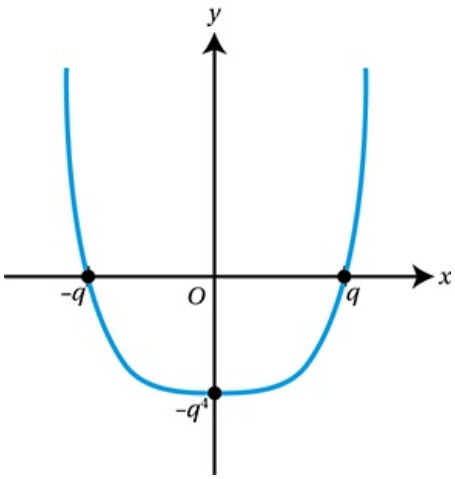
7 **a** $(x^4 - q^4) = (x^2 + q^2)(x^2 - q^2) = \left(x^2 + q^2\right)(x+q)(x-q)$

b Roots at $x = \pm q$ only, y-intercept at $-q^4$.

Positive quartic shape.

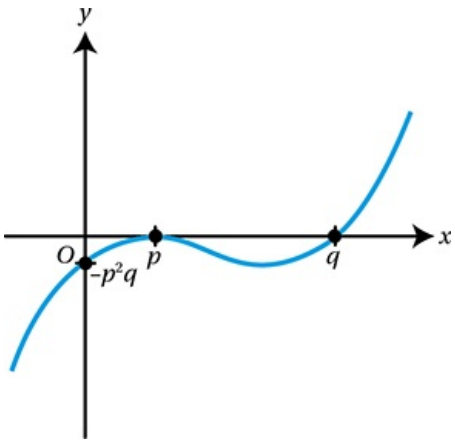
Even function (reflective y symmetry through the y-axis).

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8 **a** Repeated root at $x = p$, single root at $x = q$. y-intercept at $-p^2q$.

Positive cubic shape.



b From the graph, there will be only one solution for $y = k$ where $k > 0$.

MIXED PRACTICE 4

1 Repeated root at $x = -3 \Rightarrow$ factor of $(x+3)^2$.

Single root at $x = 1 \Rightarrow$ factor of $(x-1)$.

Single root at $x = 3 \Rightarrow$ factor of $(x-3)$.

$$y = k(x+3)^2(x-1)(x-3) = k(x^4 + 2x^3 - 12x^2 - 18x + 27)$$

$$y(0) = 27 \Rightarrow k = 1$$

$$y = x^4 + 2x^3 - 12x^2 - 18x + 27$$

$$a = 1, b = 2, c = -12, d = -18, e = 27$$

2 $\dfrac{x^3 + 2x^2 - 3x - 6}{x+2} = x^2 + bx + c$

First, we can confirm that $(x+2)$ is a factor of the cubic in the numerator:

Let $\mathrm{f}(x) = x^3 + 2x^2 - 3x - 6$

Then $\mathrm{f}(-2) = -8 + 8 + 6 - 6 = 0$

By the Factor Theorem, if $\mathrm{f}(-2) = 0$ then $(x+2)$ is a factor of $\mathrm{f}(x)$, so

$x^3 + 2x^2 - 3x - 6 = (x+2)(x^2 + bx + c)$ for some quadratic factor of the given form.

$$\begin{aligned} x^3 + 2x^2 - 3x - 6 &= (x+2)(x^2 + bx + c) \\ &= x^3 + x^2(2+b) + x(2b+c) + 2c \end{aligned}$$

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Comparing coefficients:

$x^2 : 2 = 2 + b$ so $b = 0$

$x^1 : -3 = 2b + c$ so $c = -3$

$x^0 : -6 = 2c$ is consistent with $c = -3$

3 **a** $\mathrm{f}(2) = 8 - 16 + 2 + 6 = 0$ so by the factor theorem, $(x - 2)$ is a factor of $\mathrm{f}(x)$.

b $\mathrm{f}(x) = (x - 2)(x^2 + ax + b) = x^3 + (a - 2)x^2 + (b - 2a)x - 2b$

Comparing coefficients:

$x^2 : a - 2 = -4 \Rightarrow a = -2$

$x^1 : b - 2a = 1 \Rightarrow b = -3$

$x^0 : -2b = 6$ is consistent with the value found.

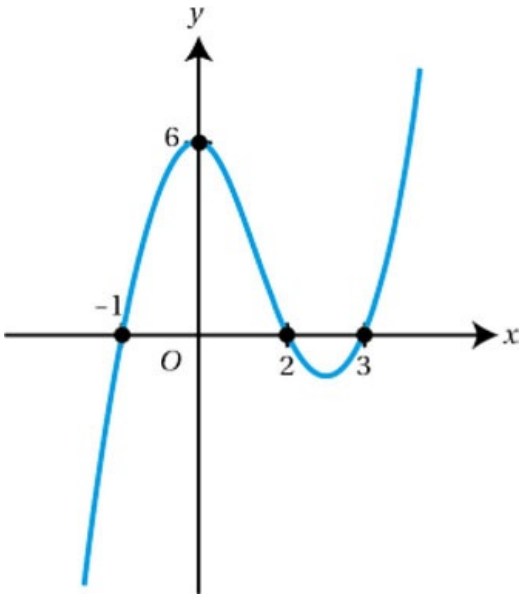
Tip

The final coefficient comparison is useful for checking the validity of the solution. Always be thorough and compare all coefficients, even if you do not write down the check as part of your answer.

$\mathrm{f}(x) = (x - 2)(x^2 - 2x - 3) = (x - 2)(x - 3)(x + 1)$

c Roots at $x = -1, x = 2, x = 3$; y-intercept at 6.

Positive cubic shape.



4 **a** By the factor theorem, if $\mathrm{f}(x)$ and $\mathrm{g}(x)$ each have a factor of $(x - 2)$ then

$\mathrm{f}(2) = \mathrm{g}(2) = 0.$

$\mathrm{f}(2) = 8 + 2(a - 3) + 2b = 2a + 2b + 2 = 0 \qquad (1)$

$\mathrm{g}(2) = 24 + 4 + 10a + 4b = 10a + 4b + 28 = 0 \qquad (2)$

$(2) - 2(1) : 6a + 24 = 0$

$a = -4$ and so $b = 3$

b So $\mathrm{f}(x) = (x - 2)(x^2 + kx + m)$ for some integers m and n.

Expanding and comparing coefficients:

$x^3 : 1 = 1$

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$x^2 : k - 2 = 0 \text{ so } k = 2$

$x^1 : m - 2k = a - 3 = -7 \text{ so } m = -3$

$x^0 : -2m = 2b = 6$ is consistent with the above.

So $\mathrm{f}(x) = (x-2)(x^2+2x-3) = (x-2)(x+3)(x-1)$

$\mathrm{g}(x) = 3x^3 + x^2 - 20x + 12 \text{ so } \mathrm{g}(-3) = -81 + 9 + 60 + 12 = 0$

Therefore, by the factor theorem, $(x+3)$ is another factor of $\mathrm{g}(x)$ which is also a factor of $\mathrm{f}(x)$.

5 **a** Using the factor theorem: the polynomial evaluated at $x = \frac{1}{2}$ and $x = -2$ should equal 0.

$2\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right) + b = 0 \Rightarrow \frac{1}{4}a + b = -\frac{9}{4}$

$2(-2)^3 + a(-2)^2 + 4(-2) + b = 0 \Rightarrow 4a + b = 24$

Solving these simultaneous equations gives $a = 7$, $b = -4$

b You need all three factors in order to sketch the graph:

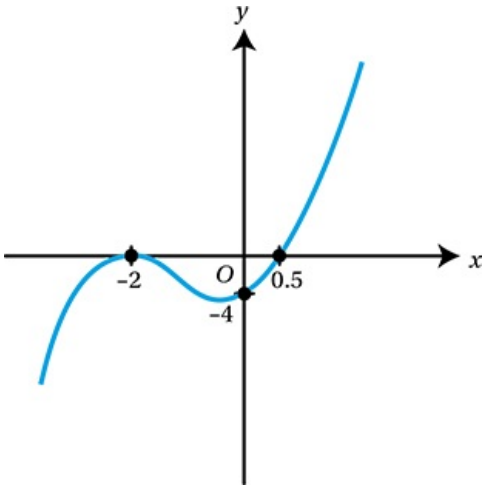
$2x^3 + 7x^2 + 4x - 4 = (2x-1)(x+2)(x-k)$

You can find k by looking at the constant term: $(-1)(2)(-k) = -4$, so $k = -2$.

Hence $2x^3 + 7x^2 + 4x - 4 = (2x-1)(x+2)(x+2)$

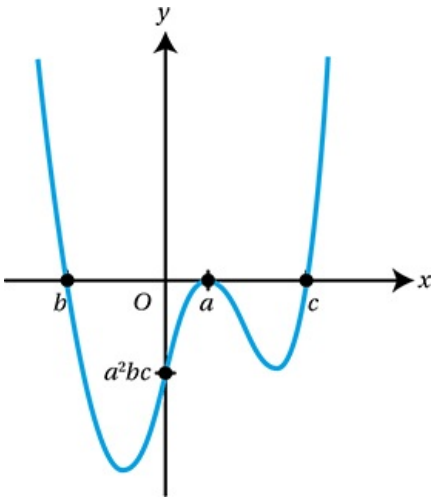
The graph is a positive cubic. There is a repeated root at -2, so the graph is tangent to the x-axis at $x = -2$. It also crosses the x-axis at $x = \frac{1}{2}$ (when $2x - 1 = 0$).

The y-intercept is -4.



6 Repeated root at $x = a$, single roots at $x = b$ and $x = c$, y-intercept at $a^2bc < 0$

Positive quartic shape.



7 a

Tip

You are given no clue as to what an easy-to-find root of $f(x)$ might be. If you have a calculator which can solve or graph the equation, you can calculate a simple root and then use the factor theorem to validate the factor in your written answer. Otherwise, test simple values such as $\pm 1, \pm 2$ or ± 0.5 until you find a root. You can be confident that there is a simple root available.

$f(2) = 8 + 4 - 22 + 10 = 0$

By the factor theorem, since $f(2) = 0$, it follows that $(x - 2)$ is a factor of $f(x)$.

b So $f(x) = (x - 2)(ax^2 + bx + c)$ for some integers a, b and c.

Expanding and comparing coefficients:

$x^3 : a = 1$

$x^2 : b - 2a = 1 \text{ so } b = 3$

$x^1 : c - 2b = -11 \text{ so } c = -5$

$x^0 : -2c = 10$ is consistent with the above.

So $f(x) = (x - 2)(x^2 + 3x - 5)$.

So the roots of $f(x)$ are 2 and $\dfrac{-3 \pm \sqrt{3^2 + 4 \times 5}}{2} = \dfrac{-3 \pm \sqrt{29}}{2}$.

8

Tip

You saw from question 11 in Exercise 4C an alternative method using the factor theorem.

$f(x)$ is a cubic with lead coefficient 1.

The other factor must be linear of the form $(x + k)$.

$f(x) = (x + k)\left(x^2 - 4x + 3\right) = x^3 + x^2(k - 4) + x(3 - 4k) + 3k$

$= x^3 + ax^2 + 27x + b$

Comparing coefficients:

$x^3 : 1 = 1$

$x^2 : a = k - 4$

$x^1 : 27 = 3 - 4k \Rightarrow k = -6$

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$x^0 : b = 3k = -18$

So $a = -10, b = -18$.

Chapter 5 worked solutions

5 Using graphs

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 5A

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3 $y = \dfrac{1}{x} = 2x$

$2x^2 = 1$ therefore $x = \pm\dfrac{1}{\sqrt{2}}$

Points of intersection are $\pm\left(\dfrac{\sqrt{2}}{2}, \sqrt{2}\right)$.

4 $\begin{cases} 3^x + 2^x = 13 \ (1) \\ 3^x - 2^x = \ 5 \ \ (2) \end{cases}$

$(1) + (2) : 2 \times 3^x = 18$

$3^x = 9$ therefore $x = 2$

Tip

Strictly, you should check the solution. The problem provides two equations for only one unknown, so there may be no valid solution; with exactly the same method you would get the same answer $x = 2$ if the equations had had 4^x in each case instead of 3^x, but $x = 2$ would not have been a true solution in that case.

Check:

$(1) : 3^2 + 2^2 = 13$ is true.

$(2) : 3^2 - 2^2 = 5$ is true.

$x = 2$ is the solution.

5 $y = 2^x$ so $y^2 = 4^x$.

Substituting the first equation into the second:

$y^2 + y = 72$ so $(y + 9)(y - 8) = 0$ therefore $y = -9$ or $y = 8$.

But $y = 2^x$ and there is no real value for x which gives $2^x < 0$ so $y = 2^x = 8,\ x = 3$.

6 **a** The two numbers must sum to 8 so can be generalised as x and $8 - x$.

$x(8 - x) = 9.75$ therefore $x^2 - 8x + 9.75 = 0$

b $(x - 1.5)(x - 6.5) = 0$ so $x = 1.5$ or 6.5

7 $\begin{cases} x(1 + y) = 0 \ (1) \\ x^2 + y^2 = 4 \ (2) \end{cases}$

From $(1) : x = 0$ or $y = -1$

From (2): if $x = 0,\ y = \pm 2$. If $y = -1,\ x = \pm\sqrt{3}$.

8 Graph of $y = (x - 2)(x - 3)^2$ crosses the x-axis at $(2, 0)$ and then touches it again at $(3, 0)$.

For intersection with $y = k$, any value of k less than zero will have only one intersection.

$m = 0$

EXERCISE 5B

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1 Line: $y = x - 6$

Substitute into curve equation:

$x^2 - 6x + (x-6)^2 - 2(x-6) + 2 = 0$ so $x^2 - 6x + x^2 - 12x + 36 - 2x + 12 + 2 = 0$

$2x^2 - 20x + 50 = 0$ therefore $x^2 - 10x + 25 = 0$ thus $(x-5)^2 = 0$.

There is a single repeated solution to the intersection problem, so the line is tangent to the curve (intersects at only one point).

2 Substitute $y = mx + 3$ into the quadratic: $mx + 3 = 3x^2 - x + 5$

$3x^2 - (m+1)x + 2 = 0$

For the line to be a tangent there must be a single repeated root.

Discriminant $\Delta = (m+1)^2 - 4 \times 3 \times 2 = 0$

$m + 1 = \sqrt{24} = \pm 2\sqrt{6}$ therefore $m = -1 \pm 2\sqrt{6}$.

3 Substitute $3y = k - 2x$ so that $9y^2 = (k-2x)^2$.

$4x^2 + (k-2x)^2 = 36$

Therefore $4x^2 + 4x^2 - 4kx + k^2 - 36 = 0$

$8x^2 - 4kx + k^2 - 36 = 0$

For the line to be a tangent there must be a single repeated root.

Discriminant $\Delta = (-4k)^2 - 4 \times 8 \times (k^2 - 36) = 0$

$16k^2 - 32k^2 = -32 \times 36$ therefore $-16k^2 = -16 \times 2 \times 36$

$k^2 = 72$ so $k = \pm 6\sqrt{2}$.

Tip

Notice that in many questions where you might choose to use a calculator to work out a high value product, keeping the component factors and then cancelling will be just as effective, and may also be faster. In this instance, you don't actually have to multiply 32×36 as you can see that there will be a factor of 16 to cancel very soon.

4 Solving for the intersections of $y = x^2$ and $y = a - (x-a)^2$ require that there be no solutions. Substituting: $x^2 = a - (x-a)^2 = a - x^2 + 2ax - a^2$

$2x^2 - 2ax + a^2 - a = 0$

For no real solutions, require discriminant $\Delta < 0$.

$\Delta = (-2a)^2 - 4 \times 2 \times (a^2 - a) < 0$ so $4a^2 - 8a^2 + 8a < 0$

$8a - 4a^2 < 0$ therefore $4a(2-a) < 0$

Negative quadratic is less than zero outside the roots so $a < 0$ or $a > 2$.

5 Substituting $y = kx + 5$ into the parabola equation: $kx + 5 = x^2 + 2$

$x^2 - kx - 3 = 0$

Discriminant $\Delta = (-k)^2 + 12 = k^2 + 12$, therefore discriminant will be positive for all values of.

Positive discriminant means there will be two roots, and therefore two intersections of the line and the parabola for all values of k.

EXERCISE 5D

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1 **a** Replace x with $(x-d)$: translation $\begin{pmatrix} d \\ 0 \end{pmatrix}$.

b Asymptotes of $\frac{a}{x}$ are $x=0$ and $y=0$. After translation, asymptotes are $x=d$ and $y=0$.

2 Original curve: $y=\frac{1}{x^2}$. Target curve $y=\frac{a}{x^2}=\left(\frac{\sqrt{a}}{x}\right)^2=\frac{1}{\left(x/\sqrt{(a)}\right)^2}$

Replace x with $\frac{x}{\sqrt{a}}$: horizontal stretch with scale factor $\sqrt{a}$

3 **a** Intersection of $y=\frac{a}{x}$ and $y=\frac{b}{x^2}$: $\frac{a}{x}=\frac{b}{x^2}$ therefore $x=\frac{b}{a}$.

$y=\frac{a}{x}=\frac{a^2}{b}$ so P has coordinates : $\left(\frac{b}{a},\frac{a^2}{b}\right)$.

b Height of rectangle: $\frac{a^2}{b}$, length of rectangle: $\frac{b}{a}$.

Area of rectangle: $\frac{a^2}{b}\times\frac{b}{a}=a$.

4 Find intersections of $y=mx+c$ and $y=\frac{a}{x}$.

Tip

This question can be solved easily using calculus, but is also approachable using more basic techniques and knowledge of quadratics.

$\frac{a}{x}=mx+c$ therefore $mx^2+cx-a=0$.

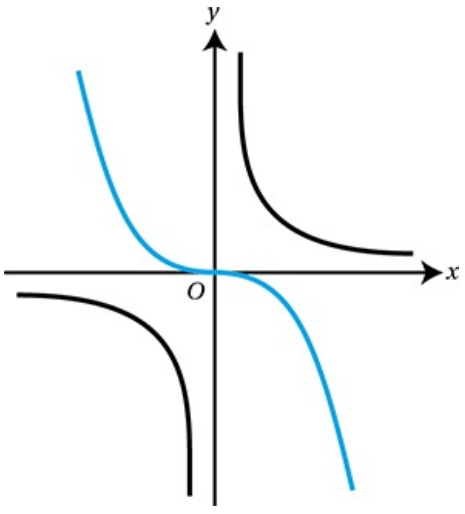
Quadratic equation: for the line to be tangent to the curve, there must be a single root to this quadratic so the discriminant $\Delta=0$.

$\Delta=c^2-4m\times(-a)=0$

$4am+c^2=0$ so $m=-\frac{c^2}{4a}$

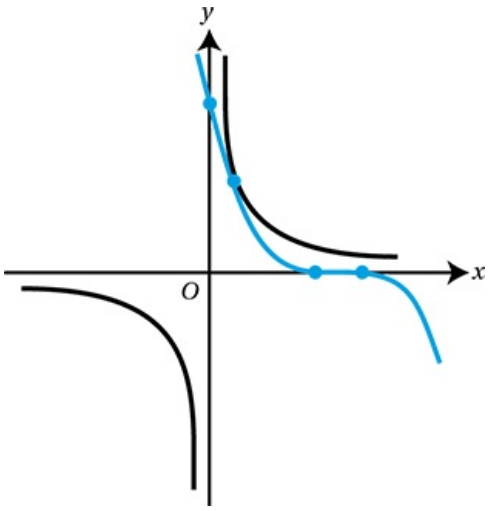
5 You do not need the exact equations of the cubics involved, simply the qualities of the cubic shape, positioned and stretched so as to intersect with the reciprocal curve the required number of times.

Starting with no intersections: the curve $y=-x^3$ exists only in the upper left and lower right quadrants so cannot intersect the reciprocal $y=\frac{1}{x}$.

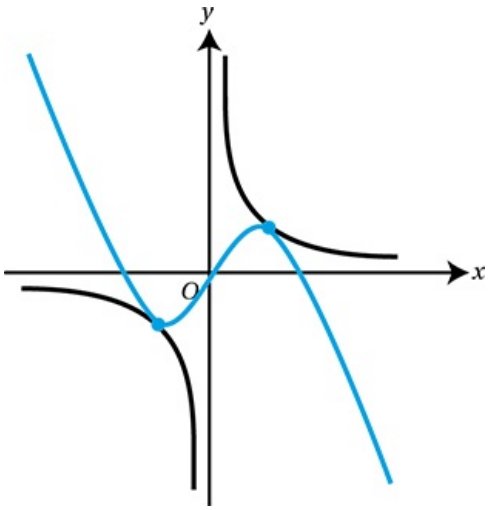


If you translate the cubic to the right exactly the correct amount it will be tangent to the reciprocal, and so intersect at exactly one point.

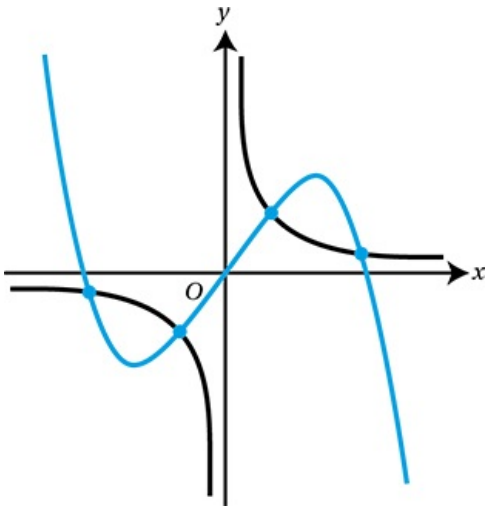
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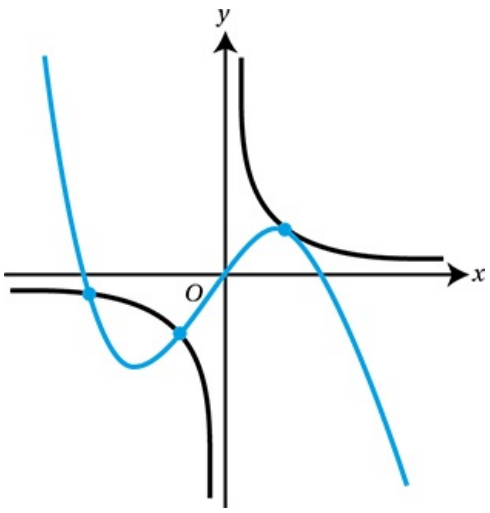
Switching to a cubic with two turning points, you can devise a symmetrical curve which touches the reciprocal graph in two points.



Stretching this cubic vertically will result in four intersections.



Translating this stretched cubic vertically until it is tangent to one branch of the reciprocal will result in three intersections.



EXERCISE 5E

5 **a** Line D connects $(1, 312)$ to $(40, 0)$.

The gradient is $\dfrac{0-312}{40-1} = -8$ so the equation is $y - 0 = -8(p - 40) \Leftrightarrow y = 320 - 8p$.

Line S connects $(10, 0)$ to $(40, 240)$.

The gradient is $\dfrac{240-0}{40-10} = 8$ so the equation is $y - 0 = 8(p - 10) \Leftrightarrow y = 8p - 80$.

b 320 is the y-intercept, so it corresponds to the point where $p = 0$. It is the number of items that would be sold if they were free. However, it is likely that more would be 'sold' if they were given away for free, so the straight line trend may not continue for very small values of p.

c The number of items is a whole number, so the minimum possible demand is when $y_D = 1$.

$1 = 320 - 8p$ and this is when $p = 39.87$

d When $y_D = y_S$: $320 - 8p = 8p - 80$, so $p = 25$

6 **a** $C_{\text{A}} = 65 + 0.03m,\ C_{\text{B}} = 0.05m$

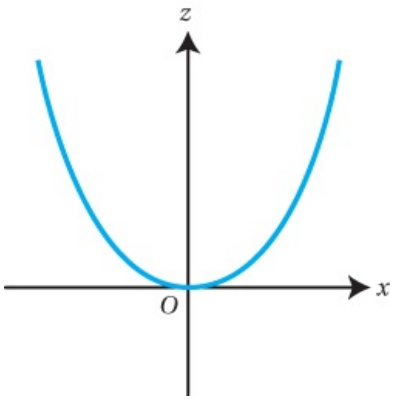
b Require $C_{\text{A}} < C_{\text{B}}$

$65 + 0.03m < 0.05m$ thus $0.02m > 65$ so $m > 3250$

After 3250 minutes, contract A becomes cheaper.

7 $y = \dfrac{k}{x^2},\ z = \dfrac{q}{y} = \dfrac{qx^2}{k} = cx^2$ where k, q and c are all constants of proportionality.

$z = cx^2$



8 Let G be the strength of the gravitational field and x be the distance from the centre of the Earth.

$G = \dfrac{k}{x^2}$

Putting the satellite into orbit: $x \to 1.1x$

$G(1.1x) = \dfrac{k}{(1.1x)^2} = \dfrac{1}{1.21} \times \dfrac{k}{x^2} = 0.826 \times \dfrac{k}{x^2}$

The gravitational field strength decreases by 17.4%.

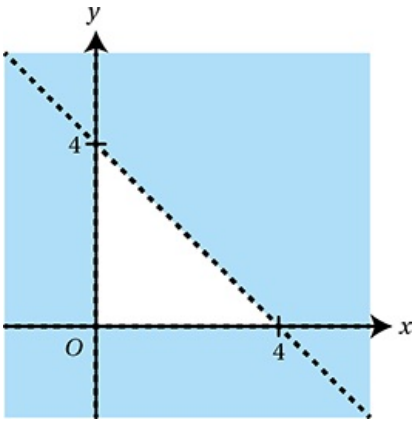
EXERCISE 5F

2 Boundary lines $x = 0,\ y = 0$ and $y = 4 - x$, all dashed.

Shade in negative x and negative y.

Test the origin and find that $y < 4 - x$ at the origin, so shade in the other side of the oblique line.

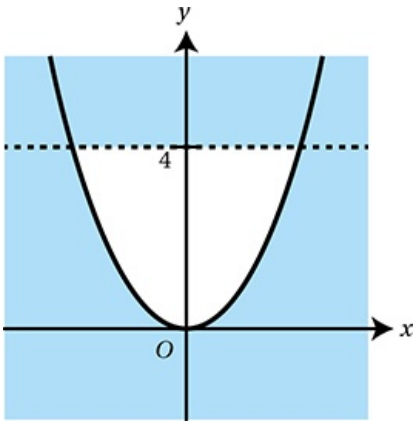
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3 Boundary curves $y = x^2$ with a full line and $y = 4$ with a dashed line.

Test point $(0, 1)$: $y > x^2$ so shade the outer area around the parabola.

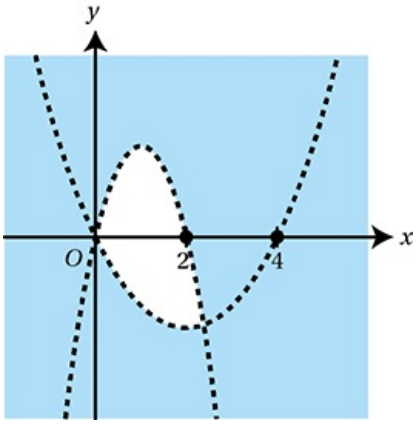
Shade in the area above $y = 4$.



4 Boundary curves $y = x(x - 4)$ and $y = x\,(2 - x)$, both dashed.

Test point $(1, 0)$: $y > x^2 - 4x$ so shade in the region outside the positive parabola.

$y < 2x - x^2$ so shade in the region outside the negative parabola.



5 Full boundary line is $y = 4 - x$.

Dotted boundary line is $y = x$.

Require the side for which $(0, 2)$ is valid.

Substituting $x = 0, y = 2$ into boundary line equations to establish direction of inequalities:

$2 < 4 - 0$ so inequality is $y \leqslant 4 - x$.

$2 > 0$ so inequality is $y > x$.

The region is defined by $y > x$.

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6 Parabola has roots at $x = 1$ and $x = 2$, so has the form $y = a(x - 1)(x - 2)$.

Its y-intercept is at 2 so $a = 1$.

Horizontal dotted line is $y = 2$.

Require the side of each inequality for which $(1, 1)$ is valid.

Substituting $x = 1, y = 1$ into boundary line equations to establish direction of inequalities:

$1 > (1 - 1)(1 - 2)$ so inequality is $y > (x - 1)(x - 2)$.

$1 < 2$ so inequality is $y < 2$.

The region is defined by $y > (x - 1)(x - 2), y < 2$.

7 Writing the inequalities as a sandwich inequality allows elimination of y:

$11x < y < 120x - 2x^2$

$109x - 2x^2 > 0$ therefore $x(109 - 2x) > 0$

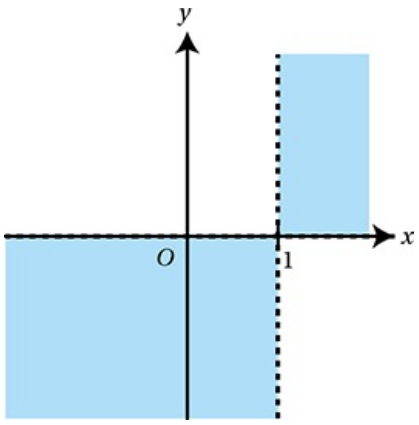
Negative quadratic is positive between the roots: $0 < x < 54.5$.

Highest integer value of x is 54.

8 $y > xy \Rightarrow y(1 - x) > 0$

For a product of two expressions to be positive, either both must be positive or both must be negative.

Solution: Either $y > 0$ and $x < 1$ or $y < 0$ and $x > 1$.



MIXED PRACTICE 5

1 Substituting $y = 7 - x$ into the circle equation:

$x^2 + (7 - x)^2 - 25 = 0$ so $2x^2 - 14x + 24 = 0$

$x^2 - 7x + 12 = 0$ therefore $(x - 3)(x - 4) = 0$

$(x, y) = (3, 4)$ or $(4, 3)$

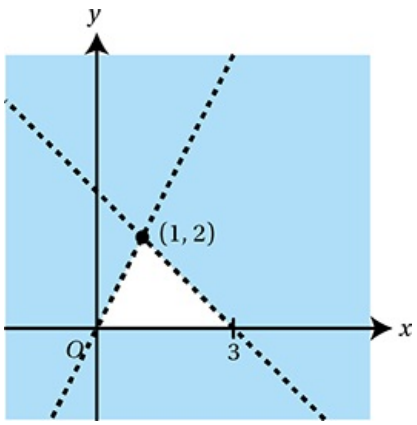
2 **a** Boundary lines $y = 3 - x$ and $y = 2x$ dashed, $y = 0$ solid.

Test point $(0, 1)$: $y < 3 - x$ so shade in the region above $y = 3 - x$.

$y > 0$ so shade in the region below $y = 0$.

$y < 2x$ so shade in the region on the side of $y = 2x$ which includes the test point.

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b From the graph, the intersection of the two oblique inequalities is at $(1, 2)$ so the upper bound on the y values is 2.

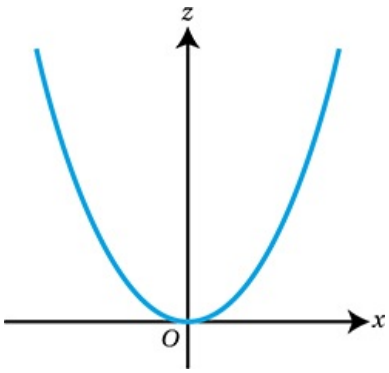
Tip

Note that an upper bound is not necessarily the maximum value. There is no maximum value for y, because $y < 2$ has no maximum value.

3 $y = (x-1)^2$ to $y = (x+2)^2$ Replace x with $x + 3$.

Translation $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

4 $z = kx^2$



5 For x kilometres:

$C_A = 1.6x$ and $C_B = 1.5 + 1.2x$

Require $C_A = C_B$:

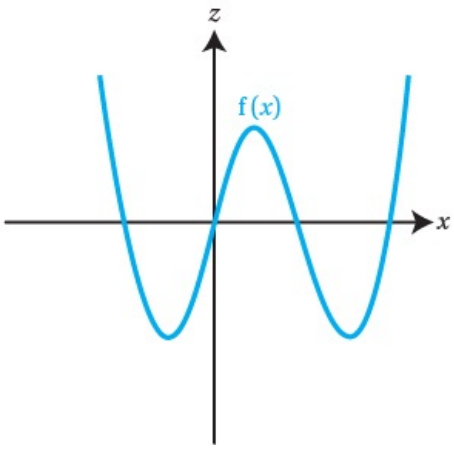
$0.4x = 1.5$ therefore $x = 3.75$

A journey of 3.75 km would cost the same from each company.

6 **a** $f(x) \to f(x-1) + 2$

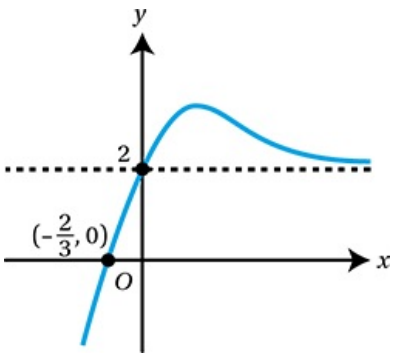
Replacing x with $x - 1$ and adding 2 outside the function: translation $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

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b Maximum of $\mathrm{f}(x)$ is at $(0, 0)$. After translation, the maximum will be at $(1, 2)$.

7 The graph is compressed by a factor of 3 in the horizontal direction. The point where the graph crosses the y-axis doesn't move, and the horizontal asymptote is still the same.



8 **a** For this translation, replace x by $x - 2$. So the equation is $y = (x - 2)^2$.

b This reflection changes y to $-y$, so the equation is: $-y = x^3 - 4 \Leftrightarrow\ y = 4 - x^3$.

9 **a** The straight line connects the points $(2, 4.1)$ and $(5, 4.8)$.

The gradient is $\dfrac{4.8 - 4.1}{5 - 2} = 0.233\ (3\ \text{s. f.}\,)$,

so the equation is $M - 4.1 = 0.233(n - 2) \Leftrightarrow M = 0.233n + 3.63$.

b $M = 3.63$ corresponds to $n = 0$, so it represents the weight at birth. 0.233 is the weight gain every week.

c There are 52 weeks in a year, so the mode predicts $M = 0.233(52) + 3.63 = 15.7$. Hence the model is not appropriate.

10 **a** Substitute y from the first equation into the second:

$10x + 2(2x^2 - 3x - 5) + 11 = 0\,\text{so}\,4x^2 + 4x + 1 = 0$

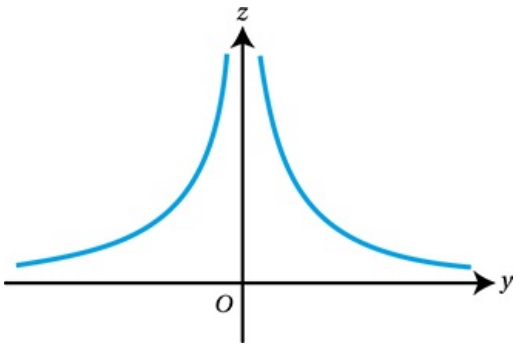
$(2x + 1)^2 = 0$ thus $x = -\dfrac{1}{2}, y = -3$ (substituting x into the first equation).

b There is only one solution, so the line is tangent to the curve.

11 x is inversely proportional to $y : x = \dfrac{k}{y}, z$ is proportional to $x^2 : z = nx^2$.

So $z = n\left(\dfrac{k}{y}\right)^2 = \dfrac{a}{y^2}$ (for some constant a).

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12 a Let $u = x^2$.

Then $u^2 - 13u + 36 = 0$ therefore $(u - 4)(u - 9) = 0$.

$(x^2 - 4)(x^2 - 9) = 0$ so $(x + 2)(x + 2)(x + 3)(x + 3) = 0\, x = \pm 2, \pm 3$.

b $x^4 - 13x^2 + 36 \leqslant 0$

Positive quartic with four roots must be negative between the first and second roots and again between the third and fourth roots.

$-3 \leqslant x \leqslant -2$ or $2 \leqslant x \leqslant 3$.

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Chapter 6 worked solutions

6 Coordinate geometry

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 6A

3 $d = \sqrt{(-2a - a)^2 + (8a - 2a)^2} = \sqrt{(-3a)^2 + (6a)^2}$

$= \sqrt{9a^2 + 36a^2} = \sqrt{45a^2} = 3\sqrt{5}a$

4 Let the coordinates of Q be (x, y).

$(1, 1) = \left(\frac{a + x}{2}, \frac{b + y}{2}\right)$

So $1 = \frac{a + x}{2} \Rightarrow x = 2 - a$ and $1 = \frac{b + y}{2} \Rightarrow y = 2 - b$

i.e. the coordinates of Q are $(2 - a, 2 - b)$

5 **a** $AB = \sqrt{(4 - 0)^2 + (4 - 1)^2} = \sqrt{16 + 9} = 5$

$BC = \sqrt{(7 - 4)^2 + (8 - 4)^2} = \sqrt{9 + 16} = 5$

$\therefore AB = BC$

b Gradient of line AC:

$m_{AC} = \frac{8 - 1}{7 - 0} = 1$

Gradient of line AB:

$m_{AB} = \frac{4 - 1}{4 - 0} = \frac{3}{4}$

Therefore, B does not lie on the line AC.

6 $3^2 = (3 - a)^2 + (1 - 2a)^2$

$9 = 9 - 6a + a^2 + 1 - 4a + 4a^2 = 5a^2 - 10a + 10$

$5a^2 - 10a + 1 = 0$

$a = \frac{10 \pm \sqrt{10^2 - 4 \times 5 \times 1}}{2 \times 5} = 1 \pm \frac{2}{\sqrt{5}}$

7 $y^2 = (x - 0)^2 + (y - 1)^2 = x^2 + y^2 - 2y + 1$

$2y = x^2 + 1$ so $y = \frac{x^2 + 1}{2}$

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8 $$\begin{aligned} AM^2 &= \left(\left(\frac{x_1+x_2}{2}\right)-x_1\right)^2+\left(\left(\frac{y_1+y_2}{2}\right)-y_1\right)^2 \\ &= \left(\frac{x_1+x_2}{2}\right)^2-2\left(\frac{x_1+x_2}{2}\right)x_1+x_1^2 \\ &\quad +\left(\frac{y_1+y_2}{2}\right)^2-2\left(\frac{y_1+y_2}{2}\right)y_1+y_1^2 \\ &= \frac{x_1^2+x_2^2+2x_1x_2}{4}-x_1^2-x_1x_2+x_1^2 \\ &\quad +\frac{y_1^2+y_2^2+2y_1y_2}{4}-y_1^2-y_1y_2+y_1^2 \\ &= \frac{\begin{array}{l}x_1^2+x_2^2+2x_1x_2-4x_1^2-4x_1x_2+4x_1^2 \\ +y_1^2+y_2^2+2y_1y_2-4y_1^2-4y_1y_2+4y_1^2\end{array}}{4} \\ &= \frac{x_1^2-2x_1x_2+x_2^2+y_1^2-2y_1y_2+y_2^2}{4} \\ &= \frac{(x_2-x_1)^2+(y_2-y_1)^2}{4} \\ &= \frac{1}{4}AB^2 \end{aligned}$$

$$\therefore AM = \frac{1}{2}AB$$

9 a i $$\begin{aligned} AB &= \sqrt{(4a-(-2))^2+(2a-4)^2} = \sqrt{(4a+2)^2+(2a-4)^2} \\ &= \sqrt{16a^2+16a+4+4a^2-16a+16} \\ &= \sqrt{20a^2+20} \\ &= \sqrt{20}\sqrt{a^2+1} \\ &= 2\sqrt{5}\sqrt{a^2+1} \end{aligned}$$

ii $$M = \left(\frac{-2+4a}{2}, \frac{4+2a}{2}\right) = (2a-1,\ a+2)$$

b $$\begin{aligned} OM &= \sqrt{(2a-1-0)^2+(a+2-0)^2} \\ &= \sqrt{4a^2-4a+1+a^2+4a+4} \\ &= \sqrt{5a^2+5} \\ &= \sqrt{5}\sqrt{a^2+1} \end{aligned}$$

$$\begin{aligned} AB : OM &= 2\sqrt{5}\sqrt{a^2+1} : \sqrt{5}\sqrt{a^2+1} \\ &= 2:1 \end{aligned}$$

which is independent of a.

EXERCISE 6B

5 a Gradient $m = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{3+1}{-4-5} = -\dfrac{4}{9}$, line equation $y - y_1 = m(x - x_1)$.

$y+1 = -\dfrac{4}{9}(x-5)$ therefore $9y+9 = -4x+20$

$4x+9y-11 = 0$

b $m = -\dfrac{4}{9}$, line equation $y - y_1 = m(x - x_1)$

$y-2 = -\dfrac{4}{9}(x+4)$ therefore $9y-18 = -4x-16$

$4x+9y-2 = 0$

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Tip

Alternatively, recognise that the equation must still have the form $4x + 9y + c = 0$ and substitute the values $x = -4$, $y = 2$ to find c.

c Substituting $(3k, k)$:

$12k + 9k - 2 = 0$ so $21k = 2$ therefore $k = \frac{2}{21}$.

6 Line equation $y - y_1 = m(x - x_1)$

$y - 5 = -\frac{2}{3}(x - 2)$ therefore $3y - 15 = -2x + 4$ so $2x + 3y = 19$

When $x = 0$, $y = \frac{19}{3}$.

When $y = 0$, $x = \frac{19}{2}$.

$P = \left(0, \frac{19}{3}\right)$, $Q = \left(\frac{19}{2}, 0\right)$

OPQ is right angled, so:

$\text{area} = \frac{1}{2} \times OP \times OQ = \frac{1}{2} \times \frac{19}{2} \times \frac{19}{3} = \frac{361}{12}$

7 **a** For l_1:

$\text{the gradient } m = \frac{\text{change in } y}{\text{change in } x} = \frac{12 + 1}{10 + 3} = 1$

The line equation is $y - y_1 = m(x - x_1)$.

$y + 1 = 1(x + 3)$

$x - y = -2$

For l_2:

the line equation is $y - y_1 = m(x - x_1)$.

$y - 12 = -\frac{3}{2}(x + 16)$

$3x + 2y = -24$

b Substituting $l_1 (y = x + 2)$ into l_2:

$$\begin{aligned} 3x + 2(x + 2) &= -24 \\ 5x &= -28 \\ x &= -5.6,\ y = -3.6 \end{aligned}$$

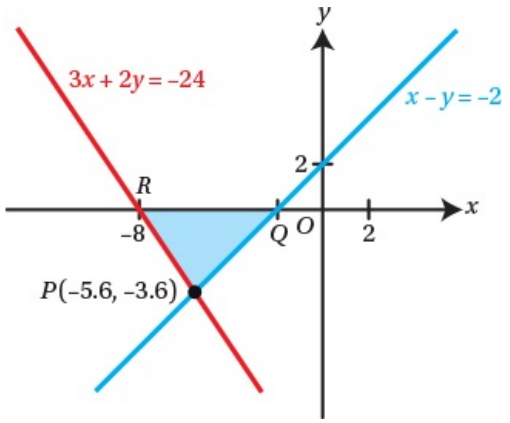
The intersection is at $(-5.6, -3.6)$.

c In l_1: when $y = 0$, $x = -2$ so Q has coordinates $(-2, 0)$.

In l_2: when $y = 0$, $x = -8$ so R has coordinates $(-8, 0)$.

Area of $PQR = 1/2 \times \text{base} \times \text{height} = {}^1\!/_2 \times 6 \times 3.6 = 10.8$

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8 **a** The gradient is $\frac{2}{3} = \frac{\text{change in } y}{\text{change in } x} = \frac{k+1}{3k-3}$

so $2(k-1) = k+1$

$k = 3$

b B has coordinates $(2, 7)$.

The line equation is $y - y_1 = m(x - x_1)$.

$y - 7 = \frac{2}{3}(x-2)$

$2x - 3y + 17 = 0.$

9 $4y - 7x = -14$

When $x = 0$, $y = -\frac{14}{4} = -\frac{7}{2}$.

When $y = 0$, $x = 2$.

$A = (2, 0)$, $B = \left(0, -\frac{7}{2}\right)$, $M = \left(1, -\frac{7}{4}\right)$

Distance $OM = \sqrt{1^2 + \left(-\frac{7}{4}\right)^2} = \sqrt{1 + \frac{49}{16}} = \frac{\sqrt{65}}{4}$

10 Line equation $y - y_1 = m(x - x_1)$ therefore $y - \frac{2}{p} = -\frac{6}{p^2}\left(x - \frac{p}{3}\right)$

$p^2y - 2p = -6x + 2p$ so $p^2y + 6x = 4p$

When $x = 0$, $y = \frac{4}{p}$.

When $y = 0$, $x = \frac{4p}{6} = \frac{2p}{3}$.

$A = \left(0, \frac{4}{p}\right)$, $B = \left(\frac{2p}{3}, 0\right)$

AOB is a right-angled triangle, so area $= \frac{1}{2} \times OA \times OB = \frac{1}{2} \times \frac{4}{p} \times \frac{2p}{3} = \frac{4}{3}$

This area is constant, and therefore independent of p.

EXERCISE 6C

3 **a** Substituting $x = 1$, $y = 4$ into the equation:

$5 \times 1 - 2 \times 4 + 3 = 5 - 8 + 3 = 0$

P satisfies the line equation so does lie on the line.

b $l_1 : y = \frac{5}{2}x + \frac{3}{2}$

IRANINOVIN

Gradient of l_1 is $\frac{5}{2}$ therefore gradient of l_2 is $-\frac{2}{5}$ (negative reciprocal).

Line equation $y - y_1 = m(x - x_1)$ so $y - 4 = -\frac{2}{5}(x - 1)$.

$5y - 20 = -2x + 2$ therefore l_2 has equation $2x + 5y - 22 = 0$

4 $OA^2 = (-2)^2 + 3^2 = 13$, $OB^2 = 1^2 + 5^2 = 26$ and $AB^2 = (-2 - 1)^2 + (3 - 5)^2 = 13$

Since $OB^2 = AB^2 + OA^2$, by Pythagoras' Theorem, $OAB = 90^\circ$.

$\text{Area} = \frac{1}{2} \times OA \times AB = \frac{13}{2}$

5 **a** Midpoint $M = \left(\frac{5-1}{2}, \frac{2+7}{2}\right) = \left(2, \frac{9}{2}\right)$.

b Gradient of AB is $m = \frac{\text{change in } y}{\text{change in } x} = \frac{2-7}{5+1} = -\frac{5}{6}$

Perpendicular gradient $-\frac{1}{m} = \frac{6}{5}$.

Line equation $y - y_1 = m(x - x_1)$.

$y - \frac{9}{2} = \frac{6}{5}(x - 2)$ so $y = \frac{6}{5}x + \frac{9}{2} - \frac{12}{5}$ therefore $y = \frac{6}{5}x + \frac{21}{10}$

6 **a** Equation must have the form $2x + 3y = c$.

Substituting $x = -4$, $y = 1$: $-8 + 3 = c = -5$.

Line has equation $2x + 3y = -5$.

b For the first line, when $y = 0$, $x = 3$: so $P = (3, 0)$

For the second line, when $y = 0$, $x = -\frac{5}{2}$, so $Q = \left(-\frac{5}{2}, 0\right)$.

$PQ = \frac{11}{2}$

7 Let Q have coordinates $(q, 0)$.

$PR^2 = (12 - 0)^2 + (4 - 7)^2 = 153$

$PQ^2 = (q - 0)^2 + (0 - 7)^2 = 49 + q^2$

$RQ^2 = (12 - q)^2 + (4 - 0)^2 = 144 - 24q + q^2 + 16 = 160 + q^2 - 24q$

For right-angle at Q, by Pythagoras' theorem, $PR^2 = PQ^2 + RQ^2$.

$153 = 49 + q^2 + 160 + q^2 - 24q$ so $2q^2 - 24q + 56 = 0$

$q^2 - 12q + 28 = 0$ therefore $q = \frac{12 \pm \sqrt{(-12)^2 - 4 \times 28}}{2} = 6 \pm 2\sqrt{2}$

$Q = (6 \pm 2\sqrt{2}, 0)$

8 Midpoint $M = \left(\frac{-4+5}{2}, \frac{3+1}{2}\right) = \left(\frac{1}{2}, 2\right)$.

Gradient of PQ $m_1 = \frac{\text{change in } y}{\text{change in } x} = \frac{1-3}{5+4} = -\frac{2}{9}$.

Perpendicular gradient $m_2 = -\frac{1}{m_1} = \frac{9}{2}$.

Line equation $y - y_1 = m_2(x - x_1)$

$y - 2 = \frac{9}{2}\left(x - \frac{1}{2}\right)$ so $4y - 8 = 18x - 9$ therefore $18x - 4y - 1 = 0$

9 **a** $AM^2 = 3^2 + (5 - p)^2 = 34 - 10p + p^2$

$BM^2 = (3 - q)^2 + 5^2 = 34 - 6q + q^2$

$AB^2 = q^2 + p^2$

IRANINOVIN

For angle $A\widehat{M}B = 90^\circ$, $AB^2 = AM^2 + BM^2$ so $68 - 10p - 6q + p^2 + q^2 = p^2 + q^2$

$68 - 10p - 6q = 0$

$5p + 3q = 34$

b When $p = 4$, $20 + 3q = 34$ therefore $q = \frac{14}{3}$.

$A = (0, 4)$, $B = \left(\frac{14}{3}, 0\right)$

$$\text{Area of triangle } AMB = \frac{1}{2}AM \times BM = \frac{1}{2} \times \sqrt{3^2 + 1^2} \times \sqrt{\left(\frac{5}{3}\right)^2 + 5^2}$$

$$= \frac{1}{2}\sqrt{10} \times 5\sqrt{\frac{10}{9}} = \frac{25}{3}$$

$$\text{Area of triangle } OAB = \frac{1}{2}OA \times OB = \frac{1}{2} \times 4 \times \frac{14}{3} = \frac{28}{3}$$

$$\text{Total area of } OAMB = \frac{53}{3}$$

10a The equation of l can be rearranged as $y = \frac{1}{2}x + \frac{3}{2}$, so its gradient is $\frac{1}{2}$.

The gradient of the perpendicular line is -2. Hence the equation is $y - 6 = -2(x + 1)$, or $y = -2x + 4$.

b The shortest distance is along the perpendicular line, so you need to find where the line $y = -2x + 4$ intersects l. Substitute y into the equation of l:

$x - 2(-2x + 4) + 3 = 0 \Rightarrow x = 1$, $y = 2$.

The distance between $(1, 2)$ and $\mathrm{P}(-1, 6)$ is $\sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$.

11



Tip

This question is easier to address using vectors (see Chapter 12) but can be answered with slightly more working by calculating the gradients of each side. Note that simply showing $AB = CD$ and $BC = AD$ is not sufficient because that does not exclude the possibility of two of these 'edges' crossing over.

a Gradient of AB is $\frac{k}{1} = k$, gradient of BC is $\frac{2}{-4} = -\frac{1}{2}$,

gradient of CD is $\frac{-k}{-1} = k$, gradient of DA is $-\frac{2}{4} = -\frac{1}{2}$.

Opposite sides are parallel, so the shape $ABCD$ is a parallelogram.

b Require that AB and BC are perpendicular:

$k \times -\frac{1}{2} = -1$ so $k = 2$

EXERCISE 6D

5 a $(x + 6)^2 + (y - 3)^2 = 117$

b When $x = 0$, $y - 3 = \pm\sqrt{117 - 36} = \pm\sqrt{81} = \pm 9$

Intersections with y-axis are at $(0, 12)$ and $(0, -6)$

6 a $\left(x - \frac{5}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 3 + \left(\frac{5}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{38}{4}$

Circle has centre $\left(\frac{5}{2}, -\frac{1}{2}\right)$ and radius $\frac{\sqrt{38}}{2}$.

IRANINOVIN

b Distance from A to the circle centre is $\sqrt{\left(\frac{7}{2}\right)^2 + \left(\frac{7}{2}\right)^2} = \sqrt{\frac{98}{4}} = \frac{\sqrt{98}}{2}$.

This is greater than the radius so A lies outside the circle.

7

Tip

It may appear that you should work out a circle equation as the first step of answering this problem, but in fact you can simply use the fact that $AP = AQ = 7$, where A is the circle centre.

Let $A(3, -5)$ be the circle centre and $(p, 0)$ be one of the intersections with the x-axis.

$AP = 7 = \sqrt{(3-p)^2 + 5^2}$

$9 - 6p + p^2 + 25 = 49$ therefore $p^2 - 6p - 15 = 0$

$p = \frac{6 \pm \sqrt{6^2 + 60}}{2} = 3 \pm 2\sqrt{6}$

Distance $PQ = 4\sqrt{6}$, the difference between the two values for p.

8 **a** $AB^2 = 10^2 + 6^2 = 136$ and $BC^2 = 9^2 + 15^2 = 306$

$CA^2 = 19^2 + 9^2 = 442 = AB^2 + BC^2$ so, by Pythagoras' theorem, $A\hat{B}C = 90^\circ$

b $AC = \sqrt{442}$

c AC must be a diameter of the circle.

Midpoint $M\left(\frac{5}{2}, -\frac{3}{2}\right)$ is the centre, radius is $\frac{\sqrt{442}}{2}$.

Circle equation is $\left(x - \frac{5}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{221}{2}$

9 **a** Require that $(0, 0)$ lies on the circle.

Substituting: $p^2 + 3^2 = 26$ therefore $p = \sqrt{17}$

b Distance squared between $(3, 2)$ and the centre:

$(3 - \sqrt{17})^2 + 5^2 = 9 + 17 - 6\sqrt{17} + 25 = 51 - 6\sqrt{17} = 26.3 > 26$

The point lies outside the circle.

10a Angle subtended by a diameter is 90°.

b PZ and QZ are perpendicular so their gradients multiply to -1.

Gradient of $PZ = \frac{y-b}{x-a}$, gradient of $QZ = \frac{y-d}{x-c}$.

$\frac{y-b}{x-a} \times \frac{y-d}{x-c} = -1$ therefore $(y-b)(y-d) = -(x-a)(x-c)$

$(x-a)(x-c) + (y-b)(y-d) = 0$

EXERCISE 6E

4 $l_1 : y = \frac{5}{2}x - \frac{7}{2}$ so gradient of l_1 is $m_1 = \frac{5}{2}$.

l_2 perpendicular to l_1 so gradient $m_2 = -\frac{1}{m_1} = -\frac{2}{5}$.

Line equation $y - y_1 = m(x - x_1)$.

$l_2 : y - 1 = -\frac{2}{5}(x + 3)$ therefore l_2 has equation $y = -\frac{2}{5}x - \frac{1}{5}$.

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Intersection: substitute l_2 into l_1:

$-\frac{2}{5}x - \frac{1}{5} = \frac{5}{2}x - \frac{7}{2}$ so $-4x - 2 = 25x - 35$ therefore $29x = 33$

$x = \frac{33}{29}$, $y = -\frac{19}{29}$ therefore intersection point is $\left(\frac{33}{29}, -\frac{19}{29}\right)$.

5 **a** Substituting the coordinates of P into the LHS of equation of the circle gives:

$(-3-1)^2 + (2+2)^2 = 32$

This equals the RH of the circle equation, hence P lies on the circle.

b $(1, -2)$

c Gradient of the radius to P is $\frac{2+2}{-3-1} = -1$.

Tangent is normal to the radius so has gradient 1.

Line equation $y - y_1 = m(x - x_1)$

$y - 2 = x + 3$ therefore tangent has equation $y = x + 5$

6 **a** $l_1 : y = -2x + 10$ so gradient of l_1 is $m_1 = -2$.

l_2 is perpendicular to l_1 so has gradient $m_2 = -\frac{1}{m_1} = \frac{1}{2}$.

Line equation $y - y_1 = m(x - x_1)$

$y - 0 = \frac{1}{2}(x + 2)$ therefore l_2 has equation $y = \frac{1}{2}x + 1$.

b Substitute l_2 into l_1 to find intersection M: $2x + \frac{1}{2}x + 1 - 10 = 0$ so $\frac{5}{2}x = 9$.

$x = \frac{18}{5}$, $y = \frac{14}{5}$ therefore intersection is $M\left(\frac{18}{5}, \frac{14}{5}\right)$.

c In l_1 when $y = 0$, $x = 5$ so B has coordinates $(5, 0)$.

In AMB there is a right-angle at M.

$$\begin{aligned}\text{Area} &= \frac{1}{2}AM \times BM \\ &= \frac{1}{2}\sqrt{\left(\frac{28}{5}\right)^2 + \left(\frac{14}{5}\right)^2} \times \sqrt{\left(\frac{7}{5}\right)^2 + \left(\frac{14}{5}\right)^2} \\ &= \frac{49}{5}\end{aligned}$$

7 **a** AB has gradient $\frac{7-3}{5-1} = 1$ and midpoint $M_{AB} = \left(\frac{1+5}{2}, \frac{7+3}{2}\right) = (3, 5)$.

Perpendicular bisector of AB has equation $y - 5 = -1(x - 3)$ so $y = -x + 8$.

AC has gradient $\frac{9-3}{2-1} = 6$ and midpoint $M_{AB} = \left(\frac{1+2}{2}, \frac{9+3}{2}\right) = \left(\frac{3}{2}, 6\right)$.

Perpendicular bisector of AC has equation $y - 6 = -\frac{1}{6}\left(x - \frac{3}{2}\right)$ so $y = -\frac{1}{6}x + \frac{25}{4}$.

b Intersection when $-x + 8 = -\frac{1}{6}x + \frac{25}{4}$.

$\frac{5}{6}x = \frac{7}{4}$ therefore $x = \frac{21}{10}$.

Intersection is at $P\left(\frac{21}{10}, \frac{59}{10}\right) = (2.1, 5.9)$.

c BC has gradient $\frac{9-7}{2-5} = -\frac{2}{3}$ and midpoint $M_{BC} = \left(\frac{5+2}{2}, \frac{9+7}{2}\right) = \left(\frac{7}{2}, 8\right)$.

Perpendicular bisector of BC has equation $y - 8 = \frac{3}{2}\left(x - \frac{7}{2}\right)$ so $y = \frac{3}{2}x + \frac{11}{4}$.

Substituting $x = 2.1 : y = 3.15 + 2.75 = 5.9$, so this line also passes through P.

d Radius is equal to $AP = BP = CP = \sqrt{(1 - 2.1)^2 + (3 - 5.9)^2} = \sqrt{9.62} = 3.10$.

8 **a** Radius equals OC therefore $OC^2 = 2^2 + 5^2 = 29$.

Circle has equation $(x - 2)^2 + (y - 5)^2 = 29$.

b Distance $AC = \sqrt{2^2 + (5 - 10)^2} = \sqrt{29}$ so A is one radius from the circle centre, so lies on the circle circumference.

c OC has gradient $\frac{5}{2}$ so AB has gradient $-\frac{2}{5}$.

AB has equation $y - 10 = -\frac{2}{5}x$ therefore $y = -\frac{2}{5}x + 10$.

Substituting into the circle equation to find B:

$(x - 2)^2 + \left(-\frac{2}{5}x + 5\right)^2 = 29$ so $x^2 - 4x + 4 + \frac{4}{25}x^2 - 4x + 25 = 29$.

$\frac{29}{25}x^2 - 8x = 0$ therefore $x\left(\frac{29}{25}x - 8\right) = 0$.

$x = 0$, which corresponds to A, or $x = \frac{200}{29}$, which must correspond to B.

B is $\left(\frac{200}{29}, \frac{210}{29}\right)$ so AB has length $\sqrt{\left(\frac{200}{29}\right)^2 + \left(10 - \frac{210}{29}\right)^2} = 7.43$.

Tip

Alternatively, find X, the intersection of OC and AB, then by symmetry $AB = 2AX$.

OC has equation $y = \frac{5}{2}x$. At X, $\frac{5}{2}x = 10 - \frac{2}{5}x$, $\frac{29}{10}x = 10$, $x = \frac{100}{29}$, $y = \frac{250}{29}$

$AX = \sqrt{\left(\frac{100}{29}\right)^2 + \left(10 - \frac{250}{29}\right)^2} = 3.714$ so $AB = 7.43$.)

9 **a** The two circles are tangential, so the point of contact and the two centres must be collinear (lie on a single line).

Gradient from C_1 to C_2 is 0 and line has equation $y = 5$.

Circle C_1 intersects this line at $(-9, 5)$ and $(5, 5.)$

So either C_2 also passes through $(-9, 5)$ and so has radius 21, or it passes through $(5, 5)$ and has radius 7.

b Require the two points for which the distance to $(-2, 5)$ equals 7 and the distance to $(12, 5)$ equals 16.

Let such a point have coordinates (x, y).

Then $(x + 2)^2 + (y - 5)^2 = 49$ and $(x - 12)^2 + (y - 5)^2 = 256$

Make a simple substitution of $(y - 5)^2 = 49 - (x + 2)^2$ into the second equation to eliminate $y : (x - 12)^2 + 49 - (x + 2)^2 = 256$

$x^2 - 24x + 144 + 49 - x^2 - 4x - 4 = 256$ so $-28x = 67$

$x = -2.39$

Solving from the substitution: $y = 5 \pm \sqrt{49 - (x + 2)^2} = 12.0$ or -1.99

IRANINOVIN

Intersections are $(-2.39, -1.99)$ and $(-2.39,\ 12.0)$.

10 Gradient of the radius to $(2,\ 6)$ is $\dfrac{6}{2} = 3$, gradient of the tangent is $-\dfrac{1}{3}$.

Equation of the tangent is $y - 6 = -\dfrac{1}{3}(x - 2)$ so $y = -\dfrac{1}{3}x + \dfrac{20}{3}$.

When $x = 0$, $y = \dfrac{20}{3}$: $P = \left(0,\ \dfrac{20}{3}\right)$ and when $y = 0$, $x = 20$: $Q = (20,\ 0)$.

Area $OPQ = \dfrac{1}{2} \times \dfrac{20}{3} \times 20 = \dfrac{200}{3}$

11 Require a single repeated root when finding the intersection of

$(x-3)^2 + (y-6)^2 = 2^2$ and $y = kx$.

Substituting: $(x-3)^2 + (kx-6)^2 = 4$ so $x^2(1+k^2) + x(-6-12k) + 41 = 0$

Repeated root when discriminant $\Delta = 0$

$\Delta = (-6-12k)^2 - 4(1+k^2) \times 41 = 0$ so $144k^2 + 144k + 36 - 164 - 164k^2 = 0$

$$20k^2 - 144k + 128 = 0$$
$$5k^2 - 36k + 32 = 0$$
$$k = \frac{36 \pm \sqrt{36^2 - 640}}{10} \approx 1.04, 6.16$$

12 Intersections of the line $3x - y = 3$ and the circle $(x-5)^2 + (y+1)^2 = r^2$:

Substitute $y = 3x - 3$ into the circle equation:

$x^2 - 10x + 25 + 9x^2 - 12x + 4 = r^2$

Collecting and rearranging: $10x^2 - 22x + 29 - r^2 = 0$

For the line to be a tangent, there must be a single point of intersection, so a single root to this quadratic equation in x.

Single root: discriminant $\Delta = 0$.

$\Delta = (-22)^2 - 40(29 - r^2) = 0$ so $29 - r^2 = 12.1$

$r^2 = 16.9$ therefore $r = \dfrac{13\sqrt{10}}{10}$ or 4.11.

13 a Circle has equation $x^2 + y^2 = 5^2$. When $x = 0$, $y = \pm 5$ so $B = (0, -5)$.

$M = \left(\dfrac{0+4}{2},\ \dfrac{-5+3}{2}\right) = (2, -1)$

b AB has gradient $\dfrac{3+5}{4-0} = 2$ so OP has gradient $-\dfrac{1}{2}$.

OP has equation $y = -\dfrac{1}{2}x$.

Substituting to find P : $x^2 + \dfrac{1}{4}x^2 = 25$

$x^2 = 20$ therefore $x = \pm 2\sqrt{5}$.

Must be in same quadrant as M so $x = 2\sqrt{5}$, $y = -\sqrt{5}$.

$P = (2\sqrt{5}, -\sqrt{5})$

$BP^2 = (2\sqrt{5})^2 + (-5 + \sqrt{5})^2 \neq 25 = OB^2$

Since $OB \neq BP$, it follows that $OAPB$ is not a rhombus.

Tip

Note that even though $OAPB$ is not a rhombus, it must at least be a kite, by symmetry through OP. In consequence, OP and AB will still be perpendicular; diagonals at

perpendicular is a quality of a rhombus but not a defining characteristic, so assessing the angle cannot be used to prove the shape is definitely a rhombus, though finding that the angle is not 90° would be sufficient to prove a shape is not a rhombus.

14 a Completing the square for the circle equation:

$(x-5)^2+(y-5)^2=25$

Centre $C(5,\,5)$ and radius 5.

$A(5,\,0)$ and $B(0,\,5)$ both lie on the circle and from the coordinates $OACB$ is clearly a square, so CA is perpendicular to the x-axis and CB is perpendicular to the y-axis; the axes are therefore tangents to the circle.

b Distance $MC=\sqrt{(8-5)^2+(9-5)^2}=5$ so M is one radius from the circle centre and so lies on the circle circumference.

c The line connecting M to the circle centre has gradient $\frac{4}{3}$ so the tangent line has gradient $-\frac{3}{4}$.

The tangent line has equation $y-9=-\frac{3}{4}(x-8)$ so $3x+4y=60$.

Intersections with the axes are at $(20,\,0)$ and $(0,\,15)$.

So the outer triangle has area $\frac{1}{2}\times 20\times 15=150$.

The circle has area $\pi\times 5^2=25\pi$.

So the exact value of the shaded area is $150-25\pi$

15 Require $x^2+y^2=1$ (the unit circle) and $(x-a)^2+(y-b)^2=r^2$ to have a single intersection; that is, the two circles must be tangent to each other.

If they are tangent, their centres and the point of intersection must lie on a single line.

The centres of the circles are $(0,\,0)$ and $(a,\,b)$ so the line has equation $y=\frac{bx}{a}$.

This line will pass through the unit circle where $x^2+\left(\frac{bx}{a}\right)^2=1$.

$x^2\frac{\left(a^2+b^2\right)}{a^2}=1$ so $x^2=\frac{a^2}{a^2+b^2}$

$y^2=\frac{b^2}{a^2}x^2=\frac{b^2}{a^2+b^2}$

So the points of intersection are $\pm\left(\frac{a}{\sqrt{a^2+b^2}},\,\frac{b}{\sqrt{a^2+b^2}}\right)$.

But these must also lie on the other circle, so $(x-a)^2+(y-b)^2=r^2$

$a^2\left(\frac{\pm1}{\sqrt{a^2+b^2}}-1\right)^2+b^2\left(\frac{\pm1}{\sqrt{a^2+b^2}}-1\right)^2=r^2$ so $\left(a^2+b^2\right)\left(\frac{\pm1}{\sqrt{a^2+b^2}}-1\right)^2=r^2$

$\frac{\left(a^2+b^2\right)}{\left(a^2+b^2\right)}\left(\pm1-\sqrt{a^2+b^2}\right)^2=r^2$ so $\left(\pm1-\sqrt{a^2+b^2}\right)^2=r^2$

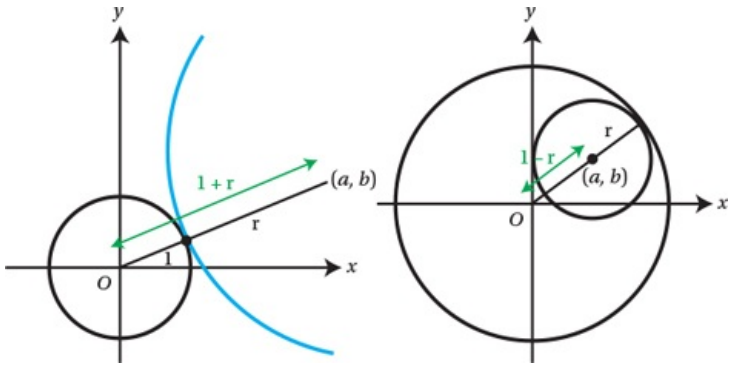
$\left(\pm1-\sqrt{a^2+b^2}\right)=r$ or $-r$ therefore $\sqrt{a^2+b^2}=\pm1-r$ or $\pm1+r$

$a^2+b^2=(1\pm r)^2$

Tip

The algebra above is unwieldy, and the same result can be much more quickly observed by drawing a diagram.

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Tip

Here you can immediately see, and argue, that the distance between the two circle centres is the hypotenuse of a triangle which is either one greater or one less than the radius of the non-unit circle.

That is, $\sqrt{a^2+b^2} = 1 \pm r$ *if* $r < 1$ *and* $r \pm 1$ *if* $r > 1$

MIXED PRACTICE 6

1 Complete the square: $(x-4)^2 + (y+3)^2 = 144 + 16 + 9 = 169 = 13^2$ so $r = 13$.

2 **a** Substituting $x = 2k$, $y = 2k + 1$: $6k - (4k + 2) + 7 = 0$

$2k = -5$ therefore $k = -2.5$.

b l_1 has gradient $\frac{3}{2}$ so require that AB has gradient $-\frac{2}{3}$.

$A(-5, 4)$, $B(-2, p)$: gradient of AB is $\frac{p+4}{-2+5}$.

$\frac{p+4}{3} = -\frac{2}{3}$ therefore $p = -6$.

c l_2 is parallel to l_1 so must have the form $3x - 2y + c = 0$

Passes through B so substitute $x = -2$, $y = -6$: $-6 + 12 + c = 0$

$c = -6$ therefore l_2 has equation $3x - 2y - 6 = 0$

d When $y = 0$, $3x - 6 = 0 \Rightarrow x = \frac{2}{2}$. C is the point $\left(\frac{2}{2}, 0\right)$.

3 **a** Completing the square: $(x-1)^2 + (y-5)^2 = 19 + 1^2 + 5^2 = 45$

Centre P is the point $(1, 5)$.

b Radius is $\sqrt{45}$.

Distance $PA = \sqrt{(7-1)^2 + (5-2)^2} = \sqrt{36+9} = \sqrt{45}$.

A is one radius distant from circle centre, so lies on the circumference.

c PA has gradient $\frac{2-5}{7-1} = -\frac{1}{2}$.

l is perpendicular to this, so has gradient 2.

l has equation $y + 1 = 2(x - 1)$ therefore $y = 2x - 3$.

PA has equation $y - 2 = -\frac{1}{2}(x - 7)$ so $y = -\frac{1}{2}x + \frac{11}{2}$.

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Intersection of these two lines is S: $2x - 3 = -\frac{1}{2}x + \frac{11}{2}$ so $\frac{5}{2}x = \frac{17}{2}$

$x = \frac{17}{5}$, $y = \frac{19}{5}$ therefore S has coordinates $\left(\frac{17}{5}, \frac{19}{5}\right)$.

4 **a** Centre $(-3, 2)$ so $(x+3)^2 - 9 + (y-2)^2 - 4 - 4 = 0$

$r^2 = 17$ therefore $r = \sqrt{17}$.

b $x^2 + (3x+4)^2 + 6x - 4(3x+4) - 4 = 0$ so $10x^2 + 18x - 4 = 0$.

$(5x-1)(x+2) = 0$ therefore $x = \frac{1}{5}$ or -2 and $y = \frac{23}{5}$ or -2.

5 Gradient of the tangent is -3.

Therefore gradient from the contact point to the centre is $\frac{1}{3}$.

$\frac{k+3+7}{k-4-4} = \frac{1}{3}$ so $3(10+k) = k - 8$

$2k = -38$ therefore $k = -19$.

6 **a** $BA^2 = (-2-3)^2 + (3-4)^2 = 26$, $AC^2 = (2-3)^2 + (9-4)^2 = 26$.
$CB^2 = (2+2)^2 + (9-3)^2 = 52$

Since $CB^2 = BA^2 + AC^2$, by Pythagoras' theorem, $B\widehat{A}C = 90^\circ$.

b The circumcircle must have BC as diameter, so has centre $\left(\frac{3+9}{2}, \frac{-2+2}{2}\right) = (6, 0)$ and radius $\frac{1}{2}\sqrt{52} = \sqrt{13}$.

Equation of the circle is $(x-6)^2 + y^2 = 13$.

c Gradient from circle centre to B is $\frac{0+2}{6-3} = \frac{2}{3}$.

Tangent at B is perpendicular to this, so has gradient $-\frac{3}{2}$, passing through $B(3, -2)$.

Equation of tangent is $(y+2) = -\frac{3}{2}(x-3)$.

$2y + 4 = -3(x-3)$ therefore $3x + 2y - 5 = 0$.

7 Circle has equation $(x-3)^2 + y^2 = 25$.

Substitute $y = 2x + k$ to find the intersections: $(x-3)^2 + (2x+k)^2 = 25$

$5x^2 + x(-6+4k) + 9 + k^2 - 25 = 0$ so $5x^2 + x(-6+4k) + (k^2 - 16) = 0$

Quadratic equation; for there to be two solutions, require the discriminant $\Delta > 0$.

$\Delta = (-6+4k)^2 - 4 \times 5 \times (k^2 - 16) > 0$ so $36 - 48k + 16k^2 - 20k^2 + 320 > 0$

$-4k^2 - 48k + 356 > 0$ therefore $k^2 + 12k - 89 < 0$

Positive quadratic is less than zero between the roots; roots are
$k = \frac{-12 \pm \sqrt{12^2 + 4 \times 89}}{2} = -6 \pm \sqrt{125}$

There are two intersections for $-6 - 5\sqrt{5} < k < -6 + 5\sqrt{5}$.

8 Radius of circle is r where $r^2 = CD = (7-4)^2 + (12-10)^2 = 13$

CD has gradient $\frac{12-10}{7-4} = \frac{2}{3}$.

Tangent at D is perpendicular to this, so has gradient $-\frac{3}{2}$ and passes through $D(4, 10)$

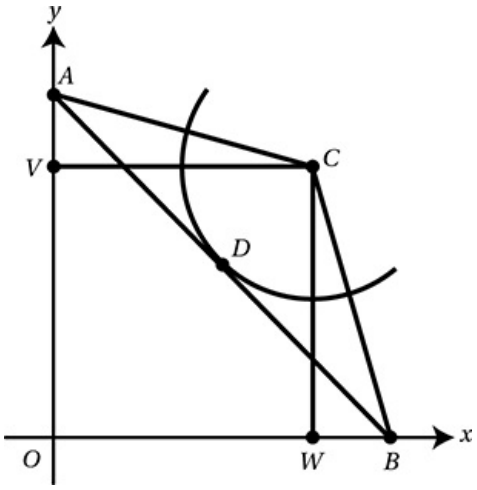
Tangent at D has equation $y - 10 = -\frac{3}{2}(x-4)$ so $y = -\frac{3}{2}x + 16$.

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Tangent intersects the axes at $A(0,\ 16)$ and $B\left(\frac{32}{3},\ 0\right)$.

a Area $AOB = \frac{1}{2} \times 16 \times \frac{32}{3} = \frac{256}{3}$.

b Let point $V : (0,\ 12)$ and $W(7,\ 0)$ be the points on the axes horizontally alongside and vertically below the circle centre C.



Then $OVCW$ is a 7×12 rectangle, area 84.

ACV is a right-angled triangle, area $\frac{1}{2} \times 4 \times 7 = 14$.

BCW is a right-angled triangle, area $\frac{1}{2} \times \frac{11}{3} \times 12 = 22$.

Triangle ABC is the area of quadrilateral $OACB$ less triangle OAB.

$OACB = OVCW + ACV + BCW = 120$

$ABC = OACB - OAB = 120 - \frac{256}{3} = \frac{104}{3}$

9 For a general point $\mathrm{P}(x,\ y)$ on the circle, angle APB must be a right angle, so the gradients of AP and PB must have a product of -1. You can use this to find the equation of the circle.

> **Tip**
>
> Either find the centre (midpoint of AB) and radius (half the distance AB) and form the usual equation, or use this method, which is faster and specific to this exact problem, using the fact that the angle in a semicircle is a right angle.

$\frac{y-7}{x+3} \times \frac{y+1}{x-5} = -1$ therefore $y^2 - 6y - 7 = -(x^2 - 2x - 15)$.

$x^2 + 2x + y^2 - 6y - 22 = 0$

10 The point on the tangent line which touches the circle must satisfy both equations:

$y = kx + 3$ and $(x-6)^2 + (y-3)^2 = 2^2$

Substituting: $(x-6)^2 + (kx)^2 = 4$ so $x^2(1+k^2) - 12x + 32 = 0$

Quadratic equation should have a single solution if the line is a tangent, so discriminant $\Delta = 0$.

$\Delta = (-12)^2 - 4(1+k^2) \times 32 = 0$ therefore $144 - 128 - 128k^2 = 0$

$128k^2 = 16$ so $k = \pm\frac{1}{\sqrt{8}} = \pm\frac{\sqrt{2}}{4}$

11 a $(x-2)^2 + (y-1)^2 = 100$ so $x^2 + y^2 - 4x - 2y - 95 = 0$

b $(5-2)^2 + (k-1)^2 = 100$ so $k^2 - 2k - 90 = 0$ therefore $k = 1 + \sqrt{91}$ as $k > 0$

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c Distance from $(-3,\ 9)$ to $(2,\ 1) = \sqrt{5^2 + 8^2} = \sqrt{89}$.

$\sqrt{89} < 10$ so point is inside.

d Gradient of radius $= \dfrac{9-1}{8-2} = \dfrac{4}{3}$.

Gradient of tangent $= -\dfrac{3}{4}$.

$y - 9 = -\dfrac{3}{4}(x - 8)$ therefore $y = -\dfrac{3}{4}x + 15$

12

Tip

You could look for the point on the line (call it P) closest to $A(-3,\ 2)$ and then use the fact that AP would be perpendicular to the line to make an equation to find P, and then find the answer AP, which would be the shortest distance from A to the line. Alternatively, envisage a circle centred at $A(-3,\ 2)$ with unknown radius, for which the line is to be a tangent. The answer to the question is then the radius.

A circle centred at $(-3,\ 2)$ with radius r has equation $(x+3)^2 + (y-2)^2 = r^2$.

Require that the line $y = \dfrac{1}{2}(19 - 3x)$ is tangent to this circle.

Taking the intersection of line and the circle: $(x+3)^2 + \left(\dfrac{19}{2} - \dfrac{3}{2}x - 2\right)^2 = r^2$

$4(x+3)^2 + (15 - 3x)^2 = 4r^2$ therefore $4x^2 + 24x + 36 + 225 - 90x + 9x^2 - 4r^2 = 0$

$13x^2 - 66x + 261 - 4r^2 = 0$.

For the line to be tangent, there must be only one solution to this quadratic, so discriminant $\Delta = 0$

$\Delta = (-66)^2 - 4 \times 13 \times (261 - 4r^2) = 0$ therefore $4356 - 13572 + 208r^2 = 0$

$r^2 = \dfrac{9216}{208}$ so $r = \sqrt{\dfrac{9216}{208}} = \dfrac{96}{4\sqrt{13}} = \dfrac{24}{\sqrt{13}} = \dfrac{24\sqrt{13}}{13}$

Alternative method

Suppose $B(p,\ q)$ is the closest point on the line to $A(-3,\ 2)$

Then AB is perpendicular to the line and AB is the shortest distance from A to the line.

Line has equation $y = \dfrac{1}{2}(19 - 3x)$ so has gradient $-\dfrac{3}{2}$.

So AB has gradient $\dfrac{2}{3}$ and passes through $A(-3,\ 2)$

AB has equation $y - 2 = \dfrac{2}{3}(x+3)$ therefore $y = \dfrac{2}{3}x + 4$.

B lies on the intersection of this line and $y = \dfrac{1}{2}(19 - 3x)$. Substituting:

$\dfrac{2}{3}x + 4 = \dfrac{1}{2}(19 - 3x)$ so $4x + 24 = 3(19 - 3x)$

$13x = 33$ therefore $x = \dfrac{33}{13}$, $y = \dfrac{74}{13}$

$$\text{Distance } AB = \sqrt{\left(-3 - \frac{33}{13}\right)^2 + \left(2 - \frac{74}{13}\right)^2}$$
$$= \frac{1}{13}\sqrt{72^2 + 48^2}$$
$$= \frac{12}{13}\sqrt{6^2 + 4^2}$$
$$= \frac{24\sqrt{13}}{13}$$

13 C_1 has equation $(x-3)^2+(y+5)^2=3^2+5^2-18=16=4^2$.

C_2 has equation $(x-7)^2+(y-3)^2=7^2+3^2-49=3^2$.

The two circle centres are $(3,-5)$ and $(7,\ 3)$ and the distance between these is $\sqrt{(7-3)^2+(3+5)^2}=\sqrt{80}$

However, the two circles have radius 4 and 3 respectively, the sum of which is less than the distance separating the centres, so each lies entirely outside the other and they do not intersect.

14 a AB forms a chord across the circle, which must be perpendicular to the radius which bisects it. Since AB is horizontal, the perpendicular bisector passing through the circle centre must be $x=5$, so the distance from A to $(5,\ 0)$ equals the distance from B to $(5,\ 0)$: B is $(8,\ 0)$ therefore $p=8$.

Distance from A to the circle centre is $\sqrt{(5-2)^2+7^2}=\sqrt{58}$.

Equation of the circle is $(x-5)^2+(y-7)^2=58$.

b When $x=0, (y-7)^2=58-5^2=33$.

$y=7\pm\sqrt{33}$ so the points are $C(0,\ 7+\sqrt{33})$, $D(0,\ 7-\sqrt{33})$.

The area of $ABCD$ equals the area of BOC less the area of AOD:

$$\frac{1}{2}\times 8\times(7+\sqrt{33})-\frac{1}{2}\times 2\times(7-\sqrt{33})$$
$$=28+4\sqrt{33}-(7-\sqrt{33})$$
$$=21+5\sqrt{33}$$

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Chapter 7 worked solutions

7 Logarithms

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 7A

6 $\log_{10}(9x+1) = 3$ so $9x + 1 = 10^3 = 1000$ therefore $x = 111$

7 $\log_8\sqrt{1-x} = \frac{1}{3}$ so $\sqrt{1-x} = 8^{\frac{1}{3}} = 2$

$1 - x = 4$ therefore $x = -3$

8 $\ln(3x-1) = 2$ so $3x - 1 = \mathrm{e}^2$ therefore $x = \frac{\mathrm{e}^2+1}{3}$

9 $3(1+\log x) = 6 + \log x$ so $3 + 3\log x = 6 + \log x$

$2\log x = 3$ therefore $x = 10^{\frac{3}{2}} = 31.6$ (3 s. f.)

10 $(\log_3 x)^2 = 4$ so $\log_3 x = \pm 2$ therefore $x = 3^{\pm 2} = 9$ or $\frac{1}{9}$

11 $\left.\begin{matrix}\log_3 x + \log_5 y = 6 \\ \log_3 x - \log_5 y = 2\end{matrix}\right\}$

Adding the two simultaneous equations:

$2\log_3 x = 8$ so $\log_3 x = 4$ and $\log_5 y = 2$

$x = 3^4 = 81$, $y = 5^2 = 25$

12 $\sqrt[6]{\pi^4 + \pi^5} = 2.718281886$ and $\mathrm{e} = 2.7182818$

That is, they are the same value, at least to six decimal places.

EXERCISE 7B

4 $3\log_b x = \log_b 64$

$\log_b x^3 = \log_b 64$

$x^3 = 64$

$x = 4$

5 $\ln x = 2\ln 9 - \ln 3 = \ln 9^2 - \ln 3 = \ln\left(\frac{81}{3}\right) = \ln 27$

$x = 27$

6 $\log(3x+6) = \log(3) + 1$

$\log\left(\frac{3x+6}{3}\right) = 1$

$\log(x+2) = 1$ so $x + 2 = 10$

$x = 8$

7 **a** $\log a^2 b = \log a^2 + \log b = 2\log a + \log b = 2x + y$

b $\log\left(\frac{100a}{\sqrt{c}}\right) = \log 100 + \log a - \log c^{\frac{1}{2}}$

$= \log 10^2 + \log a - \frac{1}{2}\log c = 2 + x - \frac{z}{2}$

8 **a** $\ln 50 = \ln 2 + \ln 25 = \ln 2 + 2\ln 5 = a + 2b$

b $\ln 0.16 = \ln\left(\frac{2^2}{5^2}\right) = 2\ln 2 - 2\ln 5 = 2a - 2b$

9 $2\ln x + \ln 9 = 3$

$\ln x^2 + \ln 9 = 3$

$\ln 9x^2 = 3$ so $9x^2 = e^3$ and $(3x)^2 = e^3$

$3x = e^{\frac{3}{2}}$

$x = \frac{1}{3}e^{\frac{3}{2}}$

$A = \frac{1}{3}, B = \frac{3}{2}$

10 $\log(x+5) - 1 = \log(x-1)$

$\log\left(\frac{x+5}{10}\right) = \log(x-1)$ so $x + 5 = 10(x-1)$

$9x = 15$

$x = \frac{5}{3}$

11 $\log_2(x+2) = 3 - \log_2 x = \log_2 8 - \log_2 x$ therefore $\log_2(x+2) = \log_2\left(\frac{8}{x}\right)$

$x + 2 = \frac{8}{x}$ so $x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$ therefore $x = -4$ or 2

However, since you require $\log_2 x$ to be valid, $x = -4$ cannot be a solution

$x = 2$ is the only solution.

Tip

Always make a quick check on any solutions to a problem involving logarithms to eliminate any invalid solutions.

12 $\log(x^2+1) = 1 + 2\log(x)$ so $\log(x^2+1) - \log(x^2) = 1$

$\log\left(\frac{x^2+1}{x^2}\right) = 1$ therefore $\frac{x^2+1}{x^2} = 10$

$10x^2 = x^2 + 1$ so $9x^2 = 1$ therefore $x = \pm\frac{1}{3}$

Check validity:

Original problem contains $\log(x)$ so negative solutions of x should be rejected

Only valid solution is $x = \frac{1}{3}$.

EXERCISE 7C

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3 $(4x+3)\log 5 = \log 28$

$$4x\log 5 = \log 28 - 3\log 5$$

$$x = \frac{\log 28 - 3\log 5}{4\log 5} = \frac{\log\left(\frac{28}{125}\right)}{\log 625} = -0.232 (3\,\text{s. f.})$$

4 $4 \times 3^{x-5} = 1$

$3^{x-5} = \frac{1}{4}$ so $x - 5 = \log_3\left(\frac{1}{4}\right) = -\log_3(4)$

$x = 5 - \log_3(4) = 3.74 (3\,\text{s. f.})$

5 $e^{3-2x} = 10$ so $3 - 2x = \ln 10$

$2x = 3 - \ln 10 = \ln(e^3) - \ln 10 = \ln\left(\frac{e^3}{10}\right)$

$x = \ln\left(\sqrt{\frac{e^3}{10}}\right)$

6 $10^x = 5 \times 2^{3x} = 5 \times 8^x$

$\left(\frac{5}{4}\right)^x = 5$

$x\log\left(\frac{5}{4}\right) = \log 5$

$$x = \frac{\log(5)}{\log\left(\frac{5}{4}\right)} = \frac{\log 5}{\log\left(\frac{10}{8}\right)} = \frac{\log 5}{1 - \log 8}$$

Tip

You could have written this answer in many equivalent forms:

$$\frac{\log 5}{1-\log 8} = \frac{\log 5}{\log 5 - \log 4} = \frac{\log 5}{\log 1.25} = \log_{1.25} 5$$

Which of these might be considered the 'simplest' form is a matter of taste or context; in this question, the form is specified, but you may generally leave your answers in any equivalently simple form.

7 $2^{3x-1} = 5^{2-x}$

$(3x-1)\ln 2 = (2-x)\ln 5$

$x(3\ln 2 + \ln 5) = \ln 2 + 2\ln 5$

$x(\ln 8 + \ln 5) = \ln 2 + \ln 25$

$x\ln 40 = \ln 50$

$$x = \frac{\ln 50}{\ln 40}$$

8 $3^x \times 4^{2x+1} = 6^{x+2}$

$3^x \times 16^x \times 4 = 6^x \times 36$

$\left(\frac{3 \times 16}{6}\right)^x = \frac{36}{4}$

$8^x = 9$ so $x\ln 8 = \ln 9$

$$x = \frac{\ln 9}{\ln 8}$$

EXERCISE 7D

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2 **a** $3^{2x} - 3^{x+2} + 20 = 0$

$(3^x)^2 - 3^2 \times 3^x + 20 = 0$

Let $y = 3^x$

$y^2 - 9y + 20 = 0$

b $(y-5)(y-4) = 0$

$y = 5$ or 4

$x = \log_3 5$ or $\log_3 4$

$x = 1.46$ or 1.26 (3 s. f.)

3 $(2^x)^2 - 10 \times 2^x + 16 = 0$

Let $y = 2^x$: $y^2 - 10y + 16 = 0$ therefore $(y-2)(y-8) = 0$

$y = 2$ or 8 so $2^x = 2$ or 8 thus $x = 1$ or 3

4 $2e^{2x} - 9e^x + 4 = 0$

Let $y = e^x$: $2y^2 - 9y + 4 = 0$ therefore $(2y-1)(y-4) = 0$.

$y = \frac{1}{2}$ or 4 so $e^x = \frac{1}{2}$ or 4 thus $x = \ln\left(\frac{1}{2}\right) = -\ln 2$ or $\ln(4) = 2\ln 2$

5 $5^{2x+1} - 14 \times 5^x - 3 = 0$

Let $y = 5^x$: $5y^2 - 14y - 3 = 0$ therefore $(5y+1)(y-3) = 0$

$y = -\frac{1}{5}$ or 3

Reject negative solution, as there is no real value of x for which $5^x < 0$

$y = 3 = 5^x$ therefore $x = \log_5 3$

6 $e^x = 8 - 15e^{-x}$

Let $u = e^x$ then $u = 8 - \frac{15}{u}$

$u^2 - 8u + 15 = (u-3)(u-5) = 0$

$u = 3$ or 5

$x = \ln 3$ or $\ln 5$

MIXED PRACTICE 7

1 $\log_5\left(\sqrt{x^2+49}\right) = 2$ so $\sqrt{x^2+49} = 5^2 = 25$

$x^2 + 49 = 25^2 = 625$ therefore $x^2 = 625 - 49 = 576$

$x = \pm\sqrt{576} = \pm 24$

2 $8^x = 16$ so $2^{3x} = 2^4$ thus $3x = 4$ therefore $x = \frac{4}{3}$.

3 **a** $\log\left(\frac{x^2\sqrt{y}}{z}\right) = \log x^2 + \log\sqrt{y} - \log z$

$= 2\log x + \frac{1}{2}\log y - \log z = 2a + \frac{b}{2} - c$

b $\log\sqrt{0.1x} = \frac{1}{2}\log 0.1x = \frac{1}{2}(\log x + \log 0.1) = \frac{1}{2}(a-1)$

c To answer this you use the rules for changing the base of a logarithm:

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$$\log_a x = \frac{\log x}{\log a}$$

$$\log_{100}\left(\frac{y}{z}\right) = \frac{\log\left(\frac{y}{z}\right)}{\log 100} = \frac{\log y - \log z}{2} = \frac{b-c}{2}$$

Tip

To answer the question directly without recourse to the change of base rule, you can give the answer as an unknown and then keep rearranging to work out its value:

Let $\log_{100}\left(\frac{y}{z}\right) = k$ so $100^k = \frac{y}{z}$ therefore $10^{2k} = \frac{y}{z}$

$2k = \log\left(\frac{y}{z}\right) = \log y - \log z$ thus $k = \frac{\log y - \log z}{2} = \frac{b-c}{2}$

4 $e^{2x+1} = \frac{17}{3}$

$2x + 1 = \ln\left(\frac{17}{3}\right)$ therefore $x = \frac{1}{2}\left(\ln\left(\frac{17}{3}\right) - 1\right) = 0.367$

5 $4\log_a x = \log_a 81$

$\log_a x^4 = \log_a 81$

$x^4 = 81$

$x = 3$

Tip

We do not include the negative root because x is the argument of a logarithm in the question, so must be positive.

6 $\log_x 4 = 9$ so $4 = x^9$ therefore $x = 4^{\frac{1}{9}} \approx 1.17$

7 $\log_a y + \log_a 7 = 4$

$\log_a 7y = 4$ so $7y = a^4$

$y = \frac{a^4}{7}$

8 $3^{2x-1} = 4$

$(2x-1)\ln 3 = \ln 4$

$2x = 1 + \frac{\ln 4}{\ln 3} = \frac{\ln 3 + \ln 4}{\ln 3} = \frac{\ln 12}{\ln 3}$

$x = \frac{\ln 12}{2\ln 3} = \frac{\ln 12}{\ln 9}$

9 $\left.\begin{matrix}\ln x + 2\ln y = 8 \\ 2\ln x + \ln y = 6\end{matrix}\right\}\begin{matrix}(1)\\(2)\end{matrix}$

$(1) - 2 \times (2) : -3\ln x = -4$

$\ln x = \frac{4}{3}$ so $x = e^{\frac{4}{3}} = 3.79$

$\ln y = 6 - 2\ln x = \frac{10}{3}$ so $y = e^{\frac{10}{3}} = 28.0$

10 $4\log_b x - \log_b 9 = 2$

$\log_b x^4 - \log_b 9 = \log_b b^2$ therefore $\log_b\left(\frac{x^4}{9}\right) = \log_b(b^2)$

$\frac{x^4}{9} = b^2$ so $b = \frac{x^2}{3}$

(Negative root is not valid since b is the base of a logarithm.)

11 $\log_5 y = 8$ so $\log_5(125y) = \log_5(125) + \log_5 y = 3 + 8 = 11$

12 $y = \ln x - \ln(x+2) + \ln(4 - x^2)$

$y = \ln\left(\dfrac{x(4-x^2)}{x+2}\right)$ so $y = \ln\left(\dfrac{x(2+x)(2-x)}{2+x}\right)$

$y = \ln(2x - x^2)$ therefore $x^2 - 2x = -\mathrm{e}^y$

Quadratic equation: $x^2 - 2x + \mathrm{e}^y = 0$

$x = \dfrac{2 \pm \sqrt{2^2 - 4\mathrm{e}^y}}{2}$ therefore $x = 1 \pm \sqrt{1 - \mathrm{e}^y}$

Tip

Since $y = \ln(2x - x^2)$, you can check this answer by considering the quadratic curve of $\mathrm{f}(x) = 2x - x^2$, which has roots at $x = 0$ and 2 and is positive between them.

As x approaches 0 or 2, $y = \ln \mathrm{f}(x)$ approaches $-\infty$, which accords with the solution above since as $y \to -\infty$, $x \to 1 \pm 1$.$\mathrm{f}(x)$ has its maximum at $\mathrm{f}(1) = 1$, which you see for $y = \ln \mathrm{f}(1) = 0$, so that $x = 1 \pm \sqrt{1-1} = 1$.

13 $3^{2x} - 3^{x+1} - 10 = 0$

$(3^x)^2 - 3 \times 3^x - 10 = 0$

$(3^x - 5)(3^x + 2) = 0$

$3^x = 5$ or -2

Reject negative solution

$x = \dfrac{\ln 5}{\ln 3} \approx 1.46$

14 $\log(x^2 + 1) = 1 + 2\log(x)$

$\log(x^2 + 1) - \log(x^2) = 1$

$\log\left(\dfrac{x^2+1}{x^2}\right) = 1$ so $\dfrac{x^2+1}{x^2} = 10$

$10x^2 = x^2 + 1$

$9x^2 = 1$

$x = \pm\dfrac{1}{3}$

Check the validity.

The original problem contains $\log(x)$ so reject negative solutions of x.

The only solution is $x = \dfrac{1}{3}$.

15 $3^{2x+1} = 5^{200}$

$(2x+1)\ln 3 = 200 \ln 5$ so $2x = \dfrac{200 \ln 5}{\ln 3} - 1$

$x = \left(\dfrac{100 \ln 5}{\ln 3}\right) - \dfrac{1}{2}$ therefore $x = 146$

16 a i $\log_a(xy) = \log_a x + \log_a y = p + q$

ii $\log_a\left(\dfrac{a^2x^3}{y}\right) = \log_a a^2 + \log_a x^3 - \log_a y = 2 + 3p - q$

b i $\log_a(x^2 - 10) - \log_{10} x = \log_{10}\left(\dfrac{x^2 - 10}{x}\right)$

ii $\log_{10}\left(\dfrac{x^2-10}{x}\right) = 2\log_{10}3 = \log_{10}9$ so $\dfrac{x^2-10}{x} = 9$

$x^2 - 10 = 9x$ therefore $x^2 - 9x - 10 = 0$

$(x-10)(x+1) = 0$ so $x = 10$ or -1

Reject negative solution since $\log x$ appears in question, so $x = 10$.

Check solution:

$$\begin{aligned}\log_{10}(x^2-10) - \log_{10}x &= \log_{10}(100-10) - \log_{10}(10)\\ &= \log_{10}(90) - \log_{10}(10) = \log_{10}(9)\end{aligned}$$

Tip

Remember it is always wise to verify that a solution is valid for the original question in any problem involving logarithms or roots.

17 $2^{3x-2} \times 3^{2x-3} = 36^{x-1}$

Since $36 = 2^2 \times 3^2$, this can be rewritten as:

$2^{3x-2} \times 3^{2x-3} = 2^{2x-2} \times 3^{2x-2}$ so $2^x = 3$

$x\ln 2 = \ln 3$ therefore $x = \dfrac{\ln 3}{\ln 2}$

18 $\dfrac{5}{4} \times 4^x = 3^{-2x} = 9^{-x}$

Collecting all instances of x to one side by multiplication and division:

$\dfrac{5}{4} = 9^{-x} \times 4^{-x} = 36^{-x}$ therefore $36^x = \dfrac{4}{5}$

$x\ln 36 = \ln\left(\dfrac{4}{5}\right)$ so $x = \dfrac{\ln\left(\dfrac{4}{5}\right)}{\ln 36}$

19 $e^x + e^{-x} = 4$

Let $y = e^x$: $y + \dfrac{1}{y} = 4$ so $y^2 - 4y + 1 = 0$ therefore $(y-2)^2 - 3 = 0$

$y - 2 = \pm\sqrt{3}$ so $y = 2 \pm \sqrt{3}$ therefore $e^x = 2 \pm \sqrt{3}$

$x = \ln(2 \pm \sqrt{3})$

20 $(\log_3 x)^2 = \log_3 x^3 - 2$

$(\log_3 x)^2 = 3\log_3 x - 2$

$(\log_3 x - 1)(\log_3 x - 2) = 0$

$\log_3 x = 1$ or $\log_3 x = 2$

$x = 3$ or $x = 9$

Chapter 8 worked solutions

8 Exponential models

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 8A

5 Gradient $= 1.5\mathrm{e}^{1.5x}$

a When $x = -2.1$, gradient $= 1.5\mathrm{e}^{1.5\times(-2.1)} = 0.0643$

b When $y = 12$, gradient $= 1.5\mathrm{e}^{1.5\times 12x} = 1.5 \times 12 = 18$

6 Gradient $= -2.3\mathrm{e}^{-2.3x} = -2.3y$

a When $y = 0.5$, gradient $= -1.15$

b When gradient $= -2.5$, $-2.3\mathrm{e}^{-2.3x} = -2.5$

$-2.3x = \ln\left(\dfrac{-2.5}{-2.3}\right)$ therefore $x = -\dfrac{1}{2.3}\ln\left(\dfrac{-2.5}{-2.3}\right) = -0.0363$

7 Gradient $= ke^{kx} = ky$

a $26 = 8k$ therefore $k = \dfrac{26}{8} = 3.25$

b When $x = -1$, gradient $= 3.25\mathrm{e}^{3.25\times(-1)} = 0.126$

8 Gradient $= ae^{ax} = ay$

$4.6a = -1.2$ therefore $a = -\dfrac{1.2}{4.6} = -0.261$

When gradient $= -5$, $ae^{ax} = -5$

$ax = \ln\left(-\dfrac{5}{a}\right)$ therefore $x = \dfrac{1}{a}\ln\left(-\dfrac{5}{a}\right) = -11.3$

9 **a** $8^x = (\mathrm{e}^{\ln 8})^x = \mathrm{e}^{x\ln 8}$ therefore $k = \ln 8$

b For $y = \mathrm{e}^{x\ln 8}$, gradient $= (\ln 8)\mathrm{e}^{x\ln 8} = (\ln 8)y$

When $x = -0.5$, gradient $= (\ln 8) \times 8^{-0.5} = 0.735$

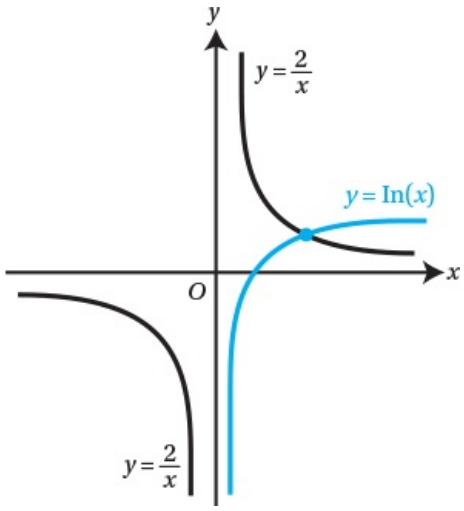
10 **a** $0.3^x = \mathrm{e}^{x\ln 0.3}$ therefore $p = \ln 0.3$

b For $y = \mathrm{e}^{x\ln 0.3}$, gradient $= (\ln 0.3)\mathrm{e}^{x\ln 0.3} = (\ln 0.3)y$

When $y = 0.065$, gradient $= (\ln 0.3) \times 0.065 = -0.0783$

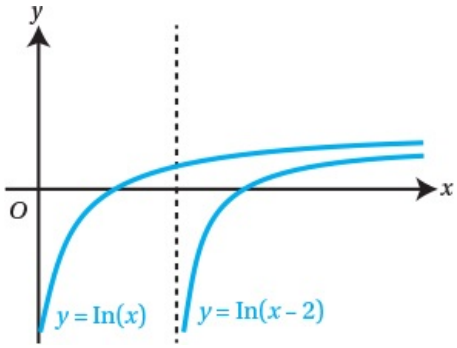
EXERCISE 8B

1 **a** Standard plot of reciprocal and logarithm graphs.



b There can be only one intersection: for $x > 0$, the reciprocal is always decreasing (towards zero) and the logarithm always increasing (without limit, from an asymptote at $x = 0$) so they must cross once and only once. The logarithm has no curve for $x < 0$.

2 Plot of logarithm (axis intercept at $(1, 0)$ and vertical asymptote at $x = 0$) and the same logarithm graph after a translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (axis intercept at $(3, 0)$ and vertical asymptote at $x = 2$).

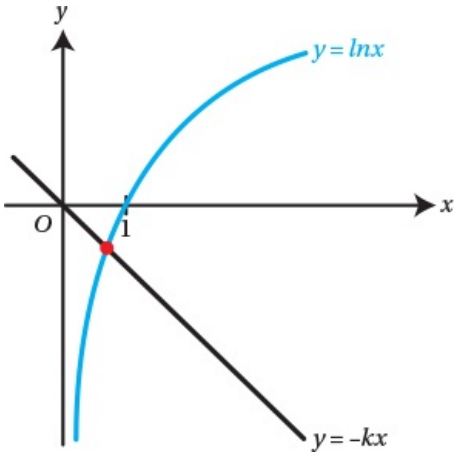


3 Draw a graph of $y = -kx$, which for positive k is a negative straight line, passing through the second and fourth quadrants.

But the logarithm graph $y = \ln(x)$ is always increasing and exists only in the fourth and first quadrants.

So there must be exactly one intersection point, in the fourth quadrant.

That is, there must be exactly one value for which $-kx = \ln(x)$, and so exactly one solution to $kx + \ln(x) = 0$.



4 $y = \ln(x)$ to $y = \ln(3x) = \ln(x) + \ln(3)$

Horizontal stretch of scale factor k causes a replacement of x with $\frac{x}{k}$ in a function, so

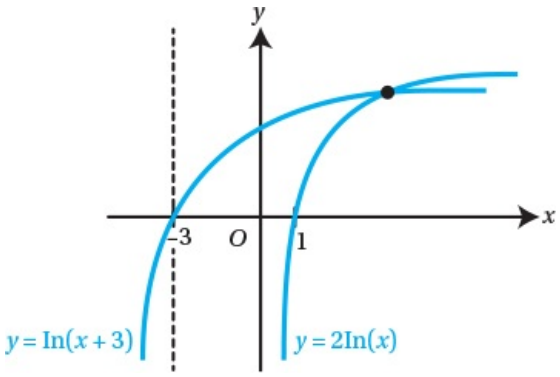
$y = \ln(x)$ to $y = \ln(3x)$ is a horizontal stretch, scale factor $\frac{1}{3}$.

A vertical translation $\begin{pmatrix} 0 \\ c \end{pmatrix}$ c causes $\mathrm{f}(x)$ to become $\mathrm{f}(x) + c$.

$y = \ln(x)$ to $y = \ln(x) + \ln(3)$ is a translation $\begin{pmatrix} 0 \\ \ln(3) \end{pmatrix}$.

5 **a** $y = 2\ln x$ is the normal logarithm function, under a vertical stretch of 2. It still has the same asymptote at $x = 0$ and root at $(1, 0)$.

$y = \ln(x + 3)$ is the logarithm function after a translation $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$, so that the asymptote is now $x = -3$ and the root at $(-2, 0)$. The graph also now has a y-intercept, at $\ln(3)$.



b Algebraically:

$2\ln(x) = \ln(x + 3)$ therefore $\ln(x^2) = \ln(x + 3)$

$x^2 = x + 3$ so $x^2 - x - 3 = 0$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \times (-3)}}{2} = \frac{1 \pm \sqrt{13}}{2}$$

Checking solutions, you find that the negative root gives rise to a negative value for x, and since $\ln x$ is an expression in the question, and $\ln x$ cannot take a negative value of x, you reject this as a false solution:

$$x = \frac{1 + \sqrt{13}}{2}$$

Tip

Any time you use algebra to solve a question which involves logarithms or roots, you must check the validity of your solutions. Very often, the act of squaring (as in this case) or exponentiating to remove logarithms will introduce a false solution which must be eliminated. It is clear from the graphs drawn in question 5a that there can be only one real solution to this question.

6 **a** $10^{\log_{10}(x)} = 10^{k\ln(x)}$ therefore $x = 10^{k\ln(x)}$.

Taking natural logarithms of both sides: $\ln(x) = \ln(10^{k\ln(x)}) = k\ln(x) \times \ln(10)$

Cancelling $\ln(x)$ and rearranging: $1 = k \times \ln(10)$ so $k = \frac{1}{\ln(10)}$.

b Transformation from $y = \ln(x)$ to $y = \frac{1}{\ln(10)}\ln(x)$ is a vertical stretch, scale factor $\frac{1}{\ln(10)}$.

EXERCISE 8C

1 **a** **i** After 1 year, the investment has had a $p\%$ increase applied once: $C\left(1 + \frac{p}{100}\right)$

ii After 1 year, the investment has had a $\frac{p}{2}\%$ increase applied twice: $C\left(1+\frac{p}{200}\right)^2$

iii After 1 year, the investment has had a $\frac{p}{4}\%$ increase applied four times: $C\left(1+\frac{p}{400}\right)^4$

b Each time the $\frac{p}{n}\%$ increase is applied, the balance is multiplied by

$\left(1+\frac{1}{n}\times\frac{p}{100}\right)=\left(1+\frac{p}{100n}\right)$

If this increase is to be applied, compounding, n times then the balance is multiplied by this factor $n$times, that is – raised to the nth power.

$V=C\left(1+\frac{p}{100n}\right)^n$

c Trying a table of values for n:

n	$\left(1+\frac{1}{n}\right)^n$
1	2
3	$2.370\,37$
10	$2.593\,742$
30	$2.674\,319$
100	$2.704\,814$
300	$2.713\,765$
1000	$2.716\,924$
3000	$2.717\,829$
$10\,000$	$2.718\,146$

The value appears to tend towards a recognised number: e.

d For $p=100$ in the original scenario, with continuous interest (effectively infinite n), each year the value would increase by a factor of e, so after x years the value would be $V=C\times\mathrm{e}^x$.

2 **a** $N(0)=100$

b $N(6)=100\mathrm{e}^{6.18}=48\,300$

c $N=1000=100\mathrm{e}^{1.03t}$

$1.03t=\ln(10)$ therefore $t=\frac{1}{1.03}\ln(10)=2.24\,\mathrm{h}$

It will take just under two and a quarter hours to reach 1000 cells.

d Growth rate (gradient) equals $1.03N$ so at that point the growth rate would be 1030 cells per hour.

3 $y(0)=k=10$

$y(3)=10\times1.1^3=13.31\,\mathrm{m}^2$

4 $N=2\mathrm{e}^{0.1t}+1$

a $N(0)=2+1$

Currently 3 billion mobile phones in the world.

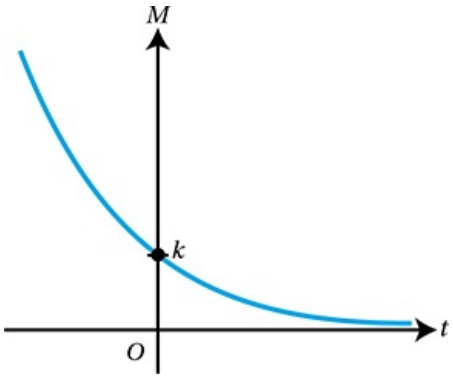
b $N(10)=2\mathrm{e}+1=6.44$

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In ten years, 6.44 billion mobile phones in the world.

5 $M = k\mathrm{e}^{-0.01t}$

a Standard negative exponential graph, y-intercept at k.



b Require $\mathrm{e}^{-0.01t} = 0.25$

$-0.01t = \ln(0.25) = -\ln(4)$ so $t = 100\ln(4) = 139$ seconds (2.3 minutes)

6 **a** Initial population size $N(0) = N_0 = 450$

$\text{gradient} = aN$ and at $t = 0$, gradient is $90\,\text{per hour} = 2160\,\text{per day}$ so $a = 4.8$

b $\text{Gradient} = aN$ so when $N = 750$, $\text{gradient} = 3600\,\text{per day}$

$(150\,\text{per hour})$.

c $450\mathrm{e}^{4.8t} = 2000$ therefore $\mathrm{e}^{4.8t} = 4.44$

$4.8t = 1.49$ so $t = 0.311\,\text{days} = 7.5\,\text{hours}$

7 As the rate of change of N is proportional to N, this is an exponential model.

You can write $N = A\mathrm{e}^{-kt}$, where N is the number of atoms after t days. The negative exponent indicates exponential decay.

A gives the initial value, so $A = 500$.

The rate of change is $-kN$, which equals $-0.4N$, so $k = 0.4$.

Hence the model is $N = 500\mathrm{e}^{-0.4t}$.

8 **a** $V = P\mathrm{e}^{kt}$, where t is measured in years.

$V(0) = 6800 = P$

$V(1) = 5440 = P\mathrm{e}^{k}$

$k = \ln\left(\dfrac{5440}{6800}\right) = \ln(0.8) = -0.223\,\text{therefore}$

$V = 6800\mathrm{e}^{-0.223t}$

b From the model, $V(10) = 730$. The model predicts a value of £730.

9 Exponential growth: $P = k \times a^n$

a $\mathrm{P}(0) = 7 \times 10^7 \Rightarrow k = 7 \times 10^7$

Annual growth factor a is $102\% = 1.02$ so $P = 7 \times 10^7 \times 1.02^n$

b Birth/death does not cover all causes of population change – net migration must also be considered.

There is no reason to believe this simple model will extend all the way to the end of the century.

10 **a** $T = ka^t$

$T(0) = k \Rightarrow k = 35$

Measuring t in minutes: $T(5) = 0.7 \times T(0)$

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$ka^5 = 0.7k \Rightarrow a = 0.7^{0.2}$

$T(7) = 35a^7 = 35 \times 0.7^{1.4} = 21.2$ so soup will be at 21.2°C

b **i** k is determined by the start temperature of the soup, so is unchanged.

ii a determines the rate of decrease of temperature; in a thermos, a would be a larger value, though still $a < 1$, since the temperature will still decrease, albeit more slowly.

11 $V = 40(1 - 3^{-0.1t})$

a $V(0) = 40(1 - 1) = 0\,\text{m s}^{-1}$

b As $t \to \infty$, $V \to 40(1 - 0) = 40\,\text{m s}^{-1}$

12 **a** $\dfrac{I(1)}{I(0)} = \dfrac{100\text{e}^{-2}}{100} = \text{e}^{-2}$

b $\dfrac{I(x+1)}{I(x)} = \dfrac{100\text{e}^{-2-2x}}{100\text{e}^{-2x}} = \text{e}^{-2}$

The factor of change every metre is always e^{-2}.

EXERCISE 8D

3 **a** $N = A\text{e}^{kt}$ so $\log N$ against t will produce a straight line.

b $N(0) = 150 = A\text{e}^0$ therefore $A = 150$

c $N(10) = 780 = 150\text{e}^{10k}$ therefore $k = 0.1 \ln\left(\dfrac{780}{150}\right) = 0.165$

d As the numbers increase, constraints of space and resources, build-up of wastes will limit the accuracy of the model.

Tip

Exponential-like growth is typically seen at the start of a population's introduction into a system, and as they approach the natural capacity of the system, growth will decrease. More complicated population models (such as the logistic) are used to predict population dynamics beyond the early stages.

4 **a** It is the initial number of bacteria. (This is because, when $t = 0$, $\text{e}^{bt} = 1$.)

b The rate of growth is bN. When $t = 0$, $N = 1500$, so:

$$1500b = 500 \Rightarrow b = \frac{1}{3}$$

c We need to find t when $N = 1\,000\,000$.

$$1500\text{e}^{\frac{1}{3}t} = 1\,000\,000$$
$$\frac{1}{3}t = \ln\left(1\,000\,\frac{000}{1500}\right)$$
$$t = 19.5\,\text{hours}$$

5 **a** Gradient of line: $\dfrac{\text{change in } \ln(M)}{\text{change in } T} = -\dfrac{0.81}{5} = -0.16$

Equation of line is $\ln M = -0.16T + c$ so $c = \ln M + 0.16T$

Substituting $(1,\ 1.83)$ gives $c = 1.99 \approx 2.0$

Line is $\ln M = 2 - 0.16T$

b $M = \text{e}^{2-0.16T} = \text{e}^2 \times \text{e}^{-0.16T}$.

$K = \text{e}^2 = 7.4$, $c = \text{e}^{-0.16} = 0.85$

c Require $M < 1$

$Kc^T < 1$ so $c^T < \dfrac{1}{K}$

$T \ln c < \ln\left(\dfrac{1}{K}\right)$ so $T > -\dfrac{\ln K}{\ln c}$

(Dividing by $\ln c < 0$ causes the inequality to reverse.)$T > -\dfrac{2}{-0.16} = 12.5$

It will take 13 seconds for the mass of the substance to fall below 1 gram.

6 **a** Line of best fit from graph should be approximately $y = 1.2x + 2$. Reasonable tolerance and follow-through marks would be expected in an examination.

b $P = \mathrm{e}^2 \times t^{1.2}$ thus $C = \mathrm{e}^2 = 7.4$, $n = 1.2$ $\ln P = 1.2 \ln t + 2 = \ln(\mathrm{e}^2 \times t^{1.2})$

c $P > 200$ so $1.2 \ln t = \ln P - 2 > \ln(200) - 2$

$\ln t > \dfrac{\ln(200) - 2}{1.2} \approx 2.75$ therefore $t > 15.6$

It would take approximately 15.6 years for the population to exceed $200\,000$.

7 **a** Using the equation of the line:

$\ln N = 5.8 + 0.16t$

$N = \mathrm{e}^{5.8+0.16t} = \mathrm{e}^{5.8} \times \mathrm{e}^{0.16t} = 330\mathrm{e}^{0.16t}$ (3 s. f.)

b The rate of growth is $0.16N$.

When $t = 8$ weeks, $N = 330\mathrm{e}^{0.16\times 8} = 1188\,(4\text{ s. f.})$.

So the rate of growth is $0.16N = 190$ mice per week (3 s. f.).

8 **a** You want to find an equation of the form $y = mt + c$. To get a term of the form mt, take a logarithm of e^{-bt}.

$$N = \frac{1}{1 + J\mathrm{e}^{-bt}}$$

$$\frac{1}{N} = 1 + J\mathrm{e}^{-bt}$$

$$\frac{1}{N} - 1 = J\mathrm{e}^{-bt}$$

$$\frac{1-N}{N} = J\mathrm{e}^{-bt}$$

Taking logarithms:

$$\ln\left(\frac{1-N}{N}\right) = \ln J - bt$$

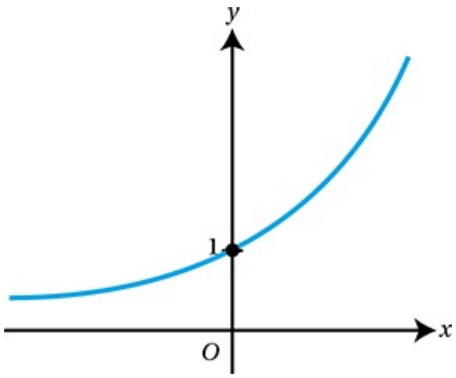
So, if you plot $y = \ln\left(\dfrac{1-N}{N}\right)$ against t, the graph will be a straight line with gradient $-b$ and y-intercept ln J.

b The logistic function predicts that the population will level off (to one unit), which is more realistic than the exponential model, which predicts unlimited population growth.

MIXED PRACTICE 8

1 **a** Positive exponential, intercept at $(0, 1)$.

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b At $x = 3$, gradient $= 0.8\mathrm{e}^{3\times 0.8} = 8.82$

c The graph $y = \mathrm{e}^{0.8x}$ is always increasing, and exists in the upper two quadrants. The graph of $y = \dfrac{1}{x}$ exists in the upper right quadrant and is always decreasing, and otherwise only exists in the lower left quadrant. There can only be one intersection, in the upper right quadrant.

2 $m = 32\mathrm{e}^{-0.14t}$

a $m(0) = 32$

There is $32\,\mathrm{g}$ of the substance at the start of the reaction.

b Gradient $= -0.14m$

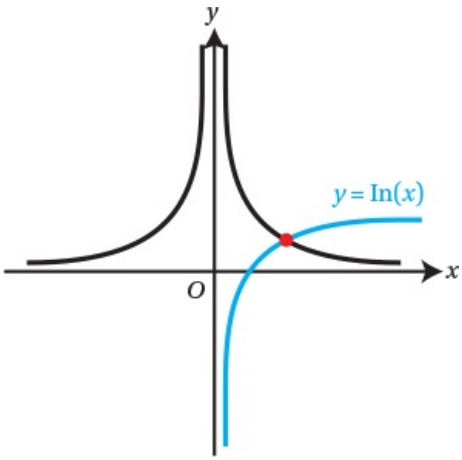
At $t = 3$, gradient $= -0.14 \times 32\mathrm{e}^{-0.14\times 3} = -2.9\,\mathrm{g\,s^{-1}}$ $(2\,\mathrm{s.f.})$

c Require $\mathrm{e}^{-0.14t} = 0.5$ therefore $t = -\dfrac{1}{0.14}\ln(0.5) = 4.95$

After $5.0\,\mathrm{s}$ the amount will have halved.

3 $y = \ln x$: standard logarithm curve, asymptote at $x = 0$ and root at $(1,\, 0)$, always increasing

$y = \dfrac{3}{x^2}$: reciprocal curve in upper left and upper right quadrants, asymptote at $x = 0$, always decreasing in upper right quadrant, always increasing in upper left.



There is a single intersection, and so a single solution to $\ln x = \dfrac{3}{x^2}$.

4 $V = 0.4 \times 2^{0.1t}$

a $V(0) = 0.4\,\mathrm{cm^3}$

b Require $2^{0.1t} = 2 \Rightarrow t = 10$.

After 10 weeks, the volume will have doubled.

c The model is for volume in a jar; projecting ten years by the model would exceed the capacity of any reasonable jar, which would make the model invalid.

Any exponential growth model will fail when food or space resources become limited, or when

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levels of by-products or waste become toxic and inhibit growth.

The model is based on weeks of growth and it would not be reasonable to expect such growth to continue indefinitely. Indeed, entering ten years into the model would project the algal blob to have a volume of approximately 2 cubic kilometres!

5 $N = Ae^{kt}$

a $N(0) = 18 = Ae^0$ therefore $A = 18$

$N(60) = 42 = 18e^{60k}$ therefore $k = \frac{1}{60}\ln\frac{42}{18} = 0.0141$ (3s. f.)

b $N(90) = 18e^{90k} = 64.2$

Rounding gives 64 people.

c $N(t) = 1200 = 18e^{kt}$ therefore $t = \frac{1}{k}\ln\frac{1200}{18} = 297.9$

After 298 minutes, i.e. at 13.58, the whole school population will know the rumour.

6 $I = 10e^{-0.05t} + 2$

a $I(0) = 10 + 2 = 12\ \mu\text{U ml}^{-1}$

b By the model, $I > 2$ at all times, ($I \to 2$ as $t \to \infty$) so the insulin level will never reach 1.8.

7 **a** $T = ka^n$

i $T(10) = 22\,000 = ka^{10}$

ii $T(0) = 37\,000 = k$

$a^{10} = \frac{22}{37}$ therefore $a = \left(\frac{22}{37}\right)^{0.1} \approx 0.949$

b $T(50) = ka^{50} = 37\,000 \times \left(\frac{22}{37}\right)^5 \approx 2750$

c $T < 1000$

$ka^n < 1000$ therefore $a^n < \frac{1}{37}$

$n \ln a < -\ln 37$ so $n > -\frac{\ln 37}{\ln a} = 69.0$

By the year 2039, the tiger population would be near extinction, according to the model.

8 **a** $N = Ae^{kt}$

Then $\ln N = \ln A + kt$, so a plot of $\ln N$ against t should give a straight line graph with vertical intercept $\ln A$ and gradient k.

b

T	0	10	20	30	40	50
$\ln N$	3.56	3.74	3.83	3. 93	4.13	4.26

c Gradient, $k \approx \frac{4.26 - 3.56}{50 - 0} = 0.014$

Tip

Once you have studied linear regression you can calculate the best fit line for these data values, but currently you can approximate using the gradient between the most distant values.

$A = e^{3.56} \approx 35$ (2 s. f.)

d Under this model, $N(260) = 35e^{0.014 \times 260} = 1330$ fish.

e The model is based on data over 50 days; extending it so far beyond the data is bound to be

unreliable; limits on population growth such as space and food resources, waste build-up, as well as seasonal (climatic and behavioural) effects which might not be observed in the small population over the two-month period could have a major impact by the time 9 months have passed.

9 $m = at^p$ so $\log m = \log a + p \log t$

Gradient $= \dfrac{5-0}{2-4} = -2.5 = p$

Line equation: $\log m - 0 = -2.5(\log t - 4)$

$\log m = -2.5 \log t + 10$ so $\log a = 10$ therefore $a = 10^{10}$.

10 $m = 240\mathrm{e}^{-0.04t}$

a When $t = T$, the mass has halved: $240\mathrm{e}^{-0.04T} = 0.5m(0) = 120$

$\mathrm{e}^{-0.04T} = 0.5$ therefore $0.04T = \ln(2)$

$T = 25 \ln(2) = 17.3$ years

b $$\text{Gradient} = -0.04 \times \text{mass}$$

$$-0.04 \times 240\mathrm{e}^{-0.04t} = -2.1 \text{ therefore } -0.04t = \ln\left(\frac{2.1}{0.04 \times 240}\right)$$

$$t = 50 \times \ln\left(\frac{0.04 \times 240}{2.1}\right) = 76.0$$

11 a $M = 40\mathrm{e}^{kt}$

When $t = 0$, $M = 40$.

You can find k from the second pair of values: $80 = 40\mathrm{e}^{21k}$

$\mathrm{e}^{21k} = 2$ so $21k = \ln 2$ therefore $k = 0.033$

The last value in the table is $M = 40\mathrm{e}^{0.033 \times 63} = 320$.

b The rate of increase is k times the value of M.

When $t = 21$, $M = 80$, so the rate of increase is $0.03 \times 80 = 2.4$.

12 Half-life of 260 years:

$$m(260) = 0.5m(0) \text{ so } m_0\mathrm{e}^{-260k} = 0.5m_0$$
$$-260k = \ln(0.5) \text{ therefore } 260k = \ln(2)$$
$$k = \frac{1}{260}\ln(2) = 0.00267$$

13 a $V(0) = 0$

b The upper bound on the speed is $V = 42\,\mathrm{m\,s^{-1}}$.

c $V = 22 = 42(1 - \mathrm{e}^{-0.2t})$ so $42\mathrm{e}^{-0.2t} = 20$

$-0.2t = \ln\left(\dfrac{20}{42}\right)$ therefore $t = 5 \ln\left(\dfrac{42}{20}\right) \approx 3.71\,\mathrm{s}$.

14 $T - 22 = C\mathrm{e}^{-kt}$

a $T(0) = 98$, $T(2) = 94$

$$76 = C\mathrm{e}^0 = C$$
$$72 = C\mathrm{e}^{-2k}$$
$$\mathrm{e}^{-2k} = \frac{72}{76} \text{ therefore } k = -\frac{1}{2}\ln\left(\frac{72}{76}\right) \approx 0.027$$

b Require $T - 22 < 56$.

$76\mathrm{e}^{-kt} < 56$ so $\mathrm{e}^{-kt} < \dfrac{56}{76}$ therefore $t > -\dfrac{1}{k}\ln\left(\dfrac{56}{76}\right) \approx 11.3$

It takes approximately 11.3 minutes for the tea to cool to $78^\circ\mathrm{C}$.

Chapter 9 worked solutions

9 Binomial expansion

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 9A

General comment: Throughout these worked solutions, the notation $\binom{n}{r}$ will be used instead of ${}^nC_r$, although questions and text in the Student Book and examination may use either. Students are free to choose their preferred notation for use in answers in examinations, irrespective of the form of the question.

Tip

It is wise always to lay out an answer using the general term of the expansion first, before evaluating indices and coefficients. This avoids simple errors with signs and the application of indices, and also makes the working easy to understand.

5 $(3+x)^3 = {}^3C_0(3)^3 + {}^3C_1(3)^2(x) + {}^3C_2(3)(x)^2 + {}^3C_3(x)^3$
$= 3^3 + 3(9)(x) + 3(3)(x^2) + x^3$
$= 27 + 27x + 9x^2 + x^3$

6 The general term of $(1-2y)^5$ has the form $\binom{5}{r}1^{5-r}(-2y)^r$.

The coefficient of this term is $\binom{5}{r}(-2)^r$.

a $\binom{5}{r}(-2)^r = 80$ which gives $r = 4$.

The term is $\binom{5}{4}(-2y)^4 = 80y^4$.

b $\binom{5}{r}(-2)^r = -80$ which gives $r = 3$.

The term is $\binom{5}{3}(-2y)^3 = -80y^3$.

7 General term of $(1-5x)^9$ has the form $\binom{9}{r}1^{9-r}(-5x)^r$

Require coefficient of x^3 so $r = 3$

Term is $\binom{9}{3}(-5x)^3 = 84 \times (-125x^3) = -10\,500x^3$. Coefficient is $-10\,500$

8 General term of $(3-2x)^7$ has the form $\binom{7}{r}(3)^{7-r}(-2x)^r$.

Require x^2 term so $r = 2$.

Term is $\binom{7}{2}(3)^5(-2x)^2 = 21 \times 243 \times 4x^2 = 20\,412x^2$.

IRANINOVIN

9 $(y+3y^2)^6 = y^6(1+3y)^6$

$= y^6[1\times 1^6 + 6\times 1^5(3y) + 15\times 1^4(3y)^2 + 20\times 1^3(3y)^3 + \ldots]$

$= y^6 + 18y^7 + 135y^8 + 540y^9 + \ldots$

10 a $(1+2x)^4 = {}^4C_0(1)^4 + {}^4C_1(1)^3(2x)$

$+{}^4C_2(1)^2(2x)^2 + {}^4C_3(1)(2x)^3 + {}^4C_4(2x)^4$

$= 1 + 4(2x) + 6(4x^2) + 4(8x^3) + (16x^4)$

$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$

b $\left(\dfrac{x+2}{x}\right)^4 = \left(1+\dfrac{2}{x}\right)^4$

So replace x in the series in part a with $\dfrac{1}{x}$:

$\left(1+\dfrac{2}{x}\right)^4 = 1 + 8\left(\dfrac{1}{x}\right) + 24\left(\dfrac{1}{x}\right)^2 + 32\left(\dfrac{1}{x}\right)^3 + 16\left(\dfrac{1}{x}\right)^4$

$= 1 + \dfrac{8}{x} + \dfrac{24}{x^2} + \dfrac{32}{x^3} + \dfrac{16}{x^4}$

11 General term of $(x+x^{-1})^8$ has the form $\binom{8}{r}(x)^{8-r}(x^{-1})^r = \binom{8}{r}x^{8-2r}$.

Require coefficient of x^2 so $r=3$.

Term is $\binom{8}{3}x^2 = 56x^2$.Required coefficient is 56.

12 a General term of $(x+ay)^n$ has the form $\binom{n}{r}(x)^{n-r}(ay)^r$.

Require $\binom{n}{r}(x)^{n-r}(ay)^r = 60x^4y^2$.

By inspecting the indices of x and y : $n = (n-r)+r = 4+2 = 6$

b By inspecting the index of x or y : $r = 2$.

Require $\binom{6}{2}x^4a^2y^2 = 60x^4y^2 \Rightarrow 15a^2 = 60$.

$a = \pm 2$

13 $\left(2z^2+3z^{-1}\right)^4$

$= \binom{4}{0}\left(2z^2\right)^4 + \binom{4}{1}\left(2z^2\right)^3\left(3z^{-1}\right)^1 + \binom{4}{2}\left(2z^2\right)^2\left(3z^{-1}\right)^2$

$+\binom{4}{3}\left(2z^2\right)^1\left(3z^{-1}\right)^3 + \binom{4}{4}\left(3z^{-1}\right)^4$

$= 1\times 16z^8 + 4\times 8z^6\times 3z^{-1} + 6\times 4z^4\times 9z^{-2}$

$+4\times 2z^2\times 27z^{-3} + 1\times 81z^{-4}$

$= 16z^8 + 96z^5 + 216z^2 + 216z^{-1} + 81z^{-4}$

14 a $\left(3x^2y+5xy^{-1}\right)^n = \binom{n}{0}\left(3x^2y\right)^n + \ldots$

$\binom{n}{0}\left(3x^2y\right)^n = 27x^6y^3 \Rightarrow n = 3$

b $\left(3x^2y+5xy^{-1}\right)^3 = \binom{3}{0}\left(3x^2y\right)^3 + \binom{3}{1}\left(3x^2y\right)^2\left(5xy^{-1}\right)^1$

$+\binom{3}{1}\left(3x^2y\right)^1\left(5xy^{-1}\right)^2 + \binom{3}{3}\left(5xy^{-1}\right)^3$

$= 27x^6y^3 + 135x^5y + 225x^4y^{-1} + 125x^3y^{-3}$

15 General term of $\left(x-2x^{-2}\right)^9$ has the form $\binom{9}{r}(x)^{9-r}(-2x^{-2})^r = \binom{9}{r}(-2)^r x^{9-3r}$.

Require $9 - 3r = 0 \Rightarrow r = 3$.

Term is $\binom{9}{3}(-2)^3 x^0 = -672$.Constant coefficient is -672.

16 General term of $(x^2 - 3x^{-1})^7$ has the form $\binom{7}{r}(x^2)^{7-r}(-3x^{-1})^r = \binom{7}{r}(-3)^r x^{14-3r}$.

Require $14 - 3r = 5 \Rightarrow r = 3$.

Term is $\binom{7}{3}(-3)^3 x^5 = -945x^5$.

17 General term of $(2x - 5x^{-2})^{12}$ has the form

$\binom{12}{r}(2x)^{12-r}(-5x^{-2})^r = \binom{12}{r}2^{12-r}(-5)^r x^{12-3r}$.

Require $12 - 3r = 0 \Rightarrow r = 4$

Term is $\binom{12}{4}2^8(-5)^4 x^0 = 79\,200\,000$. Constant coefficient is $79\,200\,000$.

EXERCISE 9B

3 **a** $(2+x)^n = \binom{n}{0}(2)^n + \binom{n}{1}(2)^{n-1}(x)^1 + \ldots = 32 + ax + \ldots$

$2^n = 32$ therefore $n = 5$

b $\binom{n}{1}2^{n-1} = a$ so $a = 5 \times 2^4 = 80$

4 **a** $(1+2x)^n = \binom{n}{0}(1)^n + \binom{n}{1}(1)^{n-1}(2x)^1 + \binom{n}{2}(1)^{n-2}(2x)^2 + \ldots$

$= 1 + 20x + ax^2 + \ldots$

$n \times 2x = 20x$ so $n = 10$

b $\frac{n(n-1)}{2}(2x)^2 = ax^2$ so $a = 45 \times 2^2 = 180$

5 $\left(1+\frac{x}{2}\right)^n = \binom{n}{0}(1)^n + \binom{n}{1}(1)^{n-1}\left(\frac{x}{2}\right)^1 + \binom{n}{2}(1)^{n-2}\left(\frac{x}{2}\right)^2 + \ldots = 1 + ax + \frac{33}{2}x^2 + \ldots$

a Comparing the x^2 terms:

$\frac{33}{2}x^2 = \frac{n(n-1)}{2} \times \frac{x^2}{4}$

$n^2 - n = 132$ so $(n-12)(n+11) = 0$ therefore $n = 12$

b Comparing x^1 term:

$ax = \binom{12}{1} \times \frac{x}{2}$ therefore $a = 6$

6 $\binom{n}{k} = \binom{n}{k+1}$

$\frac{n!}{(n-k)!k!} = \frac{n!}{(n-k-1)!\,(k+1)!}$ so $(n-k-1)!\,(k+1)! = (n-k)!k!$

$[(n-k-1)!k!]\,(k+1) = [(n-k-1)!k!]\,(n-k)$

$k+1 = n-k$ therefore $n = 2k+1$

7 **a** $(a+b)(a+b)^3 = (a+b)(1a^3 + 3a^2b + 3ab^2 + 1b^3)$

$= 1a^4 + 3a^3b + 3a^2b^2 + 1ab^3$

$+1a^3b + 3a^2b^2 + 3ab^3 + 1b^4$

The coefficient of a^2b^2 in the expansion of $(a+b)^4$ is shown to be 6.

EXERCISE 9C

1 **a** $(1+2x)^5 = 1 \times (1)^5 + 5 \times (1)^4 \times (2x) + 10 \times (1)^3 \times (2x)^2$

$+10 \times (1)^2 \times (2x)^3 + 5 \times (1)^3 \times (2x)^4 + 1 \times (2x)^5$

$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$

b Multiply each term in the above expansion by 3, then by $-x$:

$(3-x)(1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5)$

$= (3 + 30x + 120x^2 + 240x^3 + 240x^4 + 96x^5)$

$-(x + 10x^2 + 40x^3 + 80x^4 + 80x^5 + 32x^6)$

$= 3 + 29x + 110x^2 + 200x^3 + 160x^4 + 16x^5 - 32x^6$

2 **a** $(2)^7 + 7(2)^6(-5x) + 21(2)^5(-5x)^2 + 35(2)^4(-5x)^3$

$= 128 - 2240x + 16\,800x^2 - 70\,000x^3$

b The x^3 term arises in two ways: $1 \times (-70\,000x^3)$ and $2x \times 16\,800x^2$

$-70\,000x^3 + 33\,600x^3 = -36\,400x^3$

The coefficient of x^3 is $-36\,400$.

3 **a** $(3-5x)^4 = 1\times(3)^4 + 4\times(3)^3(-5x) + 6\times(3)^2(-5x)^2 + \ldots$

$= 81 - 540x + 1350x^2 + \ldots$

b

Tip

In this type of question, find the value of x which makes the first part of the question relevant to finding the approximation. In more complicated questions this may require some ingenuity.

Require that $3 - 5x = 2.995 \Rightarrow x = 0.001$

$x^0 = 1$	$81x^0 = 81$
$x^1 = 0.001$	$-540x^1 = -0.54$
$x^2 = 0.000\,001$	$1350x^2 = 0.001\,35$

Hence, approximately, $2.995^4 = 81 - 0.54 + 0.001\,35 = 80.461\,35$

Rounded to $5\,\text{s.f.}$, $2.995^4 \approx 80.461$

4 **a** $8^5 + 5(8)^4(2x) + 10(8)^3(2x)^2 = 32\,768 + 40\,960x + 20\,480x^2$

b You want $8 + 2x = 8.02$, so take $x = 0.01$.

$8.02^5 \approx 32\,768 + 40\,960 \times 0.01 + 20\,480 \times 0.01^2$

$= 32\,768 + 409.6 + 2.048 = 33\,197.648$

5 **a** $(2+3x)^7 = \binom{7}{0}(2)^7 + \binom{7}{1}(2)^6(3x)^1 + \binom{7}{2}(2)^5(3x)^2 + \ldots$

$= 128 + 1344x + 6048x^2 + \ldots$

b **i** Require that $2 + 3x = 2.3 \Rightarrow x = 0.1$

$x^0 = 1$	$128x^0 = 128$
$x^1 = 0.1$	$1344x^1 = 134.4$
$x^2 = 0.01$	$6048x^2 = 60.48$

Hence, approximately, $2.3^7 = 128 + 134.4 + 60.48 = 322.88$

ii Require that $2 + 3x = 2.03 \Rightarrow x = 0.01$

$x^0 = 1$	$128x^0 = 128$
$x^1 = 0.01$	$1344x^1 = 13.44$
$x^2 = 0.0001$	$6048x^2 = 0.6048$

c Hence, approximately, $2.03^7 = 128 + 13.44 + 0.6048 = 142.0448$

The approximation in question 5bii will be more accurate, both on an absolute and on a relative basis, since the discarded terms reduce more rapidly, with high powers of x being less significant to the total.

6 **a** $4^7 + 7(4^6)(-x) + 21(4^5)(-x)^2 = 16\,384 - 28\,672x + 21\,504x^2$

b There are three ways to make an x^2 term:

$2(21\,504x^2) + 2x(-28\,672x) - x^2(16\,384) = -30\,720x^2$.

The coefficient of x^2 is $-30\,720$.

7 **a** General term of $(\mathrm{e} + 2\mathrm{e}^{-1})^5$ has the form $\binom{5}{r}(\mathrm{e})^{5-r}(2\mathrm{e}^{-1})^r = \binom{5}{r}2^r\mathrm{e}^{5-2r}$.

$(e + 2e^{-1})^5 = 1\times e^5 + 5\times 2e^3 + 10\times 4e^1$

$+10\times 8e^{-1} + 5\times 16e^{-3} + 1\times 32e^{-5}$

$(\mathrm{e} + 2\mathrm{e}^{-1})^5 = \mathrm{e}^5 + 10\mathrm{e}^3 + 40\mathrm{e} + \frac{80}{\mathrm{e}} + \frac{80}{\mathrm{e}^3} + \frac{32}{\mathrm{e}^5}$

b Alternate terms will cancel if the signs are opposite or double if the signs are the same:

$$(e + 2e^{-1})^5 + (e - 2e^{-1})^5 = 2e^5 + 80e + \frac{160}{e^3}$$

8 **a** $(1+x)^n(1-x)^n = (1-x^2)^n$

b $$\begin{aligned}(1-x^2)^{10} &= \binom{10}{0}(1)^{10} + \binom{10}{1}(1)^9(-x^2)^1 \\ &\quad + \binom{10}{2}(1)^8(-x^2)^2 + \ldots \\ &= 1 - 10x^2 + 45x^4 + \ldots\end{aligned}$$

9 **a** $(3x^2y + 5xy^{-1})^n = \binom{n}{0}(3x^2y)^n + \ldots \binom{n}{0}(3x^2y)^n = 27x^6y^3 \Rightarrow n = 3$

b $$\begin{aligned}(3x^2y + 5xy^{-1})^3 &= \binom{3}{0}(3x^2y)^3 + \binom{3}{1}(3x^2y)^2(5xy^{-1})^1 \\ &\quad + \binom{3}{1}(3x^2y)^1(5xy^{-1})^2 + \binom{3}{3}(5xy^{-1})^3 \ldots \\ &= 27x^6y^3 + 135x^5y + 225x^4y^{-1} + 125x^3y^{-3}\end{aligned}$$

$27x^6y^3 + 135x^5y + 225x^4y^{-1} + 125x^3y^{-3}$

10 General term of $(1+x)^7$ has the form $\binom{7}{r}x^r$.

Term in x^5 in $(1+3x)(1+x)^7$ will be $1 \times \binom{7}{5}x^5 + 3x \times \binom{7}{4}x^4 = 126x^4$

Coefficient of x^5 is 126.

MIXED PRACTICE 9

1 n is the sum of the two values whose factorials are seen in the denominator: $n = 8$.

2 Term in x^5 is $\binom{12}{7}(2)^7(-x)^5 = 792 \times 128 \times (-x^5) = -101376x^5$. Coefficient x^5 is $-101\,376$.

3 $$\begin{aligned}&\binom{3}{0}(2x^{-1})^3 + \binom{3}{1}(2x^{-1})^2(5y) + \binom{3}{2}(2x^{-1})(5y)^2 + \binom{3}{3}(5y)^3 \\ &= 8x^{-3} + 3(4x^{-2})(5y) + 3(2x^{-1})(25y^2) + 125y^3 \\ &= 8x^{-3} + 3(4x^{-2})(5y) + 3(2x^{-1})(25y^2) + 125y^3 \\ &= 8x^{-3} + 60x^{-2}y + 150x^{-1}y^2 + 125y^3\end{aligned}$$

4 General term of $(3x + 2y^2)^5$ has the form $\binom{5}{r}(3x)^{5-r}(2y^2)^r$.

Require coefficient of x^2y^6 so $r = 3$.

Term is $\binom{5}{3}(3x)^2(2y^2)^3 = 10 \times 9x^2 \times 8y^6 = 720x^2y^6$. Coefficient is 720.

5 $$\begin{aligned}(2-\sqrt{2})^5 &= \binom{5}{0}(2)^5 + \binom{5}{1}(2)^4(-\sqrt{2})^1 \\ &\quad + \binom{5}{2}(2)^3(-\sqrt{2})^2 + \binom{5}{3}(2)^2(-\sqrt{2})^3 \\ &\quad + \binom{5}{4}(2)^1(-\sqrt{2})^4 + \binom{5}{5}(-\sqrt{2})^5 \\ &= 232 - 164\sqrt{2}\end{aligned}$$

6 General term is $\binom{4}{r}(x^3)^r(-2x^{-1})^{4-r} = \binom{4}{r}(-2)^{4-r}x^{4r-4}$

IRANINOVIN

Term in x^0 occurs when $r = 1$, so the coefficient is $\binom{4}{1}(-2)^3 = -32$.

7 $\binom{4}{0}(x^2)^4 + \binom{4}{1}(x^2)^3(-2x^{-1}) + \binom{4}{2}(x^2)^2(-2x^{-1})^2$
$+\binom{4}{3}(x^2)^1(-2x^{-1})^3 + \binom{4}{4}(-2x^{-1})^4$
$= x^8 - 8x^5 + 24x^2 - 32x^{-1} + 16x^{-4}$

8 General term of $(x^2 + ax^{-4})^9$ has the form $\binom{9}{r}(x^2)^{9-r}(ax^{-4})^r = \binom{9}{r}(a)^r x^{18-6r}$

Require $18 - 6r = 0 \Rightarrow r = 3$

Term is $\binom{9}{3}(a)^3 = 84a^3 = -\frac{28}{9}$

$a^3 = -\frac{1}{27}$ therefore $a = -\frac{1}{3}$

9 Term in d^{11} arises from $2\binom{14}{11}d^{11} + 5d\binom{14}{10}d^{10}$.

Coefficient is $2 \times 364 + 5 \times 1001 = 5733$.

10 Term in x^6 arises from $(-x^2)\binom{5}{4}x^4 = -5x^6$. Coefficient is -5.

11 a $(2x+5)^4 = (2x)^4 + 4(2x)^3 5 + 6(2x)^2 5^2 + 4(2x)5^3 + 5^4$
$= 16x^4 + 160x^3 + 600x^2 + 1000x + 625$

b In the second expansion, every other term will have a negative coefficient.

$(2x+5)^4 - (2x-5)^4 =$

$(16x^4 + 160x^3 + 600x^2 + 1000x + 625)$
$-(16x^4 - 160x^3 + 600x^2 - 1000x + 625)$
$= 320x^3 + 2000x$

c $9^4 - (-1)^4 = 6560$ and $7360 - 800 = 6560$, so $x = 2$ is a solution.

$320x^3 - 1680x + 800 = 0$ so $4x^3 - 21x + 10 = 0$

$(x-2)(4x^2 + 8x - 5) = 0$ therefore $(x-2)(2x-1)(2x+5) = 0$

Hence $x = \frac{1}{2}$, $x = -2\frac{1}{2}$.

12 a General term is $\binom{n}{r}(2x)^{n-r}(ay)^r$.

Total of indices is $n = 5$.

b y index is $2 \Rightarrow r = 2$

Term is $\binom{5}{2}(2x)^3(ay)^2 = 80a^2x^3y^2$

$\Rightarrow a^2 = \frac{1}{4} \Rightarrow a = \frac{1(a>0)}{2}$

c $\binom{5}{0}(2x)^5 + \binom{5}{1}(2x)^4\left(\frac{1}{2}y\right)$
$+\binom{5}{2}(2x)^3\left(\frac{1}{2}y\right)^2 + \binom{5}{3}(2x)^2\left(\frac{1}{2}y\right)^3$
$= 32x^5 + 40x^4y + 20x^3y^2 + 5x^2y^3$

d Require $2x + \frac{1}{2}y = 20.05$; use $x = 10, y = 0.1$

$$40x^4\,y = 40\,000$$
$$20x^3y^2 = 200$$
$$5x^2y^3 = 0.5$$
$$\therefore 20.05^5 \approx 3\,240\,200$$

13 General term of $\left(2x + x^{-\frac{1}{2}}\right)^5$ has the form

$$\binom{5}{r}(2x)^{5-r}\left(x^{-\frac{1}{2}}\right)^r = \binom{5}{r}2^{5-r}x^{5-\frac{3r}{2}}.$$

Require $5 - \dfrac{3r}{2} = 2 \Rightarrow r = 2$.

Term is $\binom{5}{2}2^3x^2 = 80x^2$. Coefficient of x^2 is 80.

14 $(1+ax)^n = \binom{n}{0}(1)^n + \binom{n}{1}(1)^{n-1}(ax)^1 + \binom{n}{2}(1)^{n-2}(ax)^2 + \ldots = 1 + 10x + 40x^2 + \ldots$

Comparing coefficients:

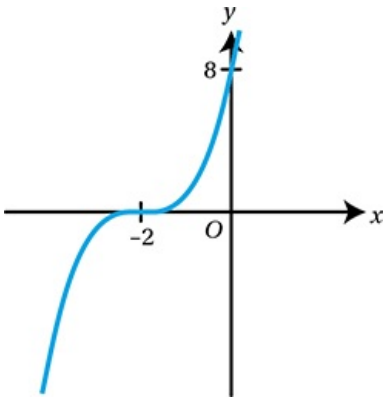
$\binom{n}{1}ax = anx = 10x \Rightarrow a = \dfrac{10}{n}$

$\binom{n}{2}a^2x^2 = \dfrac{a^2n\,(n-1)}{2}x^2 = 40x^2$

$\dfrac{50\,(n-1)}{n} = 40$ so $50n - 50 = 40n$ therefore $10n = 50$.

$n = 5, a = 2$

15 a Graph of $y = x^3$, after a translation $\begin{pmatrix}-2\\0\end{pmatrix}$. Axis intercepts $(-2, 0)$ and $(0, 8)$



b $\binom{3}{0}(x)^3 + \binom{3}{1}(x)^2(2) + \binom{3}{2}(x)(2)^2 + \binom{3}{3}(2)^3$

$= x^3 + 6x^2 + 12x + 8$

c Evaluate for $x = 0.01$

$x^3 =$	$0.000\,001$
$6x^2 =$	$0.000\,6$
$12x =$	0.12
$8 =$	8

Total: $8.12\,06\,01$

d $x^3 + 6x^2 + 12x + 16 = (x+2)^3 + 8$ so $(x+2)^3 = -8$

$x + 2 = -2$ therefore $x = -4$

16 a $6k^2a^2 = 24$ so $k^2a^2 = 4$ therefore $ak = 2$

b $4k^3a = 128$ so $4k^3(\frac{2}{k}) = 128$ therefore $k^2 = 16$

$k = 4, a = \frac{1}{2}$

c $4 \times 4 \times (\frac{1}{2})^3 = 2$ so coefficient of x^3 is 2.

Chapter 10 worked solutions

10 Trigonometric functions and equations

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 10A

7 $\cos(180 + x) + \cos(180 - x) = -\cos x - \cos x = -2\cos x$

8 $\cos x = \sin(x + 90^\circ) = a$

$\sin(x - 90^\circ) = \sin(x + 90^\circ - 180^\circ) = -\sin(x + 90^\circ) = -a$

$k = -1$

9 These standard results can be applied:

$\sin x = \sin(x + 360^\circ)$
$\sin(x + 180^\circ) = -\sin x$
$\sin(x + 90^\circ) = \cos x$

To establish a similar result for $\sin(x + 270^\circ)$:

$\sin(x + 270^\circ) = \sin((x + 180^\circ) + 90^\circ) = \cos(x + 180^\circ) = -\cos x$

So, $\sin x + \sin(x + 90) + \sin(x + 180) + \sin(x + 270) + \sin(x + 360)$

$= \sin x + \cos x - \sin x - \cos x + \sin x$

$= \sin x$

10 Using $\cos x = \sin(90^\circ - x)$, the equation becomes $\sin(90^\circ - x) = \sin(\ln 3)$.

$-360^\circ < x < 360^\circ \Rightarrow -270^\circ < 90^\circ - x < 450^\circ$

Using the properties of sin, for $-270^\circ < 90^\circ - x < 450^\circ$:

$\sin(\ln 3) = \sin(180 - \ln 3) = \sin(\ln 3 + 360) = \sin((180^\circ - \ln 3) - 360^\circ)$

Tip

$\ln 3 - 360^\circ$ is not in the interval.

$\therefore 90 - x = \ln 3,\ 180 - \ln 3,\ \ln 3 + 360,\ -\ln 3 - 180$
$x = 90 - \ln 3,\ \ln 3 - 90,\ -270 - \ln 3,\ 270 + \ln 3$

EXERCISE 10B

5 $\tan(x + 360^\circ) - \tan(180^\circ - x) = \tan x - \tan(-x) = 2\tan x$

6 $\tan x = \tan \mathrm{e}$

$x = \mathrm{e}$ is one obvious solution.

$\tan$ is periodic with period 180° so add and subtract multiples of 180° to get other solutions.

$180^\circ > \mathrm{e} - 180^\circ > -180^\circ$ so this is another solution.

But $\mathrm{e} + 180^\circ > 180^\circ$ so lies outside the interval and is not included.

Similarly for $\mathrm{e} - 180n$ where n is an integer.

So the only solutions are $x = \mathrm{e},\ \mathrm{e} - 180^\circ$

7 $\dfrac{\tan(360^\circ - x)}{\tan(270^\circ - x)} = \dfrac{-\tan x}{\left(\dfrac{1}{\tan x}\right)} = -t^2$

EXERCISE 10C

4 **a** $\sin 60^\circ + \tan 30^\circ = \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{3}} = \dfrac{3+2}{2\sqrt{3}} = \dfrac{5}{2\sqrt{3}} = \dfrac{5\sqrt{3}}{6}$

b $1 - 2\sin^2 60^\circ = 1 - 2\left(\dfrac{\sqrt{3}}{2}\right)^2 = 1 - \dfrac{3}{2} = -\dfrac{1}{2}$

$\cos^{-1}\left(-\dfrac{1}{2}\right) = 120^\circ$

$A = 120$

5 **a** $\cos 150^\circ - \cos 120^\circ = -\dfrac{\sqrt{3}}{2} + \dfrac{1}{2} = \dfrac{1\sqrt{3}}{2}$

b Almost any values will suffice; a very simple example is $A = B = 0^\circ$

$\cos 0^\circ - \cos 0^\circ = 0 \neq 1 = \cos(0^\circ - 0^\circ)$

6 $\dfrac{\sin 30^\circ + \sin 45^\circ}{\cos 30^\circ + \cos 45^\circ} = \dfrac{\dfrac{1}{2} + \dfrac{\sqrt{2}}{2}}{\dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{2}}{2}} = \dfrac{1+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

$= \left(\dfrac{1+\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right)\left(\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right)$

$= \dfrac{\sqrt{3}+\sqrt{6}-\sqrt{2}-2}{3-2}$

$= \sqrt{3}+\sqrt{6}-\sqrt{2}-2$

$a = 2,\ b = -2$

EXERCISE 10D

5 **a** $\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-\dfrac{4}{9}} = \pm\dfrac{\sqrt{5}}{3}$

For $270^\circ < \theta < 360^\circ$, $\sin\theta < 0$

$\sin\theta = -\dfrac{\sqrt{5}}{3}$

b $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = -\dfrac{\sqrt{5}}{2}$

6 $\tan\theta = \dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}} = 3$

$\dfrac{\sin^2\theta}{1-\sin^2\theta} = 9$ so $\sin^2\theta = 9 - 9\sin^2\theta$

$\sin^2\theta = \dfrac{9}{10}$ therefore $\sin\theta = \pm\dfrac{3}{\sqrt{10}}$

7 For x in the interval $90^\circ < x < 180^\circ, -1 < \cos x < 0$

$\cos^2 x = 1 - \sin^2 x = 1 - s^2$

Selecting the negative root: $\cos x = -\sqrt{1-s^2}$

8 **a** $3 - 2\tan^2 x = \dfrac{3\cos^2 x - 2\sin^2 x}{\cos^2 x} = \dfrac{5\cos^2 x - 2}{\cos^2 x} = 5 - \dfrac{2}{\cos^2 2x}$

b $\dfrac{1}{1+\tan^2 x} = \dfrac{1}{1+\left(\dfrac{\sin^2 x}{\cos^2 x}\right)} = \dfrac{\cos^2 x}{\cos^2 x + \sin^2 x} = \cos^2 x$

9 $\left(\dfrac{1}{\sin\ x} - \dfrac{1}{\tan\ x}\right)\left(\dfrac{1}{\sin\ x} + \dfrac{1}{\tan\ x}\right)$

$= \dfrac{1}{\sin^2 x} - \dfrac{1}{\tan^2 x}$

$= \dfrac{1}{\sin^2 x} - \dfrac{\cos^2 x}{\sin^2 x}$

$= \dfrac{\sin^2 x}{\sin^2 x}$

$= 1$

10 $2\tan^2 2x - \dfrac{2}{\cos^2 2x} = 2\left(\dfrac{\sin^2 2x}{\cos^2 2x} - \dfrac{1}{\cos^2 2x}\right) = -2\left(\dfrac{1-\sin^2 2x}{\cos^2 2x}\right) = -2$

$k = -2$

11 $\sin^2 x + \cos^2 x = 1$ so $\tan^2 x + 1 = \dfrac{1}{\cos^2 x}$

a $\cos^2 x = \dfrac{1}{1+t^2}$

b $\sin^2 x = 1 - \cos^2 x = \dfrac{t^2}{1+t^2}$

c $\cos^2 x - \sin^2 x = \dfrac{1-t^2}{1+t^2}$

d $\dfrac{2}{\sin^2 x} + 1 = \dfrac{2(1+t^2)}{t^2} + 1 = \dfrac{2+3t^2}{t^2}$

12 **a** $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \dfrac{1}{\sin\theta\cos\theta}$

b $\dfrac{\cos\theta}{1-\sin\theta} = \dfrac{\cos\theta}{1-\sin\theta} \times \dfrac{1+\sin\theta}{1+\sin\theta}$

$= \dfrac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$

$= \dfrac{1+\sin\theta}{\cos\theta}$

$= \dfrac{1}{\cos\theta} + \tan\theta$

EXERCISE 10E

5 $2\sin x + 1 = 0$

$\sin x = -\frac{1}{2}$

Primary solution: $x_1 = -30^\circ$

Secondary solution: $x_2 = 180^\circ - x_1 = 210^\circ$

Adjusted by 360° to lie within the required interval $x = -30^\circ, -150^\circ$.

6 $3\tan x = -5$

Primary solution: $x_1 = \arctan\left(-\frac{5}{3}\right) = -59.0^\circ$

Periodic solutions: $x_2 = x_1 + 180 = 121,\ x_3 = x_1 + 360 = 301.$

$x = 121^\circ,\ 301^\circ$

7 $\cos x = \frac{\sqrt{3}}{2}$

Primary solution: $x_1 = \arccos\left(\frac{\sqrt{3}}{2}\right) = 30^\circ$

Secondary solution: $x_2 = 360^\circ - x_1 = 330^\circ$

Adjusted by 360° to find further solutions within the required interval $x = \pm 30^\circ, \pm 330^\circ$.

8 $\sin\theta(3\cos\theta - 2) = 0$

$\sin\theta = 0$ or $\cos\theta = \frac{2}{3}$

$\sin\theta = 0 : \theta = 0^\circ,\ 180^\circ$ or 360°

$\cos\theta = \frac{2}{3} : \theta = 48.2^\circ$ or $360^\circ - 48.2^\circ = 312^\circ$

$\theta = 0^\circ,\ 48.2^\circ,\ 180^\circ,\ 312^\circ$ or 360°

9 $(\tan x + 1)(5\sin x - 2) = 0$

$\tan x = -1$ or $\sin x = \frac{2}{5}$

$\tan x = -1 : x = -45^\circ$ or 135° or 315°

$\sin x = \frac{2}{5} :\ x = 23.6^\circ$ or $180^\circ - 23.6^\circ = 156^\circ$

In the required interval, $x = 23.6^\circ,\ 135^\circ,\ 156^\circ$ or 315°.

10 For example: $x = 1$ gives:

$\tan^{-1} x = 45^\circ$

$\sin^{-1} x = 90^\circ$

$\cos^{-1} x = 0^\circ$

Clearly in this case, $\tan^{-1} x \neq \frac{\sin^{-1} x}{\cos^{-1} x}$.

The false idea being disproved by counter example here is that division 'passes through' the inversion of a function – that for a function $\mathrm{h}(x) = \frac{\mathrm{f}(x)}{\mathrm{g}(x)}$ it should follow that $\mathrm{h}^{-1}(x) = \frac{\mathrm{f}^{-1}(x)}{\mathrm{g}^{-1}(x)}$; this is generally not the case.

11 $\sin^{-1} x$ always returns an angle between -90° and 90°, so any input x outside this interval will work as a counter example.

For example: $\sin^{-1}(\sin 180^\circ) = \sin^{-1}(0) = 0^\circ \neq 180^\circ$.

EXERCISE 10F

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3 $0^\circ < x < 180^\circ$ so $0^\circ < 3x < 540^\circ$

$\tan 3x = \sqrt{3}$

$3x = \tan^{-1}(\sqrt{3}) = 60^\circ + 180n^\circ$

Within the given interval, $3x = 60^\circ, 240^\circ, 420^\circ$ therefore $x = 20^\circ, 80^\circ, 140^\circ$

4 $x \in [-180^\circ, 180^\circ]$ so $2x \in [-360^\circ, 360^\circ]$

$2\cos(2x) + 1 = 0$

$\cos(2x) = -\frac{1}{2}$

$2x = 120^\circ + 360n^\circ$ or $240^\circ + 360n^\circ$

Within the given interval, $2x = 120^\circ, 240^\circ, -120^\circ$ or -240°

So $x = 60^\circ, 120^\circ, -60^\circ$ or -120°

5 $-360^\circ < \theta < 360^\circ$ so $-180^\circ < \frac{\theta}{2} < 180^\circ$

$\sin\left(\frac{\theta}{2}\right) = -\frac{2}{3}$

$\frac{\theta}{2} = -41.8^\circ + 360n^\circ$ or $221^\circ + 360n^\circ$

Within the given interval, $\frac{\theta}{2} = -41.8^\circ$ or 138° therefore $\theta = -83.6^\circ$ or -276°.

6 $0^\circ < \theta < 540^\circ$ so $0^\circ < \frac{\theta}{3} < 180^\circ$

$\tan\left(\frac{\theta}{3}\right) = 2.5$

$\frac{\theta}{3} = 68.2^\circ + 180n^\circ$

Within the given interval, $\frac{\theta}{3} = 68.2^\circ$ therefore $\theta = 205^\circ$

7 $-360^\circ < x < 360^\circ$ so $-720^\circ < 2x < 720^\circ$ and then $-690^\circ < 2x + 30^\circ < 750^\circ$

$\sin(2x + 30^\circ) = -\frac{\sqrt{3}}{2}$

$2x + 30^\circ = -60^\circ + 360n^\circ$ or $240^\circ + 360n^\circ$

Within the given interval,

$2x + 30^\circ = -420^\circ, -60^\circ, 300^\circ, 660^\circ$ or $-480^\circ, -120^\circ, 240^\circ, 600^\circ$

$2x = -450^\circ, -90^\circ, 270^\circ, 630^\circ$ or $-510^\circ, -150^\circ, 210^\circ, 570^\circ$

$x = -225^\circ, -45^\circ, 135^\circ, 315^\circ, -255^\circ, -75^\circ, 105^\circ, 285^\circ$

(ordered): $x = -255^\circ, -225^\circ, -75^\circ, -45^\circ, 105^\circ, 135^\circ, 285^\circ, 315^\circ$

8 $-90^\circ < x < 90^\circ$ so $-270^\circ < 3x < 270^\circ$ and then $-320^\circ < 3x - 50^\circ < 220^\circ$

$\cos(3x - 50^\circ) = \frac{\sqrt{2}}{2}$

$3x - 50^\circ = 45^\circ + 360n^\circ$ or $315^\circ + 360n^\circ$

Within the given interval, $3x - 50^\circ = -315^\circ, 45^\circ$ or -45°

$3x = -265^\circ, 95^\circ$ or 5° therefore $x = -88.3^\circ, 31.7^\circ$ or 1.67°

(ordered) $x = -88.3^\circ, 1.67^\circ, 31.7^\circ$

9 $x \in (-180^\circ, 180^\circ)$ so $2x \in (-360^\circ, 360^\circ)$

IRANINOVIN

$\cos 2x + 0.5 = 2 - 2\cos 2x$

$3\cos 2x = 1.5$ therefore $\cos 2x = \frac{1}{2}$

$2x = 60^\circ + 360n^\circ$ or $300^\circ + 360n^\circ$

Within the given interval, $2x = -300^\circ$, 60° or -60°, 300° so $x = \pm 30^\circ, \pm 150^\circ$.

10 $-\sqrt{180^\circ} < x < \sqrt{180^\circ}$ so $0 < x^2 < 180^\circ$

$\sin(x^2) = \frac{1}{2}$

$x^2 = 30^\circ + 360n^\circ$ or $150^\circ + 360n^\circ$

Within the given interval, $x^2 = 30^\circ$ or 150° so $x = \pm\sqrt{30^\circ}, \pm\sqrt{150^\circ} = \pm 5.48^\circ, \pm 12.2^\circ$.

EXERCISE 10G

3 $2\sin x \cos x = \cos x$

$\cos x(2\sin x - 1) = 0$ so so $\cos x = 0$ or $\sin x = \frac{1}{2}$.

$x = 90^\circ, 270^\circ$ or $x = 30^\circ, 150^\circ$ therefore $x = 30^\circ, 90^\circ, 150^\circ, 270^\circ$

4 **a** $2\sin^2 x - 3\sin x - 2 = 0$

$(2\sin x + 1)(\sin x - 2) = 0$ therefore $\sin x = -\frac{1}{2}$ or $\sin x = 2$.

b Reject values of $\sin x$ outside $-1 \leqslant \sin x \leqslant 1$.

$\sin x = -\frac{1}{2}$

Primary solution: $x_1 = -30^\circ$

Secondary solution: $x_2 = 180^\circ - x_1 = 210^\circ$

Adjusted by period 360° to lie within the required interval: $x = 210^\circ, 330^\circ$.

5 $\tan^2 x - \tan x - 12 = 0$

$(\tan x + 3)(\tan x - 4) = 0$ so $\tan x = -3$ or 4

$x = -71.6^\circ, 108.4^\circ$ or $76.0^\circ, -104.0^\circ$ therefore (ordered)

$x = -104.0^\circ, -71.6^\circ, 76.0^\circ, 108.4^\circ$

6 $0^\circ < \theta < 360^\circ$ so $0^\circ < 2\theta < 720^\circ$

$2\sin 2\theta = \sqrt{3}\sin 2\theta\cos 2\theta$ therefore $\sin 2\theta\left(2 - \sqrt{3}\cos 2\theta\right) = 0$

$2\theta = 0^\circ + 180n^\circ$ or $\cos 2\theta = \frac{2}{\sqrt{3}} > 1$ (no real solutions)

Within the given interval, $2\theta = 180^\circ, 360^\circ, 540^\circ$ so $\theta = 90^\circ, 180^\circ, 270^\circ$.

7 $x \in [0^\circ, 180^\circ]$ so $3x \in [0^\circ, 540^\circ]$

$3\cos^2 3x + 7\cos 3x + 2 = 0$ therefore $(3\cos 3x + 1)(\cos 3x + 2) = 0$

$\cos 3x = -\frac{1}{3}$ or $\cos 3x = -2 < -1$ (no real solutions)

$3x = 109^\circ + 360n^\circ$ or $251^\circ + 360n^\circ$

Within the given interval, $3x = 109^\circ, 469^\circ$ or 251° so $x = 36.4^\circ, 156^\circ$ or 83.5°

(ordered) $x = 36.4^\circ, 83.5^\circ, 156^\circ$.

EXERCISE 10H

3 $0^\circ < x < 360^\circ$ so $0^\circ < 2x < 720^\circ$

$\sin 2x = -\cos 2x \ \tan 2x = -1$

$2x = -45^\circ + 180n^\circ$

In the required interval, $2x = 135^\circ, 315^\circ, 495^\circ, 675^\circ$ so $x = 67.5^\circ, 158^\circ, 248^\circ, 338^\circ$

4 $\dfrac{\cos\theta}{\sin\theta} - 2 = 0$ therefore $\tan\theta = \dfrac{1}{2}$

$\theta = 26.6^\circ$ or $26.6^\circ - 180^\circ = -153^\circ$

5 $\sin x\,(1 + \tan x) = 0$ so $\sin x = 0$ or $\tan x = -1$

$\sin x = 0 : x = 0^\circ, 180^\circ, 360^\circ$

$\tan x = -1 : \ x = -45^\circ, 135^\circ, 315^\circ$

$x = 0^\circ, 135^\circ, 180^\circ, 315^\circ, 360^\circ$

6 $\sin x\tan x = \sin^2 x$, $-180^\circ \leqslant x \leqslant 180^\circ$

$\dfrac{\sin^2 x}{\cos x} = \sin^2 x$

$\sin^2 x = \sin^2 x\cos x$

$\sin^2 x(\cos x - 1) = 0$ so $\sin x = 0$ or $\cos x = 1$

$x = -180^\circ, 0^\circ, 180^\circ$

7 $5\sin^2\theta = 4(1 - \sin^2\theta)$ so $9\sin^2\theta = 4\text{tsin}\ \theta = \pm\dfrac{2}{3}$

For $\sin\theta = \dfrac{2}{3}$:

Primary solution: $\theta_1 = 41.8^\circ$

Secondary solution: $\theta_2 = 180^\circ - \theta_1 = 138.2^\circ$

For $\sin\theta = -\dfrac{2}{3}$, the solutions will be the negative equivalents, which also lie in the desired interval.

$\theta = \pm 41.8^\circ, \pm 138.2^\circ$

8 $2\cos^2 t - \sin t - 1 = 0$, $0^\circ \leqslant t \leqslant 360^\circ$

$2 - 2\sin^2 t - \sin t - 1 = 0$ so $2\sin^2 t + \sin t - 1 = 0$

$(2\sin t - 1)(\sin t + 1) = 0$ therefore $\sin t = \dfrac{1}{2}, -1$

Primary solutions: $t_1 = 30^\circ, -90^\circ$

Secondary solution: $t_2 = 180^\circ - t_1 = 150^\circ, 270^\circ$

Restricting to the required interval, solutions are $t = 30^\circ, 150^\circ, 270^\circ$.

9 $4\cos^2 x - 5\sin x - 5 = 0, -180^\circ < x < 180^\circ$

$4 - 4\sin^2 x - 5\sin x - 5 = 0$

$4\sin^2 x + 5\sin x + 1 = 0$

$(4\sin x + 1)(\sin x + 1) = 0$

$\sin x = -\dfrac{1}{4}, -1$

Primary solutions: $x_1 = -14.5^\circ, -90^\circ$

Secondary solutions: $x_2 = -180^\circ - x_1 = -166^\circ, -90^\circ$

Solutions: $x = -166^\circ, -90^\circ, -14.5^\circ$

10 a $\cos^2 t + 5\cos t = 2\sin^2 t$ so $\cos^2 t + 5\cos t = 2 - 2\cos^2 t$

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$3\cos^2 t + 5\cos t - 2 = 0$

$(3\cos t - 1)(\cos t + 2) = 0$

Reject solution $\cos t = -2 < -1$

$\cos t = \frac{1}{3}$

b $t = 70.5^\circ$ or $360^\circ - 70.5^\circ = 289^\circ$

11 a $6\sin^2 x + \cos x = 4$ so $6 - 6\cos^2 x + \cos x - 4 = 0$

$6\cos^2 x - \cos x - 2 = 0$ therefore $(2\cos x + 1)(3\cos x - 2) = 0$

$\cos x = -\frac{1}{2}, \frac{2}{3}$

$0^\circ \leqslant x \leqslant 360^\circ$

b Primary solutions: $x_1 = 120^\circ, 48.2^\circ$

Secondary solution: $x_2 = 360^\circ - x_1 = 240^\circ, 312^\circ$

Solutions: $x = 48.2^\circ, 120^\circ, 240^\circ, 312^\circ$

12 a $2\sin^2 x - 3\sin x\cos x + \cos^2 x = 0$

Divide through by $\cos^2 x$: $2\tan^2 x - 3\tan x + 1 = 0$

$-180^\circ < x < 180^\circ$

$(2\tan x - 1)(\tan x - 1) = 0$ therefore $\tan x = \frac{1}{2}, 1$

Primary solutions: $x_1 = 26.6^\circ, 45^\circ$

Secondary solutions: $x_2 = x_1 - 180^\circ = -153^\circ, -135^\circ$

Solutions: $x = -153^\circ, -135^\circ, 26.6^\circ, 45^\circ$

MIXED PRACTICE 10

1 $\cos(x + 180^\circ) = -\cos x$

So if $\cos(x + 180^\circ) = a$ then $\cos x = -a$.

2 From GDC: $x = -31.8^\circ, 148.2^\circ$

3 $\sin\theta = -\frac{1}{\sqrt{2}}$

Primary solution: $\theta_1 = -45^\circ$

Secondary solution: $\theta_2 = 180^\circ - \theta_1 = 225^\circ$

Further periodic solutions within the given interval: $\theta_1 + 360^\circ = 315^\circ$, $\theta_2 - 360^\circ = -135^\circ$

Solutions: $\theta = -135^\circ, -45^\circ, 225^\circ, 315^\circ$

4 $0^\circ < x < 720^\circ$ so $0^\circ < \frac{1}{2}x < 360^\circ$ and then $45^\circ < \frac{1}{2}x + 45^\circ < 405^\circ$

$\cos\left(\frac{1}{2}x + 45^\circ\right) = \frac{\sqrt{3}}{2}$

Primary solution: $\frac{1}{2}x_1 + 45^\circ = 30^\circ$ (not in the required interval)

Secondary solution: $\frac{1}{2}x_2 + 45^\circ = 360^\circ - 30^\circ = 330^\circ$

Periodic solution within the given interval: $\frac{1}{2}x_3 + 45^\circ = 360^\circ + 30^\circ = 390^\circ$

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Solutions: $\frac{1}{2}x + 45^\circ = 330^\circ, 390^\circ$

$\frac{1}{2}x = 285^\circ, 345^\circ$ therefore $x = 570^\circ, 690^\circ$

5 Using $\cos^2\theta = 1 - \sin^2\theta$: $7\sin^2\theta = 9 - 9\sin^2\theta$

$\sin^2\theta = \frac{9}{16}$

$\sin\theta = \pm\frac{3}{4}$

Primary solutions $\theta_1 = \pm 48.6^\circ$

Secondary solutions $\theta_2 = \pm 180^\circ - \theta_1 = \pm 131^\circ$

Solutions $\theta = \pm 48.6^\circ, \pm 131^\circ$

6 **a** $2\sin^2 x = 5\cos x - 1$

Using $\sin^2 x = 1 - \cos^2 x$: $2 - 2\cos^2 x = 5\cos x - 1$ so $2\cos^2 x + 5\cos x - 3 = 0$

b $(2\cos x - 1)(\cos x + 3) = 0$

$\cos x = \frac{1}{2}$ or $\cos x = -3$ (reject since $\cos x$ cannot be less than -1).

$x = 60^\circ, 300^\circ$

7 **a** Using $\sin^2\theta = 1 - \cos^2\theta$: $\cos\theta - 2 + 2\cos^2\theta + 2 = 0$

$\cos\theta + 2\cos^2\theta = 0$

b $\cos\theta(2\cos\theta + 1) = 0$ therefore $\cos\theta = -\frac{1}{2}$ or $\cos\theta = 0$

$\theta = 120^\circ, 240^\circ, 90^\circ, 270^\circ$ or (ordered): $\theta = 90^\circ, 120^\circ, 240^\circ, 270^\circ$

8 $\sin^2 2x = \frac{1}{4}$ therefore $\sin 2x = \frac{1}{2}$ or $\sin 2x = -\frac{1}{2}$

Two solutions to each equation in each period.

Period of $\sin 2x$ is 180° so the interval contains two periods.

Therefore there are a total of 8 solutions to the original equation in the given interval.

9 Maximum at $\left(\frac{90^\circ}{b}, a\right) = (2, 5) \Rightarrow a = 5, b = 45^\circ$

10 Using $\sin^2 x = 1 - \cos^2 x$: $6 - 6\cos^2 x + \cos x = 4$

$6\cos^2 x - \cos x - 2 = 0$ so $(2\cos x + 1)(3\cos x - 2) = 0$

$\cos x = -\frac{1}{2}$ or $\frac{2}{3}$.

$x = 120^\circ, 240^\circ, \cos^{-1}\left(\frac{2}{3}\right), 360 - \cos^{-1}\left(\frac{2}{3}\right)$

$x = 120^\circ, 240^\circ, 48.2^\circ, 312^\circ$ Ordered: $x = 48.2^\circ, 120^\circ, 240^\circ, 312^\circ$

11 $\frac{2}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} = \frac{2 - \sin^2 x}{\cos^2 x} = \frac{2\left(\cos^2 x + \sin^2 x\right) - \sin^2 x}{\cos^2 x}$

$= \frac{2\cos^2 x + \sin^2 x}{\cos^2 x} = 2 + \tan^2 x$



Tip

While working from left side to right (or right side to left) is usually the best option, you are permitted to rearrange the problem first if you choose, but take care to be extremely clear about your reasoning.

Alternative approach:

When proving an identity $A \equiv B$ it is equivalent to prove that $B - A \equiv 0$.

$\left(2 + \tan^2 x\right) - \left(2\sec^2 x - \tan^2 x\right) = 2 + 2\tan^2 x - 2\sec^2 x = 0$

(using the standard identity $1 + \tan^2 x = \sec^2 x$)

Therefore $2 + \tan^2 x = 2\sec^2 x - \tan^2 x$ as required.

12 a $2\sin x = \dfrac{4\cos x - 1}{\tan x}$

$2\sin x = (4\cos x - 1) \times \dfrac{\cos x}{\sin x}$ therefore $2\sin^2 x = 4\cos^2 x - \cos x$

Using $\sin^2 x = 1 - \cos^2 x$

$2 - 2\cos^2 x = 4\cos^2 x - \cos x$ therefore $6\cos^2 x - \cos x - 2 = 0$

b $(3\cos x - 2)(2\cos x + 1) = 0$ therefore $\cos x = \dfrac{2}{3}$ or $\cos x = -\dfrac{1}{2}$

$\cos x = \dfrac{2}{3}$:

Primary solution: $x_1 = \cos^{-1}\left(\dfrac{2}{3}\right) = 48.2^\circ$

Secondary solution: $x_2 = 360^\circ - x_1 = 312^\circ$

$\cos x = -\dfrac{1}{2}$:

Primary solution $x_1 = \cos^{-1}\left(-\dfrac{1}{2}\right) = 120^\circ$

Secondary solution: $x_2 = 360^\circ - x_1 = 240^\circ$

Solutions are $x = 48.2^\circ, 120^\circ, 240^\circ, 312^\circ$

13 $\dfrac{1}{1+\cos x} + \dfrac{1}{1-\cos x} = \dfrac{1 - \cos x + 1 + \cos x}{(1+\cos x)(1-\cos x)} = \dfrac{2}{1-\cos^2 x} = \dfrac{2}{\sin^2 x}$

14 i. $\dfrac{\sin^2 x - \cos^2 x}{1 - \sin^2 x} = \dfrac{\sin^2 x - \cos^2 x}{\cos^2 x} = \dfrac{\sin^2 x}{\cos^2 x} - \dfrac{\cos^2 x}{\cos^2 x} = \tan^2 x - 1$

ii. $\tan^2 x - 1 = 5 - \tan x$

$\tan^2 x + \tan x - 6 = 0 = (\tan x - 2)(\tan x + 3)$

$\tan\ x = 2$ or -3.

Primary solutions: $x_1 = \tan^{-1}(2) = 63.4^\circ$ or $\tan^{-1}(-3) = -71.6^\circ$

Periodic solutions: $x_2 = x_1 + 180^\circ = 243^\circ$ or 108° and $x_3 = x_1 + 360^\circ = 423^\circ$ or 288°.

Solutions within the given interval: $x = 63.4^\circ, 108^\circ, 243^\circ$ or 288°.

15 $\text{LHS} = \tan x + \dfrac{1}{\tan x} = \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = \dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \dfrac{1}{\sin x \cos x}$

Since you want to get $\sin^2 x$ at the bottom of the fraction, multiply top and bottom by $\sin x$.

16 $-90^\circ < x < 90^\circ$ so $-180^\circ < 2x < 180^\circ$

$6\cos^2 2x = \sin 2x + 4$

Using $\cos^2 2x = 1 - \sin^2 2x$

$6 - 6\sin^2 2x = \sin 2x + 4$ so $6\sin^2 2x + \sin 2x - 2 = 0$

$(2\sin 2x - 1)(3\sin 2x + 2) = 0$ therefore $\sin 2x = \dfrac{1}{2}$ or $\sin 2x = -\dfrac{2}{3}$

$\sin 2x = \dfrac{1}{2}$: $2x = 30^\circ$ or 150° or $\sin 2x = -\dfrac{2}{3}$: $2x = -41.8^\circ$ or -138°

$x = -69.1^\circ, -20.9^\circ, 15^\circ, 75^\circ$

17 a Repeated root: discriminant $\Delta = 0$.

$k^2 - 16 = 0$ to $k = \pm 4$

b Using $\sin^2\theta = 1 - \cos^2\theta$:

$4 - 4\cos^2\theta = 5 - k\cos\theta$

$4\cos^2\theta - k\cos\theta + 1 = 0$

c $\mathrm{f}_k(\theta) = 4\cos^2\theta - k\cos\theta + 1$

i From part **a**, for $k = 4$ there is a repeated root.

$\mathrm{f}_4(\theta) = (2\cos^2\theta - 1)^2$

$\cos\theta = 0.5$ for $\mathrm{f}_4(\theta) = 0$, so there is a single value for $\cos\theta$.

ii $\cos\theta = 0.5$ so the solutions in the given interval are $\theta = \pm 60^\circ$ and $\pm 300^\circ$.

iii Substituting $x = 1 : 4 - k + 1 = 0$ so $k = 5$.

iv $\mathrm{f}_5(\theta) = 4\cos^2\theta - 5\cos\theta + 1 = (4\cos\theta - 1)(\cos\theta - 1) = 0$

So $\cos\theta = 0.25$ or 1.

$\cos\theta = 0.25$ has four solutions in $[-360^\circ, 360^\circ]$ and $\cos\theta = 1$ has three solutions. So there are seven solutions in total.

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Chapter 11 worked solutions

11 Triangle geometry

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 11A

Tip

In questions on geometric shapes, if no diagram is given it is usually wise to draw a quick sketch diagram. This reduces the opportunity for errors of interpretation and makes it easier to check for the sense of an answer.

Always remember when checking for sense that in a triangle, the widest angle lies opposite the longest side, and the narrowest angle lies opposite the shortest side.)

4 $AB = 6,\ BC = 8, \angle ACB = 35^\circ$

By the sine rule: $\angle CAB = \sin^{-1}\left(\dfrac{BC\sin\angle ACB}{AB}\right) = 49.9^\circ$.

Since this angle is greater than the known angle, the secondary solution is also variable:

$x_2 = 180^\circ - x_1 = 130^\circ$

$AC = \dfrac{AB\sin\angle ABC}{\sin\angle ACB}$

Two possible triangles exist:

$35^\circ - 49.9^\circ - 95.1^\circ$, with $AC = 10.4$ cm

$35^\circ - 130^\circ - 14.9^\circ$, with $AC = 2.69$ cm

5 $AB = 6,\ AC = 8,\ AD = 5, \angle ADB = 75^\circ$

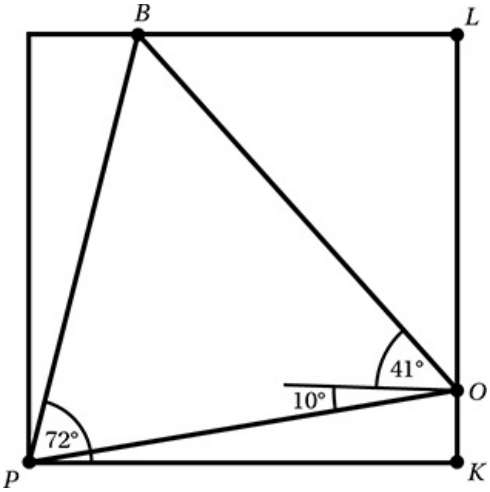
By the sine rule in triangle ABC : $\angle ABD = \sin^{-1}\left(\dfrac{AD\sin\angle ADB}{AB}\right) = 53.6^\circ$.

By the sine rule in triangle ABC : $\angle ACB = \sin^{-1}\left(\dfrac{AB\sin\angle ABC}{AC}\right) = 37.1^\circ$.

Then $\angle BAC = 180^\circ - 53.6^\circ - 37.1^\circ = 89.3^\circ$

By the sine rule in triangle ABC : $BC = \dfrac{AC\sin\angle BAC}{\sin\angle ABC} = 9.94$ cm.

6 Let observer be at O, balloon at B and peg at P. Let the point K lie below O such that PK is horizontal and let L lie above O such that LB is horizontal.



Then $\angle KPB = 72^\circ, \angle KPO = 10^\circ, \angle BOL = 49^\circ$

$\therefore \angle KOP = 80^\circ \Rightarrow \angle POB = 180^\circ - 80^\circ - 49^\circ = 51^\circ$

Also, $\angle OPB = 62^\circ \therefore \angle PBO = 180^\circ - 62^\circ - 51^\circ = 67^\circ$

Then by the sine rule in POB : $OP = \dfrac{BP \sin \angle OBP}{\sin \angle POB} = \dfrac{20 \sin 67^\circ}{\sin 51^\circ} = 23.7\,\text{m}$

Then by trigonometry in POB : $PK = OP \cos 10^\circ = 23.3\,\text{m}$

7 By the sine rule: $\sin \angle ACB = \dfrac{AB \sin \angle ABC}{AC} = 1.097$

But since $\sin x < 1$ for any angle x, this is not possible.

EXERCISE 11B

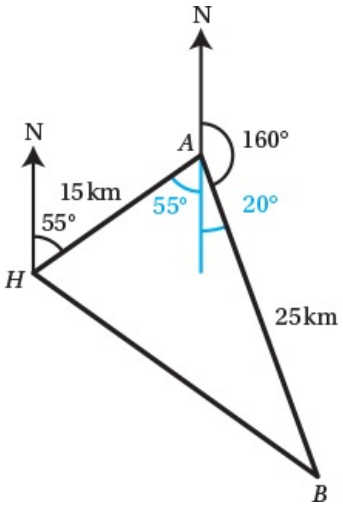
4 $BPS = 115^\circ$

By the cosine rule: $BS = \sqrt{(BP)^2 + (PS)^2 - (BP)(PS)\cos \angle BPS} = 7.08\,\text{km}$

5 If N represents the direction of north and S represents south, then:

$\angle NHA = 55^\circ = \angle HAS$ (alternate angles)

$\angle NAB = 160^\circ$ so $\angle SAB = 20^\circ$



In total, $\angle HAB = 75^\circ$

By the cosine rule:

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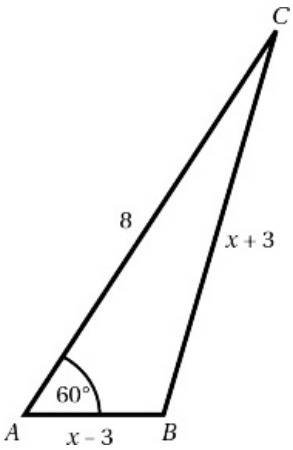
$$HB = \sqrt{HA^2 + AB^2 - 2(HA)(AB)\cos\angle HAB}$$
$$= \sqrt{15^2 + 25^2 - 2 \times 15 \times 25 \times \cos 75^\circ}$$
$$= 25.6 \text{ km } (3 \text{ s. f. })$$

6 By the cosine rule in ACD : $\angle ADC = \cos^{-1}\left(\dfrac{(AD)^2 + (CD)^2 - (AC)^2}{2(AD)(CD)}\right) = 100.3^\circ$

$\therefore \angle BDC = 180^\circ - \angle ADC = 79.7^\circ$

By the sine rule in BCD : $BC = \dfrac{DC \sin \angle BDC}{\sin \angle DBC} = 7.95$

7



By the cosine rule: $BC^2 = (AB)^2 + (AC)^2 - 2(AB)(AC)\cos\angle BAC$

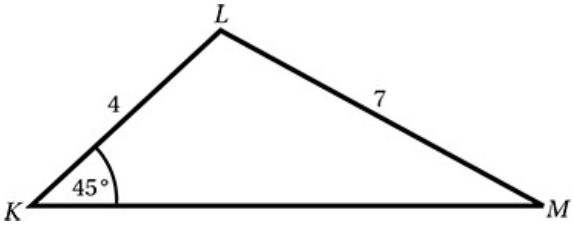
$$\therefore (x+3)^2 = (x-3)^2 + 64 - 8(x-3) \Rightarrow$$
$$x^2 + 6x + 9 = x^2 - 6x + 9 + 64 - 8x + 24$$
$$\Rightarrow 20x = 88$$
$$\Rightarrow x = 4.4$$

8 Using the cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$(3x-10)^2 = (x+2)^2 + (x-2)^2 - 2(x+2)(x-2)\cos 120^\circ$$
$$9x^2 - 60x + 100 = (x^2 + 4x + 4) + (x^2 - 4x + 4) - 2(x^2 - 4)\left(-\tfrac{1}{2}\right) = 3x^2 + 4$$
$$6x^2 - 60x + 96 = 0$$
$$x^2 - 10x + 16 = 0$$
$$(x-8)(x-2) = 0$$

$\therefore x =$ 8 cm ($x = 2$ is not a valid solution to the problem)

9



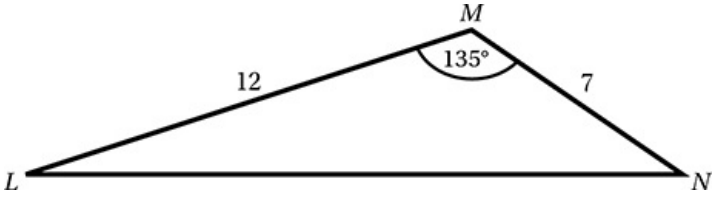
By the cosine rule: $(LM)^2 = (KL)^2 + (KM)^2 - 2(KL)(KM)\cos\angle LKM$

$$\therefore 49 = 16 + x^2 - \frac{8x}{\sqrt{2}}$$
$$\Rightarrow x^2 - 4x\sqrt{2} - 33 = 0$$
$$\Rightarrow x = 2\sqrt{2} \pm \sqrt{41}$$

Since the length must be positive, $KM = 2\sqrt{2} + \sqrt{41}$.

EXERCISE 11C

3



By cosine rule: $LN = \sqrt{(LM)^2 + (MN)^2 - 2(LM)(MN)\cos\angle LMN} = 17.7\,\text{cm}$.

By the sine rule for the area of a triangle: $\text{Area} = \frac{1}{2}(LM)(MN)\sin\angle LMN = 29.7\,\text{cm}^2$.

4 Let the length of a side of the equilateral triangle be x.

The altitude of the triangle is $x\sin(60^\circ) = \frac{\sqrt{3}}{2}x$

The area of the triangle is $\frac{1}{2} \times x \times \frac{\sqrt{3}}{2}x = \frac{\sqrt{3}}{4}x^2 = 25\sqrt{3}$

$x^2 = 100$ so $x = 10\,\text{cm}$

5 By the area of a triangle: $\text{area} = \frac{1}{2}(AB)(BC)\sin\angle ABC$

$$\frac{1}{2}(x+3)(x)\sin 150^\circ = 10$$
$$x^2 + 3x = 40$$
$$x^2 + 3x - 40 = (x+8)(x-5) = 0$$

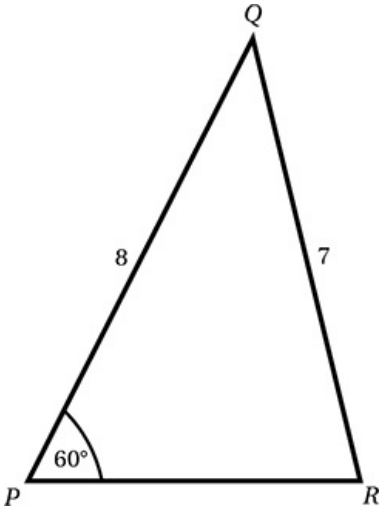
$x = 5$ (Reject the negative solution.)

6 By the area of a triangle: $\text{area} = \frac{1}{2}(AB)(BC)\sin\angle ABC$

$$\frac{1}{2}(2x-3)(x+4)\sin 120^\circ = 39\sqrt{3}$$
$$\frac{\sqrt{3}}{4}(2x^2 + 5x - 12) = 39\sqrt{3}$$
$$2x^2 + 5x - 12 = 156$$
$$2x^2 + 5x - 168 = (2x+21)(x-8) = 0$$

$x = 8$ (Reject the negative solution.)

7



By the cosine rule: $(RQ)^2 = (PQ)^2 + (PR)^2 - 2(PQ)(PR)\cos\angle RPQ$

$$49 = 64 + x^2 - 8x$$
$$\Rightarrow x^2 - 8x + 15 = 0$$
$$\Rightarrow (x-3)(x-5) = 0$$

$PR = 3$ or $5\,\text{cm}$

By the area of a triangle: Area difference $= \frac{1}{2}PQ\sin\angle RPQ(PR_1 - PR_2)$

$\therefore$ Area difference $= 4 \times \frac{\sqrt{3}}{2}(5-3)4\sqrt{3}\,\text{cm}^2$

MIXED PRACTICE 11

1 Using the sine rule:

$\sin\angle ABC = AC \times \frac{\sin\angle ACB}{AB} = \frac{8.7\sin 37.5^\circ}{6.2} = 0.854$

$\angle ABC = 58.7^\circ$ or 121° (3 s. f.)

2 **a** $AB = \frac{12}{\tan 56^\circ} = 8.09\text{ m}$(3 s. f.).

b ABM is isosceles, with $AB = BM$. Using the cosine rule:

$AM = \sqrt{AB^2 + BM^2 - 2(AB)(BM)\cos\angle ABM}$

$AM = 8.09\sqrt{2 - 2\cos 48^\circ} = 6.58\,\text{m}$

3 **a** Using the cosine rule: $BC^2 = 9^2 + 17^2 - 2 \times 9 \times 17\cos(40^\circ) = 135.6$

$BC = \sqrt{135.6} = 11.6\,\text{cm}$

b Area $= \frac{1}{2}(9)(17)\sin(40^\circ) = 49.2\,\text{cm}^2$

c Angle $\angle ABD = 180 - 40 - 63 = 77^\circ$.

Using the sine rule: $\frac{BD}{\sin 40^\circ} = \frac{9}{\sin 63^\circ} \Rightarrow BD = \frac{9\sin 40^\circ}{\sin 63^\circ} = 6.49\,\text{cm}$

4 **a** The largest angle is opposite the longest side, which is $11.3\,\text{cm}$

Using the cosine rule: $\cos\theta = \frac{6.4^2 + 7.0^2 - 11.3^2}{2 \times 6.4 \times 7.0} = -0.421$

$\theta = \cos^{-1}(-0.421) = 115^\circ$

b The angle found above is between the 6.4 and 7.0 sides.

Area $= \frac{1}{2}(6.4 \times 7.0)\sin 115 = 20.3\text{ cm}^2$

5



By the cosine rule: $BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos(BAC)$

$BC = \sqrt{12 + 100 - 40\sqrt{3}\left(-\frac{\sqrt{3}}{2}\right)} = \sqrt{112 + 60} = \sqrt{172} = 2\sqrt{43}$

6 Using the sine rule:

$\frac{\sin(\angle LKM)}{6.1} = \frac{\sin(42^\circ)}{4.2}$

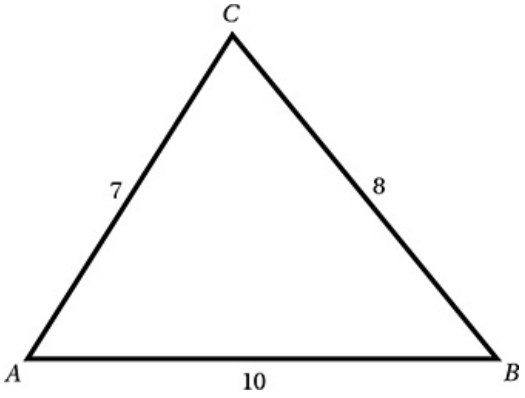
$\angle LKM = 76.37^\circ$ or $180 - 76.37 = 103.63^\circ$

Since you are told that the triangle is obtuse-angled,

$\angle LKM = 103.63^\circ$,so $\angle LMK = 180 - 42 - 103.63 = 34.37^\circ$

Hence Area $= \frac{1}{2}(LM)(MK)\sin\angle LMK = 7.23\,cm^2$(3s. f.)

7



a By the cosine rule: $\cos\angle ABC = \dfrac{AB^2 + BC^2 - AC^2}{2(AB)(BC)} = \dfrac{64 + 100 - 49}{160} = \dfrac{115}{160} = \dfrac{23}{32}$

b $\sin\angle ABC = \sqrt{1 - \left(\dfrac{23}{32}\right)^2} = \dfrac{1}{32}\sqrt{1024 - 529} = \dfrac{1}{32}\sqrt{495}$

$= \dfrac{3}{32}\sqrt{55}$

c Area of $ABC = \dfrac{1}{2}(AB)(BC)\sin\angle ABC = 40 \times \dfrac{3}{32}\sqrt{55} = \dfrac{15}{4}\sqrt{55}$

8 **a** $MB^2 = AM^2 + AB^2 - 2(AM)(AB)\cos\angle MAB = \dfrac{x^2}{4}$

$+25 - 5x\cos\theta$

b $BC^2 = AC^2 + AB^2 - 2\,(AB)\,(BC)\cos\angle BAC = x^2 + 25 - 10x\cos\theta$

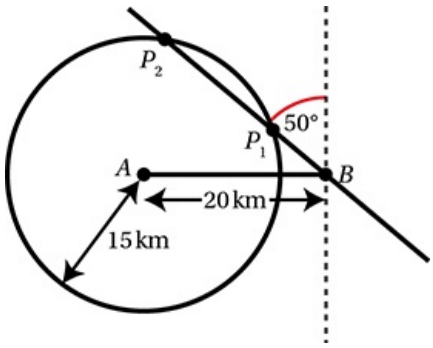
$BC = MB \Rightarrow x^2 + 25 - 10x\cos\theta = \dfrac{x^2}{4} + 25 - 5x\cos\theta$

$\dfrac{3x^2}{4} = 5x\cos\theta \Rightarrow \cos\theta = \dfrac{3x}{20}$

c $x = 5$

$\theta = \cos^{-1}\left(\dfrac{3}{4}\right) = 41.4^\circ\ (3\text{ s. f. })$

9 **a** Let ship be at point P.



For triangle ABP, $AB = 20$, $AP = 15$, angle $\angle ABP = 40^\circ$.

By the sine rule, $\angle APB = \sin^{-1}\left(AB \times \dfrac{\sin(\angle ABP)}{AP}\right) = 59.0^\circ$ or 121°

Then the bearing of the ship from A, which equals $90^\circ - \angle PAB = 90^\circ - (180^\circ - \angle APB - \angle ABP)$ is 071° or 009°.

b AP_1P_2 is an isosceles triangle, with $AP_1 = AP_2 = 15\,\text{km}$.

The angle at $A = 71.0^\circ - 9.0^\circ = 62.0^\circ$

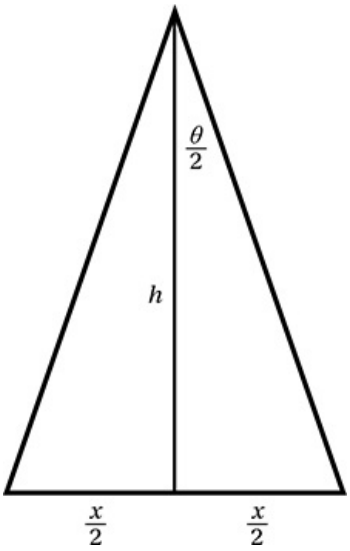
By the cosine rule, $P_1P_2 = \sqrt{{AP_1}^2 + {AP_2}^2 - 2 \times AP_1 \times AP_2 \times \cos(\angle P_1AP_2)}$

$P_1P_2 = 15.5\,\text{km}$

10 Let the side length equal x.

The pentagon can be split into five isosceles triangles, each with base length x, apex angle 72° and area $40\,\text{cm}^2$.

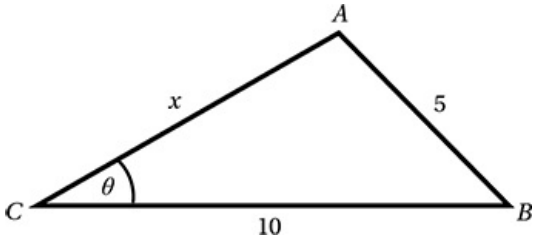
Area of an isosceles with apex angle θ and base x has altitude $h = \dfrac{x}{2\tan\left(\dfrac{\theta}{2}\right)}$



Triangle area $= \dfrac{xh}{2} = \dfrac{x^2}{4\tan\left(\dfrac{\theta}{2}\right)}$ so $\dfrac{x^2}{4\tan(36^\circ)} = 40$

$x = \sqrt{160\tan(36^\circ)} = 10.8\,\text{cm}$

11 a By the cosine rule: $AB^2 - AC^2 - BC^2 + 2(AC)(BC)\cos\angle ACB = 0$



$25 - x^2 - 100 + 20x\cos\theta = 0 \Rightarrow x^2 - 20x\cos\theta + 75 = 0$

b Real solutions $\Rightarrow \Delta \geqslant 0$ so $(20\cos\theta)^2 - 300 \geqslant 0$

$20\cos\theta \leqslant -10\sqrt{3}$ or $20\cos\theta > 10\sqrt{3} \Rightarrow -1 \leqslant \cos\theta \leqslant -\frac{\sqrt{3}}{2}$ *or* $\frac{\sqrt{3}}{2} \leqslant \cos\theta \leqslant 1$

c $150^\circ \leqslant \theta < 180^\circ$ or $0^\circ < \theta \leqslant 30^\circ$

However, since side AB is shorter than BC, it follows that angle $\angle ACB$ must be smaller than angle $\angle CAB$, so triangle ACB cannot be obtuse.

Hence $0^\circ < \theta \leqslant 30^\circ$

Tip

The rejected solution arises from $-1 \leqslant \cos\theta \leqslant -\dfrac{\sqrt{3}}{2}$, which when substituted into the quadratic in question 11a produces a quadratic with all positive coefficients, which therefore would only have negative solutions for x. Since x represents a length, such solutions would be rejected.

Chapter 12 worked solutions

12 Vectors

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 12A

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5 $|(2c)\mathbf{i} + (5c)\mathbf{j}| = \sqrt{(2c)^2 + (5c)^2} = \sqrt{29c^2} = \pm c\sqrt{29}$

Require $\pm c\sqrt{29} = 1 \therefore c = \pm\dfrac{1}{\sqrt{29}}$

Tip

It is an easy mistake to calculate that since the magnitude of $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is $\sqrt{29}$, $c = \dfrac{1}{\sqrt{29}}$, forgetting that in fact $|c| = \dfrac{1}{\sqrt{29}}$.

6 $\left|\begin{pmatrix} 3k \\ -2k \end{pmatrix}\right| = \sqrt{(3k)^2 + (-2k)^2} = \sqrt{13k^2} = \pm k\sqrt{13}$

Require $\pm k\sqrt{13} = 1 \therefore k = \pm\dfrac{1}{\sqrt{13}}$

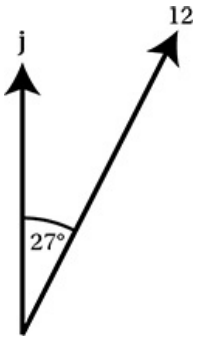
7 $\left|\begin{pmatrix} 4t + 10 \\ -2t \end{pmatrix}\right| = 2\sqrt{10}$

$\sqrt{(4t+10)^2 + (-2t)^2} = 2\sqrt{10} \Rightarrow (4t+10)^2 + (-2t)^2 = 40$

$16t^2 + 80t + 100 + 4t^2 = 40 \Rightarrow 20t^2 + 80t + 60 = 0$

$t^2 + 4t + 3 = 0 \Rightarrow (t+1)(t+3) = 0$ therefore $t = -1$ *or* -3

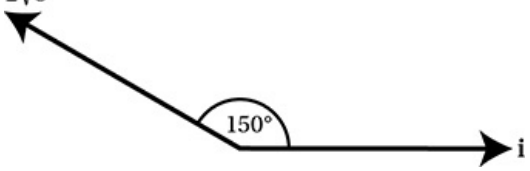
8



a 63°

b $\mathrm{v} = 12\begin{pmatrix} \sin 27^\circ \\ \cos 27^\circ \end{pmatrix} = \begin{pmatrix} 5.45 \\ 10.7 \end{pmatrix} = 5.45\mathrm{i} + 10.7\mathrm{j}$

9 $2\sqrt{3}$



$\mathrm{b} = 2\sqrt{3}\begin{pmatrix} \cos 150^\circ \\ \sin 150^\circ \end{pmatrix} = 2\sqrt{3}\begin{pmatrix} -\dfrac{\sqrt{3}}{2} \\ \dfrac{1}{2} \end{pmatrix} = \begin{pmatrix} -3 \\ \sqrt{3} \end{pmatrix}$

EXERCISE 12B

6 $3\mathbf{a} + 4\mathbf{x} = \mathbf{b}$

$3\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 4\mathbf{x} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \Rightarrow 4\mathbf{x} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} \Rightarrow \mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

7 $\mathbf{a} + t\mathbf{b} = \mathbf{c}$

$\begin{pmatrix} 3 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow t\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \Rightarrow t = -2$

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8 Require that for some t,

$\mathbf{a}+p\mathbf{b}=t\begin{pmatrix}3\\2\end{pmatrix}\Rightarrow\begin{pmatrix}2\\0\end{pmatrix}+p\begin{pmatrix}3\\1\end{pmatrix}=t\begin{pmatrix}3\\2\end{pmatrix}$

$\begin{cases}2+3p=3t\,(1)\\0+p=2t\ \ (2)\end{cases}$

$2\times(1)-3\times(2):\ 4+3p=0\Rightarrow p=-\frac{4}{3}$

9 Require that for some t,

$\lambda\mathbf{x}+\mathbf{y}=t\mathbf{j}\Rightarrow\lambda\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}4\\1\end{pmatrix}=t\begin{pmatrix}0\\1\end{pmatrix}$

$\begin{cases}2\lambda+4=0\ (1)\\3\lambda+1=t\,(2)\end{cases}$

$(1):\lambda=-2$

10 Require that for some t,

$p\mathbf{a}+\mathbf{b}=t\,(\mathbf{i}+\mathbf{j})\Rightarrow p\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}2q\\1\end{pmatrix}=t\begin{pmatrix}1\\1\end{pmatrix}$

$\begin{cases}\ p+2q=t\,(1)\\-p+1=t\,(2)\end{cases}$

$(1)-(2):$

$2p+2q-1=0\Rightarrow q=\dfrac{1-2p}{2}$

11 Require that for some t,

$\begin{pmatrix}3k+1\\k-3\end{pmatrix}=t\begin{pmatrix}-4\\1\end{pmatrix}$

$\begin{cases}3k+1=-4t\,(1)\\\ k-3=t\quad\ (2)\end{cases}$

$(1)+4\times(2):7k-11=0\Rightarrow k=\dfrac{11}{7}$

12 $\left|k\begin{pmatrix}6\\-8\end{pmatrix}\right|=20$

$\pm k\sqrt{6^2+(-8)^2}=20\Rightarrow\pm10k=20\Rightarrow k=\pm2$

Vectors are $\pm\begin{pmatrix}12\\-16\end{pmatrix}$.

13 $\left|k\begin{pmatrix}4\\2\end{pmatrix}\right|=3$

$\pm k\sqrt{4^2+2^2}=3\Rightarrow\pm2\sqrt{5}k=3\Rightarrow k=\pm\dfrac{3}{2\sqrt{5}}$

Vectors are $\pm\begin{pmatrix}\frac{6}{\sqrt{5}}\\\frac{3}{\sqrt{5}}\end{pmatrix}$.

14 $\left|k\begin{pmatrix}-2\\1\end{pmatrix}\right|=1$

$\pm k\sqrt{(-2)^2+1^2}=1\Rightarrow\pm k\sqrt{5}=1\Rightarrow k=\pm\dfrac{1}{\sqrt{5}}$.

The unit vectors are $\pm\begin{pmatrix}\frac{-2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{pmatrix}$.

15 a $\mathbf{u} = \begin{pmatrix} t^2 - 1 \\ 2t^2 + 3t + 1 \end{pmatrix} = \begin{pmatrix} (t+1)(t-1) \\ (t+1)(2t+1) \end{pmatrix}$

$= (t+1)\begin{pmatrix} t-1 \\ 2t+1 \end{pmatrix} = (t+1)\mathbf{v}$

$\mathbf{u}$ is a scalar multiple of $\mathbf{v}$ for all values of t and hence is parallel to $\mathbf{v}$.

b $\mathbf{u} = \mathbf{v}$

$(t+1)\mathbf{v} = \mathbf{v} \Rightarrow t + 1 = 1 \Rightarrow t = 0$

EXERCISE 12C

5 **a** $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = 5\mathbf{i} - \mathbf{j}$

b $\overrightarrow{SP} = -\overrightarrow{PR} - \overrightarrow{RS} = -7\mathbf{i} - 4\mathbf{j}$

6 **a** $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

$\overrightarrow{DB} = \overrightarrow{OB} - \overrightarrow{OD} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 7 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}$

$\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

b E is $(0, 3)$

7 $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 1 \\ -11 \end{pmatrix}, \overrightarrow{BC} = \begin{pmatrix} -3 \\ -7 \end{pmatrix}$

$\overrightarrow{BC} + 2x\overrightarrow{AB} + 3y\overrightarrow{AC} = \overrightarrow{AB} + 2x\overrightarrow{AC} + 3y\overrightarrow{BC}$

$\begin{pmatrix} -3 \\ -7 \end{pmatrix} + 2x\begin{pmatrix} 4 \\ -4 \end{pmatrix} + 3y\begin{pmatrix} 1 \\ -11 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix} + 2x\begin{pmatrix} 1 \\ -11 \end{pmatrix} + 3y\begin{pmatrix} -3 \\ -7 \end{pmatrix}$

$\begin{pmatrix} -3 + 8x + 3y \\ -7 - 8x - 33y \end{pmatrix} = \begin{pmatrix} 4 + 2x - 9y \\ -4 - 22x - 21y \end{pmatrix}$

$\begin{cases} 6x + 12y = 7 & (1) \\ 14x - 12y = 3 & (2) \end{cases}$

$(1) + (2) : 20x = 10 \Rightarrow x = \frac{1}{2} \therefore y = \frac{1}{3}$

Substituting:

$\overrightarrow{AD} = \overrightarrow{BC} + 2x\overrightarrow{AB} + 3y\overrightarrow{AC} = \begin{pmatrix} -3 + 8x + 3y \\ -7 - 8x - 33y \end{pmatrix} = \begin{pmatrix} 2 \\ -22 \end{pmatrix}$

$\overrightarrow{AD} = \mathbf{d} - \mathbf{a}$

$\begin{pmatrix} 2 \\ -22 \end{pmatrix} = \mathbf{d} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \mathbf{d} = \begin{pmatrix} 3 \\ -19 \end{pmatrix}$

$D = (3, -19)$

8 **a** $\overrightarrow{AC} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$

b $\overrightarrow{BP} = \overrightarrow{AP} - \overrightarrow{AB} = \begin{pmatrix} p - 6 \\ q + 3 \end{pmatrix}$

$\overrightarrow{CP} = \overrightarrow{AP} - \overrightarrow{AC} = \begin{pmatrix} p - 8 \\ q + 4 \end{pmatrix}$

c $AP^2 = 4BP^2 \Rightarrow p^2 + q^2 = 4\left((p-6)^2 + (q+3)^2\right)$

$p^2 + q^2 = 4p^2 + 4q^2 - 48p + 24q + 180 \Rightarrow 3p^2 + 3q^2 - 48p$

$+24q = -180$

$p^2 + q^2 - 16p + 8q = -60 \qquad (1)$

$CP^2 = (p-8)^2 + (q+4)^2 = p^2 - 16p + q^2 + 8q + 80$

$= -60 + 80 = 20 \qquad$ (using equation (1))

$CP\sqrt{20} = 2\sqrt{5}$

EXERCISE 12D

4 The position vector of the midpoint is

$$\frac{a+b}{2} = \frac{(7\mathrm{i}+18\mathrm{j})+(-3\mathrm{i}+6\mathrm{j})}{2}$$
$$= 2\mathrm{i}+12\mathrm{j}$$

5 $\begin{pmatrix} -9 \\ 21 \end{pmatrix} = -3\begin{pmatrix} 3 \\ -7 \end{pmatrix} \therefore \overrightarrow{CA} = -3\overrightarrow{AB}$

So A, B, C are collinear.

$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\overrightarrow{AB} - \overrightarrow{CA} = \begin{pmatrix} 6 \\ -14 \end{pmatrix} = 2\overrightarrow{AB}$

$\therefore AB : BC = 1 : 2$

6 **a** $\overrightarrow{BC} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

Require that $\overrightarrow{AD} = \overrightarrow{BC}$

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AB}$

$\mathbf{d} = \mathbf{a} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$

b $AB = \left|\begin{pmatrix} -4 \\ 9 \end{pmatrix}\right| = \sqrt{97}$ and $AD = \left|\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right| = \sqrt{26}$

Adjacent sides are not the same length so $ABCD$ is not a rhombus.

7 **a** $\mathbf{e} = \frac{1}{2}(\mathbf{b}+\mathbf{c}) = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

b $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{\mathrm{DE}} = \mathbf{e} - \mathbf{d} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Therefore $ABED$ is a parallelogram.

8 Midpoint of PQ is $A\left(\frac{3}{2}, -1\right)$ and midpoint of QR is $B\left(\frac{11}{2}, -\frac{3}{2}\right)$.

Midpoint of RS is $C\left(\frac{5}{2}, \frac{5}{2}\right)$ and midpoint of SP is $D\left(-\frac{3}{2}, 3\right)$.

$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -\frac{1}{2} \end{pmatrix}$ and $\overrightarrow{DC} = \begin{pmatrix} 4 \\ -\frac{1}{2} \end{pmatrix}$

Two sides of the quadrilateral are parallel and of the same length, so $ABCD$ is a parallelogram.

9 $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$\overrightarrow{AC} = \frac{2}{5}\overrightarrow{AB} = \frac{2}{5}\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} \Rightarrow c = a + \overrightarrow{AC} = \begin{pmatrix} 0.8 \\ 1.8 \end{pmatrix}$

10 $\mathbf{m} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$, $\mathbf{n} = \frac{1}{2}(\mathbf{b}+\mathbf{c})$

$\overrightarrow{MN} = \mathbf{n} - \mathbf{m} = \frac{1}{2}(\mathbf{c}-\mathbf{a})$

$\mathbf{p} = \frac{1}{2}(\mathbf{c}+\mathbf{d})$, $\mathbf{q} = \frac{1}{2}(\mathbf{d}+\mathbf{a})$

$\overrightarrow{QP} = \mathbf{p} - \mathbf{q} = \frac{1}{2}(\mathbf{c}-\mathbf{a})$

$\overrightarrow{MN} = \overrightarrow{QP}$ so $MNPQ$ is a parallelogram.

11 **a** $\overrightarrow{\mathrm{AB}} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$

$\overrightarrow{AC} = x\overrightarrow{AB} = \begin{pmatrix} -8x \\ 6x \end{pmatrix}$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \begin{pmatrix} 10 \\ 1 \end{pmatrix} + \begin{pmatrix} -8x \\ 6x \end{pmatrix} = \begin{pmatrix} 10 - 8x \\ 1 + 6x \end{pmatrix}$

$C = (10 - 8x, 1 + 6x)$

b $\overrightarrow{CD} = \begin{pmatrix} -7 + 8x \\ 1 - 6x \end{pmatrix}$

$CD^2 = 26 \Rightarrow (-7 + 8x)^2 + (1 - 6x)^2 = 26 \Rightarrow 100x^2 - 124x$

$+24 = 0$

$25x^2 - 31x + 6 = 0 \Rightarrow (25x - 6)(x - 1) = 0$

$x = \dfrac{6}{25} = 0.24 \quad (0 < x < 1)$

12 a $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \frac{1}{2}(\mathbf{p} + \mathbf{q})$

b $ABCD$ is a parallelogram so

$\overrightarrow{AD} = \overrightarrow{BC} = q$

Let N be the midpoint of BD.

Then:

$\overrightarrow{AN} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BD} = \overrightarrow{AB} + \frac{1}{2}\left(\overrightarrow{BA} + \overrightarrow{AD}\right) = \mathbf{p} + \frac{1}{2}(-\mathbf{p} + \mathbf{q}) = \frac{1}{2}(\mathbf{p} + \mathbf{q})$

$\therefore \overrightarrow{\mathrm{AN}} = \overrightarrow{\mathrm{AM}}$

So M is also the midpoint of diagonal BD.

13 a $\overrightarrow{AB} = \begin{pmatrix} k \\ k + 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} k - 2 \\ k + 2 \end{pmatrix}$

$\overrightarrow{DC} = \begin{pmatrix} 2k - 3 \\ 3k + 2 \end{pmatrix} - \begin{pmatrix} k - 1 \\ 2k \end{pmatrix} = \begin{pmatrix} k - 2 \\ k + 2 \end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{DC}$

so $ABCD$ is a parallelogram.

b For $ABCD$ to be a rhombus, require:

$AB = BC$

$\left|\overrightarrow{AB}\right| = \sqrt{(k - 2)^2 + (k + 2)^2}$

$\overrightarrow{BC} = \begin{pmatrix} 2k - 3 \\ 3k + 2 \end{pmatrix} - \begin{pmatrix} k \\ k + 1 \end{pmatrix} = \begin{pmatrix} k - 3 \\ 2k + 1 \end{pmatrix}$

$\left|\overrightarrow{BC}\right| = \sqrt{(k - 3)^2 + (2k + 1)^2}$

$AB^2 = BC^2 \Rightarrow (k - 2)^2 + (k + 2)^2 = (k - 3)^2 + (2k + 1)^2$

$2k^2 + 8 = 5k^2 - 2k + 10 \Rightarrow 3k^2 - 2k + 2 = 0$

Discriminant $\Delta = (-2)^2 - 4 \times 3 \times 2 = -20 < 0$

So there are no roots to the quadratic; hence there is no real value k for which $ABCD$ is a rhombus.

MIXED PRACTICE 12

1 $AB = |\mathbf{b} - \mathbf{a}| = \left|\begin{pmatrix} -3 \\ 3 \end{pmatrix}\right| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

2 **a** $\overrightarrow{PQ} = 6\mathbf{i} - 5\mathbf{j}$

$\overrightarrow{RQ} = \mathbf{q} - \mathbf{r} \Rightarrow \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} - \mathbf{r} \Rightarrow \mathbf{r} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b $R = (0, 1)$

3 **a** $\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \begin{pmatrix} 4.5 \\ -1 \end{pmatrix}$

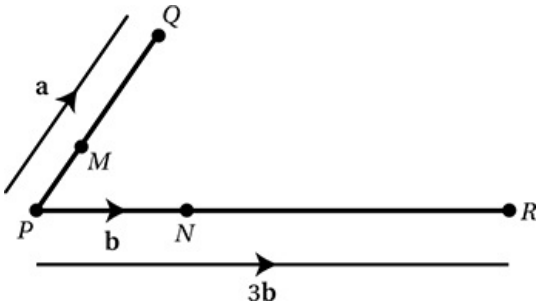
b $BM = |\mathbf{m} - \mathbf{b}| = \left|\begin{pmatrix} 4.5 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ 5 \end{pmatrix}\right| = \left|\begin{pmatrix} 7.5 \\ -6 \end{pmatrix}\right| = \sqrt{7.5^2 + (-6)^2} = \frac{3}{2}\sqrt{41}$

4 $ABCD$ is a parallelogram so $\overrightarrow{CD} = \overrightarrow{BA}$.

$\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = 2\mathbf{i} - 3\mathbf{j}$

$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$\mathbf{d} = c + \overrightarrow{BA} = 6\mathbf{i} - 2\mathbf{j}$



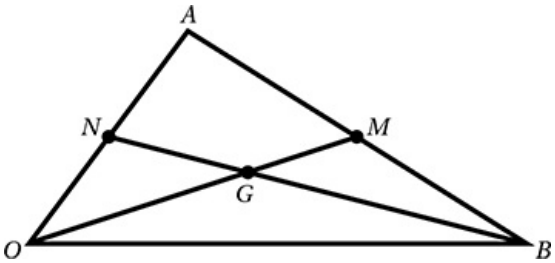
5 $\overrightarrow{PM} = \frac{1}{3}\mathbf{a}$

$\overrightarrow{MN} = \overrightarrow{PN} - \overrightarrow{PM} = \mathbf{b} - \frac{1}{3}\mathbf{a}$

$\overrightarrow{QR} = \overrightarrow{PR} - \overrightarrow{PQ} = 3\mathbf{b} - \mathbf{a} = 3\overrightarrow{MN}$

$\overrightarrow{QR}$ is a multiple of $\overrightarrow{MN}$, so $\overrightarrow{QR}$ QR and $\overrightarrow{MN}$ are parallel.

6



$\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

$\overrightarrow{OG} = \frac{2}{3}\overrightarrow{OM} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$

$\overrightarrow{ON} = \frac{1}{2}\mathbf{a}$

$\overrightarrow{NG} = \overrightarrow{OG} - \overrightarrow{ON} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} - \frac{1}{2}\mathbf{a} = \frac{1}{3}\mathbf{b} - \frac{1}{6}\mathbf{a}$

$\overrightarrow{GB} = \overrightarrow{OB} - \overrightarrow{OG} = \mathbf{b} - \left(\frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\right) = \frac{2}{3}\mathbf{b} - \frac{1}{3}\mathbf{a}$

So $\overrightarrow{GB} = 2\overrightarrow{NG}$, and therefore NG and GB are parallel.

Since both pass through G, B, G and N are collinear.

7 $\left|\begin{pmatrix} 3\sin\theta \\ 5\cos\theta \end{pmatrix}\right| = \sqrt{(3\sin\theta)^2 + (5\cos\theta)^2} = \sqrt{9\sin^2\theta + 25\cos^2\theta}$

$= \sqrt{9\sin^2\theta + 9\cos^2\theta + 16\cos^2\theta} = \sqrt{9(\sin^2\theta + \cos^2\theta) + 16\cos^2\theta}$

$= \sqrt{9 + 16\cos^2\theta}$

8 $\overrightarrow{MN} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

$\left|\overrightarrow{MN}\right| = \sqrt{9^2 + 4^2} = \sqrt{97}$

So the two unit vectors parallel to $\overrightarrow{MN}$ are $\pm\frac{1}{\sqrt{97}}\begin{pmatrix} 9 \\ 4 \end{pmatrix}$

9 a $\overrightarrow{PQ} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$ and $\overrightarrow{PN} = \frac{1}{3}\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\overrightarrow{ON} = \overrightarrow{OP} + \overrightarrow{PN} = \begin{pmatrix} 1 \\ -8 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$

$N = (4, -6)$

b $\overrightarrow{OP} = \begin{pmatrix} 1 \\ -8 \end{pmatrix} \Rightarrow \left|\overrightarrow{OP}\right| = \sqrt{1^2 + 8^2} = \sqrt{65}$

$\left|\overrightarrow{ON}\right| = \sqrt{4^2 + 6^2} = \sqrt{52}$ and $\left|\overrightarrow{PN}\right| = \sqrt{3^2 + 2^2} = \sqrt{13}\,(\sqrt{65})^2 = (\sqrt{52})^2 + (\sqrt{13})^2$

So $OP^2 = ON^2 + NP^2$ and therefore the angle ONP is 90°.

10 a $\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and $\overrightarrow{ON} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$

$\overrightarrow{OP} = \frac{1}{2}(\mathbf{c} + \mathbf{d})$ and $\overrightarrow{OQ} = \frac{1}{2}(\mathbf{d} + \mathbf{a})$

$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \frac{1}{2}(\mathbf{c} - \mathbf{a})$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{1}{2}(\mathbf{a} - \mathbf{c}) = -\overrightarrow{MN}$

b The two opposite sides MN and PQ are parallel and equal length, so $MNPQ$ is a parallelogram.

11 a $\mathbf{p} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{\mathrm{AP}} = \mathbf{p} - \mathbf{a}$

$= \mathbf{b}$

b $\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AP} \Rightarrow \mathbf{m} = \mathbf{a} + \frac{1}{2}\mathbf{b}$

c $\overrightarrow{BQ} = \overrightarrow{BO} + \overrightarrow{OQ} = -\mathbf{b} + t\overrightarrow{OP} = -\mathbf{b} + t\,(\mathbf{a} + \mathbf{b}) = t\mathbf{a} + (t-1)\,\mathbf{b}$

$\overrightarrow{BM} = \overrightarrow{BO} + \overrightarrow{OM} = -\mathbf{b} + \mathbf{a} + \frac{1}{2}\mathbf{b} = \mathbf{a} - \frac{1}{2}\mathbf{b}$

Require $\overrightarrow{BM}$ and BQ parallel,so for some k.

$\overrightarrow{BM} = k\overrightarrow{BQ}$

$\mathbf{a} - \frac{1}{2}\mathbf{b} = k(t\mathbf{a} + (t-1)\mathbf{b})$

Comparing coefficients of vectors:

$\mathbf{a}:\quad kt = 1 \qquad (1)$

$\mathbf{b}:\quad k(t-1) = -\frac{1}{2} \qquad (2)$

Substituting (1) into (2):

$k\left(\frac{1}{k} - 1\right) = -\frac{1}{2} \Rightarrow 1 - k = -\frac{1}{2} \Rightarrow k = \frac{3}{2}$

$t = \frac{1}{k} = \frac{2}{3}$

12 a $\overrightarrow{OH} = \overrightarrow{OP} + \overrightarrow{PH} = \mathbf{p} + t\overrightarrow{PQ} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$

$$= \begin{pmatrix} 5 \\ 2 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 - 3t \\ 2 + t \end{pmatrix}$$

$$OH = \left|\begin{pmatrix} 5 - 3t \\ 2 + t \end{pmatrix}\right| = \sqrt{(5 - 3t)^2 + (2 + t)^2} = \sqrt{10t^2 - 26t + 29}$$

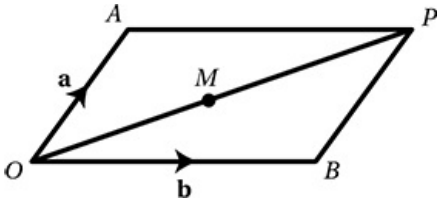
b Completing the square under the square root:

$$OH = \sqrt{10(t^2 - 2.6t) + 29} = \sqrt{10(t - 1.3)^2 - 1.69) + 29}$$

$$= \sqrt{10(t - 1.3)^2 + 12.1}$$

Minimum OH when $t = 1.3$ is $\sqrt{12.1} = 1.1\sqrt{10} = \dfrac{11\sqrt{10}}{10}$.

13



a $\overrightarrow{OM} = \dfrac{1}{2}\overrightarrow{OP} = \dfrac{1}{2}(\mathbf{a} + \mathbf{b})$

$$\overrightarrow{AM} = \overrightarrow{OM} - \overrightarrow{OA} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) - \mathbf{a} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$$

$$\overrightarrow{MB} = \overrightarrow{OB} - \overrightarrow{OM} = \mathbf{b} - \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \overrightarrow{\mathrm{AM}}$$

Therefore lines MB and AM are parallel and of equal length. M is the midpoint of AB so that $AM : MB = 1 : 1$.

b M is the midpoint of both diagonals of the parallelogram; the diagonals of a parallelogram bisect each other.

Chapter 13 worked solutions

13 Differentiation

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 13A

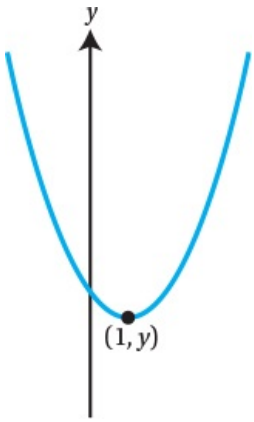
2 The main features to show on the sketch are:

- Where the derivative is positive, the original function is increasing; where the derivative is negative the function is decreasing.
- Where the derivative is zero (crosses the x-axis) the original function has a turning point.

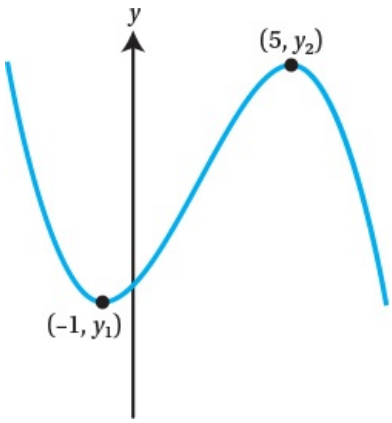
Tip

In parts **c** and **d**, there is a point where the gradient is zero but the function does not change from increasing to decreasing. This is called a point of inflection; you will learn more about these in Student Book 2.

a

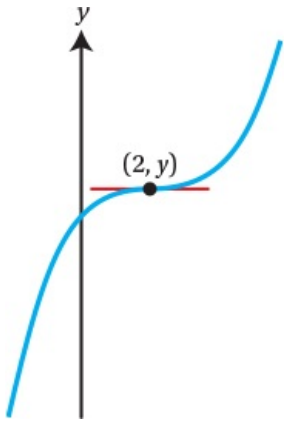


b

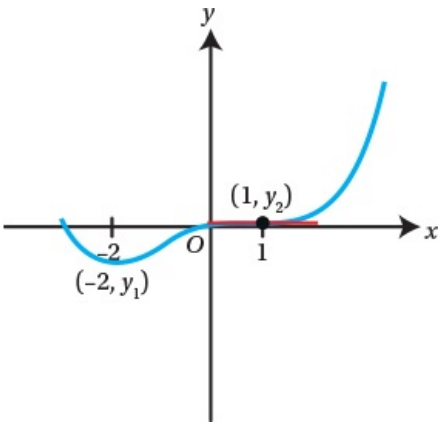


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c



d



3 **a** Sometimes true: gradient indicates the slope of the curve, not its position.

For example, $y = 2x$ has a constant positive gradient 2, both at point $(1,\ 2)$ and at $(-1, -2)$.

b Sometimes true: as for question 3a. For example, $(1, -2)$ lies on line $y = -2x$ (negative gradient) and also on $y = x - 3$ (positive gradient).

c Always true. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ is a defining property of a stationary point.

d Sometimes true: the possibility of a horizontal inflection point is not given.

For example, when $x = 0$, $\dfrac{\mathrm{d}(x^3)}{\mathrm{d}x} = 0$ but $(0,\ 0)$ is neither a local maximum nor a local minimum of the curve $y = x^3$.

e Sometimes true: if the derivative is always positive then the curve is always increasing, but not necessarily without limit.

For example, the curve $y = -e^{-x}$ has a positive gradient at all points but $y < 0$ for all values of x.

f Sometimes true: if the function reaches its minimum at a turning point (such as for $y = x^2$) then the derivative is zero there. However, if the function either has an incomplete domain (for example, $y = x$ over $x \geqslant 0$) or has a non-continuous derivative function (for example, $y = |x|$) then the minimum value will not have a zero derivative, as the derivative may not be defined at the minimum. Some functions (such as $y = x$) have no minimum value, yet have a known non-zero derivative at all points.

EXERCISE 13B

1 $y = 8$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h \to 0}\left\{\frac{8-8}{h}\right\} = \lim_{h \to 0}\{0\} = 0$$

2 $\mathrm{f}(x) = -4x$

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$$\mathrm{f}'(x) = \lim_{h\to 0}\left\{\frac{-4\left(x+h\right)-\left(-4x\right)}{h}\right\} = \lim_{h\to 0}\left\{\frac{-4x-4h+4x}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{-4h}{h}\right\} = \lim_{h\to 0}\left\{-4\right\} = -4$$

3 $y = 3x^2$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0}\left\{\frac{3(x+h)^2-3x^2}{h}\right\} = \lim_{h\to 0}\left\{\frac{3x^2+6xh+3h^2-3x^2}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{6xh+3h^2}{h}\right\} = \lim_{h\to 0}\left\{6x+3h\right\} = 6x$$

4 $y = x^2+1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0}\left\{\frac{\left(x+h\right)^2+1-\left(x^2+1\right)}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{x^2+2xh+h^2+1-x^2-1}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{2xh+h^2}{h}\right\} = \lim_{h\to 0}\left\{2x+h\right\} = 2x$$

5 $\mathrm{f}(x) = x^2-6x$

$$\mathrm{f}'(x) = \lim_{h\to 0}\left\{\frac{\left(x+h\right)^2-6\left(x+h\right)-\left(x^2-6x\right)}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{x^2+2xh+h^2-6x-6h-x^2+6x}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{2xh+h^2-6h}{h}\right\} = \lim_{h\to 0}\left\{2x+h-6\right\} = 2x-6$$

6 $y = x^2-3x+4$

$$y' = \lim_{h\to 0}\left\{\frac{\left(x+h\right)^2-3\left(x+h\right)+4-\left(x^2-3x+4\right)}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{x^2+2xh+h^2-3x-3h+4-x^2+3x-4}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{2xh+h^2-3h}{h}\right\} = \lim_{h\to 0}\left\{2x+h-3\right\} = 2x-3$$

7 **a** $(x+h)^3 = \binom{3}{0}x^3+\binom{3}{1}x^2h+\binom{3}{2}xh^2$
$+\binom{3}{3}h^3 = x^3+3x^2h+3xh^2+h^3$

b $y = x^3$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0}\left\{\frac{\left(x+h\right)^3-x^3}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{3x^2h+3xh^2+h^3}{h}\right\}$$
$$= \lim_{h\to 0}\left\{3x^2+3xh+h^2\right\} = 3x^2$$

8 $\mathrm{f}(x) = 3x^3+2$

$$\begin{aligned}
\mathrm{f}'(x) &= \lim_{h\to 0}\left\{\frac{3(x+h)^3+2-(3x^3+2)}{h}\right\}\\
&= \lim_{h\to 0}\left\{\frac{3x^3+9x^2h+9xh^2+3h^3+2-3x^3-2}{h}\right\}\\
&= \lim_{h\to 0}\left\{\frac{9x^2h+9xh^2+3h^3}{h}\right\}\\
&= \lim_{h\to 0}\left\{9x^2+9xh+3h^2\right\} = 9x^2
\end{aligned}$$

9 **a** $(x+h)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3h + \binom{4}{2}x^2h^2 + \binom{4}{3}xh^3 + \binom{4}{4}h^4$

$= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

b $\mathrm{f}(x) = x^4$

$$\begin{aligned}
\mathrm{f}'(x) &= \lim_{h\to 0}\left\{\frac{(x+h)^4-x^4}{h}\right\}\\
&= \lim_{h\to 0}\left\{\frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-x^4}{h}\right\}\\
&= \lim_{h\to 0}\left\{\frac{4x^3h+6x^2h^2+4xh^3+h^4}{h}\right\}\\
&= \lim_{h\to 0}\left\{4x^3+6x^2h+4xh^2+h^3\right\} = 4x^3
\end{aligned}$$

10 **a** The y-coordinates of A and B satisfy $y = 4x^3$, so

for $A: x = a,\ y = 4a^3$

for $B: x = a+h,\ y = 4(a+h)^3 = 4a^3 + 12a^2h + 12ah^2 + 4h^3$

Thus the gradient of AB is:

$$\frac{(4a^3+12a^2h+12ah^2+h^3)-4a^3}{(a+h)-a} = \frac{12a^2h+12ah^2+h^3}{h}$$

$= 12a^2 + 12ah + h^2$

b As h tends to zero, only the first term remains, so the limit is $12a^2$.

c The limit of the gradient of the chord is the gradient of the tangent to the curve at A.

11 $y = k\mathrm{f}(x)$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{h\to 0}\left\{\frac{k\mathrm{f}(x+h)-k\mathrm{f}(x)}{h}\right\} = \lim_{h\to 0}\left\{k\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}\right\}\\
&= k\lim_{h\to 0}\left\{\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}\right\} \qquad \text{since } k \text{ is a constant}\\
&= k\mathrm{f}'(x)
\end{aligned}$$

12 $y = \mathrm{f}(x) + \mathrm{g}(x)$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{h\to 0}\left\{\frac{\mathrm{f}(x+h)+\mathrm{g}(x+h)-(\mathrm{f}(x)+\mathrm{g}(x))}{h}\right\}\\
&= \lim_{h\to 0}\left\{\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}+\frac{\mathrm{g}(x+h)-\mathrm{g}(x)}{h}\right\}\\
&= \lim_{h\to 0}\left\{\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}\right\}+\\
&= \lim_{h\to 0}\left\{\frac{\mathrm{g}(x+h)-\mathrm{g}(x)}{h}\right\} = \mathrm{f}'(x)+\mathrm{g}'(x)
\end{aligned}$$

Tip

Strictly, you have to require that both the limits are finite for the step in the third line to be valid. If you are assuming that $\mathrm{f}'(x)$ and $\mathrm{g}'(x)$ are both well-defined then this is a legitimate

assumption.

EXERCISE 13C

4 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x^{2-1} - \dfrac{3}{2}x^{\frac{1}{2}-1} = 2x - \dfrac{3}{2}x^{-\frac{1}{2}}$

5 $\mathrm{f}'(x) = 4 \times 3x^{3-1} - 3 \times 2x^{2-1} + 2\left(-\dfrac{3}{2}\right)x^{-\frac{3}{2}-1} = 12x^2 - 6x - 3x^{-\frac{5}{2}}$

6 $\mathrm{f}'(x) = 12\left(-\dfrac{1}{3}\right)x^{-\frac{1}{3}-1} + \dfrac{5 \times \frac{2}{5}x^{\frac{2}{5}-1}}{6} = -4x^{-\frac{4}{3}} + \dfrac{1}{3}x^{-\frac{3}{5}}$

EXERCISE 13D

4 $y = x^2(3x-4) = 3x^3 - 4x^2$

$y' = 9x^2 - 8x$

5 $y = 2\sqrt{x}\left(x^3+4\right) = 2x^{\frac{1}{2}}\left(x^3+4\right) = 2x^{\frac{7}{2}} + 8x^{\frac{1}{2}}$

$y' = 7x^{\frac{5}{2}} + 4x^{-\frac{1}{2}}$

6 $y = \sqrt[5]{x^4} = x^{\frac{4}{5}}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{4}{5}x^{-\frac{1}{5}}$

7 $\mathrm{f}(x) = \dfrac{8}{3\sqrt[4]{x^3}} = \dfrac{8}{3x^{\frac{3}{4}}} = \dfrac{8}{3}x^{-\frac{3}{4}}$

$\mathrm{f}'(x) = \dfrac{8}{3}\left(-\dfrac{3}{4}\right)x^{-\frac{7}{4}} = -2x^{-\frac{7}{4}}$

8 **a** $y = \dfrac{3x^5 - 2x}{x^2} = \dfrac{3x^5}{x^2} - \dfrac{2x}{x^2} = 3x^3 - 2x^{-1}$

b $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 9x^2 + 2x^{-2}$

9 $\mathrm{f}(x) = \dfrac{(x+1)(x+9)}{x} = \dfrac{x^2+10x+9}{x} = \dfrac{x^2}{x} + \dfrac{10x}{x} + \dfrac{9}{x} = x + 10 + 9x^{-1}$

$\mathrm{f}'(x) = 1 - 9x^{-2}$

$= \dfrac{x^2-9}{x^2}$

$= \dfrac{(x-3)(x+3)}{x^2}$

10 $\mathrm{f}(x) = \dfrac{9x^2+3}{2\sqrt[3]{x}} = \dfrac{9x^2}{2\sqrt[3]{x}} + \dfrac{3}{2\sqrt[3]{x}} = \dfrac{9}{2}x^{\frac{5}{3}} + \dfrac{3}{2}x^{-\frac{1}{3}}$

$\mathrm{f}'(x) = \dfrac{9}{2} \times \dfrac{5}{3}x^{\frac{2}{3}} + \dfrac{3}{2}\left(-\dfrac{1}{3}\right)x^{-\frac{4}{3}} = \dfrac{15}{2}x^{\frac{2}{3}} - \dfrac{1}{2}x^{-\frac{4}{3}}$

11 $y = \dfrac{\left(2\sqrt{x}-3\right)^2}{\sqrt{x^3}} = \dfrac{4x - 12x^{\frac{1}{2}} + 9}{x^{\frac{3}{2}}} = \dfrac{4x}{x^{\frac{3}{2}}} - \dfrac{12x^{\frac{1}{2}}}{x^{\frac{3}{2}}} + \dfrac{9}{x^{\frac{3}{2}}}$

$= 4x^{-\frac{1}{2}} - 12x^{-1} + 9x^{-\frac{3}{2}}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2x^{-\frac{3}{2}} + 12x^{-2} - \dfrac{27}{2}x^{-\frac{5}{2}}$

EXERCISE 13E

8 **a** $y = \dfrac{3}{\sqrt{x}} = 3x^{-\frac{1}{2}}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{3}{2}x^{-\frac{3}{2}}$

When $x = 9$,

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{3}{2} \times \dfrac{1}{27} = -\dfrac{1}{18}$

b $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{9}{4}x^{-\frac{5}{2}} = \dfrac{9}{4}(9)^{-\frac{5}{2}} = \dfrac{1}{108} > 0$

$\therefore$ the gradient is increasing at this point.

9 $\mathrm{f}(x) = \sqrt{x} + \dfrac{1}{x} = x^{\frac{1}{2}} + x^{-1}$

$\mathrm{f}'(x) = \dfrac{1}{2}x^{-\frac{1}{2}} - x^{-2}$

$\mathrm{f}''(x) = -\dfrac{1}{4}x^{-\frac{3}{2}} + 2x^{-3}$

$\mathrm{f}''(4) = -\dfrac{1}{4}(4)^{-\frac{3}{2}} + 2(4)^{-3} = -\dfrac{1}{4} \times \dfrac{1}{8} + \dfrac{2}{64} = 0$

i.e. rate of change of gradient at $x = 4$ is 0.

10 $y = 3x^3 - 4x + 1$ so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 9x^2 - 4$

Function is decreasing where gradient is negative:

$9x^2 - 4 < 0$ so $\dfrac{-2}{3}x < \dfrac{2}{3}$

11 $y = \dfrac{1}{3}x^3 - x^2 - 12x + 1$ so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2 - 2x - 12$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3$

$x^2 - 2x - 12 = 3$ so $x^2 - 2x - 15 = 0$ therefore $(x - 5)(x + 3) = 0$

$x = 5$ or $x = -3$

12 $y = x^4 - 12x^2 + 3x - 1$ so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 24x + 3$

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 24$

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$

$12x^2 - 24 = 0$ therefore $x^2 - 2 = 0$ so $x = \pm\sqrt{2}$

13 $y = x^3 + kx + c$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + k$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} \geqslant k \quad$ for all x (since $x^2 \geqslant 0$)

So for $k > 0$, the gradient is always positive.

Therefore the curve is always increasing.

14 $y = x^2 - 2x + 1$ so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x - 2$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = y$

$2x - 2 = x^2 - 2x + 1$ thereofore $x^2 - 4x + 3 = 0$ so $(x - 1)(x - 3) = 0$

$x = 1$ or 3

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The points are $(1, 0)$ and $(3, 4)$.

15 $\mathrm{f}(x) = x^3 - 6x^2 + 5$ therefore $\mathrm{f}'(x) = 3x^2 - 12x$

Function is increasing where gradient is positive:

$3x^2 - 12x > 0$ so $3x(x - 4) > 0$

$\therefore x < 0$ or $x > 4$

16 $\mathrm{f}(x) = 12x - 2x^2 - \frac{1}{3}x^3$ therefore $\mathrm{f}'(x) = 12 - 4x - x^2 > 0$

Function is increasing where gradient is positive:

$12 - 4x - x^2 > 0$ therefore $x^2 + 4x - 12 < 0$ so $(x + 6)(x - 2) < 0$

$\therefore -6 < x < 2$

$a = -6,\ b = 2$

17 $y = 7x - x^2 - x^3$ therefore $\frac{\mathrm{d}y}{\mathrm{d}x} = 7 - 2x - 3x^2$ then $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2 - 6x$

Gradient is decreasing where $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$:

$-2 - 6x < 0$ therefore $6x > -2$ so $x > -\frac{1}{3}$

18 $y = \frac{1}{4}x^4 + x^3 - \frac{1}{2}x^2 - 3x + 6$ therefore $\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 + 3x^2 - x - 3$ then $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3x^2 + 6x - 1$

Gradient is decreasing where $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0 : 3x^2 + 6x - 1 < 0$

Roots are:

$$x = \frac{-6 \pm \sqrt{6^2 + 12}}{6}$$
$$= -1 \pm \frac{2\sqrt{3}}{3}$$

A positive quadratic is less than zero between its roots

Gradient is decreasing for $-1 - \frac{2\sqrt{3}}{3} < x < -1 + \frac{2\sqrt{3}}{3}$.

19 $\mathrm{f}(x) = x^3 - 6x^2 + 9x + 2$ therefore $\mathrm{f}'(x) = 3x^2 - 12x + 9$

Function is decreasing where gradient is negative:

$3x^2 - 12x + 9 < 0$ therefore $3(x - 1)(x - 3) < 0$

$\therefore 1 < x < 3$

$\mathrm{f}''(x) = 6x - 12$

Gradient is increasing where $\mathrm{f}''(x) > 0$: $6x - 12 > 0$ so $x > 2$

$1 < x < 3$ and $x > 2 \Rightarrow 2 < x < 3$

20 $$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^n = n\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}x^{n-1} = n(n-1)\frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}x^{n-2}$$
$$= n(n-1)(n-2)\frac{\mathrm{d}^{n-3}}{\mathrm{d}x^{n-3}}x^{n-3}$$
$$\vdots$$
$$= n(n-1)(n-2)\ldots(2)\frac{\mathrm{d}}{\mathrm{d}x}x^1 = n!$$

MIXED PRACTICE 13

1 $y = \dfrac{3x-2}{\sqrt{x}} = 3x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3}{2}x^{-\frac{1}{2}} + x^{-\frac{3}{2}}$

2 $y = (4x^2-1)(3-x) = -4x^3 + 12x^2 + x - 3$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -12x^2 + 24x + 1$

3 $\mathrm{f}(x) = \dfrac{x^2-4}{2x} = \dfrac{1}{2}x - 2x^{-1}$ therefore $\mathrm{f}'(x) = \dfrac{1}{2} + 2x^{-2}$

$\mathrm{f}''(x) = -4x^{-3}$ so $\mathrm{f}''(2) = -4(2)^{-3} = -\dfrac{1}{2}$

4 **a** $\mathrm{f}(x) = 3\sqrt{x} - \dfrac{2}{\sqrt{x}} = 3x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$ therefore $\mathrm{f}'(x) = \dfrac{3}{2}x^{-\frac{1}{2}} + x^{-\frac{3}{2}}$

b $\mathrm{f}'(4) = \dfrac{3}{2}(4)^{-\frac{1}{2}} + (4)^{-\frac{3}{2}} = \dfrac{3}{2} \times \dfrac{1}{2} + \dfrac{1}{8} = \dfrac{7}{8}$

5 $\mathrm{f}(x) = x^2 + bx + c$

$\mathrm{f}(1) = 2$ so $1 + b + c = 2$ therefore $b + c = 1$

$\mathrm{f}'(x) = 2x + b$

$\mathrm{f}'(2) = 12$ so $4 + b = 12$ therefore $b = 8$

Substituting into $b + c = 1$: $c = -7$

6 **a** $y = 3\sqrt{x} - 2 = 3x^{\frac{1}{2}} - 2$

Crosses the x-axis when $y = 0$:

$0 \quad = 3\sqrt{x} - 2$

$\sqrt{x} = \dfrac{2}{3}$ therefore $x = \dfrac{4}{9}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3}{2}x^{-\frac{1}{2}} = \dfrac{3}{2}\left(\dfrac{4}{9}\right)^{-\frac{1}{2}} = \dfrac{9}{4}$

Gradient at x-intercept is $\dfrac{9}{4}$

b Gradient > 0 so the curve is increasing at this point.

7 $y = 3x^2 - 4x$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6x - 4$

Function is increasing where gradient is positive $6x - 4 > 0$ so $x > \dfrac{2}{3}$.

8 $y = x^2 - 2\sqrt{x} = x^2 - 2x^{\frac{1}{2}}$

therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x - x^{-\frac{1}{2}}$ then $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 + \dfrac{1}{2}x^{-\frac{3}{2}}$

At $x = 9$:

$\dfrac{d^2y}{dx^2} = 2 + \dfrac{1}{2}(9)^{-\frac{3}{2}} = 2 + \dfrac{1}{54} = \dfrac{109}{54}$

Rate of change of gradient at $x = 9$ is $\dfrac{109}{54}$.

9 $y = \dfrac{5}{x^2} - \dfrac{1}{4x} + x = 5x^{-2} - \dfrac{1}{4}x^{-1} + x$

a $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -10x^{-3} + \dfrac{1}{4}x^{-2} + 1 = -\dfrac{10}{x^3} + \dfrac{1}{4x^2} + 1$

b $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 30x^{-4} - \dfrac{1}{2}x^{-3} = \dfrac{30}{x^4} - \dfrac{1}{2x^3}$

10 $y = x^2 + ax - 7$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x + a$

Function is increasing where gradient is positive:

$2x + a > 0$ so $x > -\dfrac{a}{2}$

$\therefore -\dfrac{a}{2} = 5$

$a = -10$

11 $y = x^3 + 4x^2 - 2x + 1$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 8x - 2$ then $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x + 8$

When $x = \dfrac{1}{2}$: $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{3}{2} + 8 = \dfrac{19}{2}$.

12 At A, $\mathrm{f}'(x) = 0$ and $\mathrm{f}''(x) = 0$, with $\mathrm{f}'(x)$ at a local minimum.

However, since $\mathrm{f}'(x) > 0$ either side of A, it is neither a local minimum nor a local maximum for $\mathrm{f}(x)$.

The value of $\mathrm{f}(x)$ is not known solely from the graph of $\mathrm{f}'(x)$.

Only option D is definitely true.

13 a $y = x^2 + 5$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}(1) = 2$

Gradient at A is 2

b $\dfrac{\left(a^2 + 5 - 6\right)}{a - 1} = 2.3 \,\text{therefore}\, a^2 - 1 = 2.3a - 2.3$

$a^2 - 2.3a + 1.3 = 0 \,\text{so}\, (a - 1.3)(a - 1) = 0$

$a = 1.3$

c Tangent has gradient 2 and chord AB has gradient 2.3 so chord AC must have a gradient between 2 and 2.3.

14 $y = x^3 - 5x$

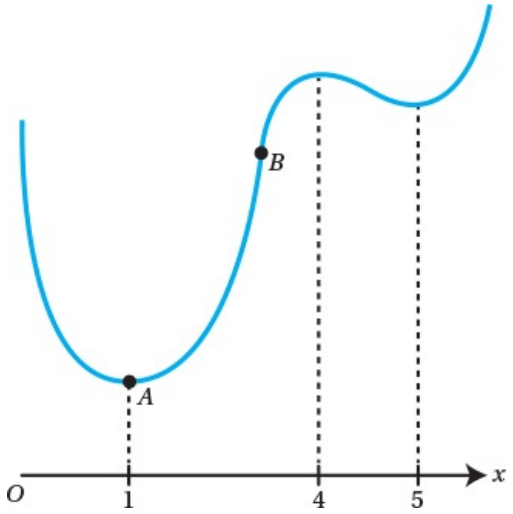
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{h \to 0} \left\{ \frac{(x+h)^3 - 5(x+h) - \left(x^3 - 5x\right)}{h} \right\} \\
&= \lim_{h \to 0} \left\{ \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 5x - 5h - x^3 + 5x}{h} \right\} \\
&= \lim_{h \to 0} \left\{ \frac{3x^2h + 3xh^2 + h^3 - 5h}{h} \right\} \\
&= \lim_{h \to 0} \left\{ 3x^2 + 3xh + h^2 - 5 \right\} \\
&= 3x^2 - 5
\end{aligned}$$

15 a A is a root of the derivative curve, so $\mathrm{f}'(x) = 0$ at A.

b $\mathrm{f}'(x) > 0$ at B, so $\mathrm{f}(x)$ is increasing for that value of x.

c The three roots of $\mathrm{f}'(x)$ represent stationary points of $\mathrm{f}(x)$.

At $x = 1$ and $x = 5$, $\mathrm{f}'(x)$ moves from negative to positive $(\mathrm{f}''(x) > 0)$ so these are minima on $\mathrm{f}(x)$. At $x = 4$, by similar reasoning, there is a maximum of $\mathrm{f}(x)$.



16 $y = \sqrt{x} + 3x = x^{\frac{1}{2}} + 3x$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{2}x^{-\frac{1}{2}} + 3$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 5$

$\dfrac{1}{2}x^{-\frac{1}{2}} + 3 = 5$ therefore $\dfrac{1}{2}x^{-\frac{1}{2}} = 2$ so $x^{-\frac{1}{2}} = 4$

$x = \dfrac{1}{16}$

When $x = \dfrac{1}{16}$: $y = \dfrac{1}{4} + \dfrac{3}{16} = \dfrac{7}{16}$

The point is $\left(\dfrac{1}{16}, \dfrac{7}{16}\right)$.

17 $y = \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2}x^{-\frac{1}{2}}$

When $y = 3$:

$3 = \frac{1}{2\sqrt{x}}$

$\sqrt{x} = \frac{1}{6}$ therefore $x = \frac{1}{36}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{4}x^{-\frac{3}{2}} = -\dfrac{1}{4}\left(\dfrac{1}{36}\right)^{-\frac{3}{2}} = -54$

Gradient is -54 at the point $\left(\dfrac{1}{36}, 3\right)$.

18 $\mathrm{f}(x) = ax^3 + bx^{-2}$ therefore $\mathrm{f}'(x) = 3ax^2 - 2bx^{-3}$ then $\mathrm{f}''(x) = 6ax + 6bx^{-4}$

$\mathrm{f}'(1) = 18$ therefore $3a - 2b = 18 \quad (1)$

$\mathrm{f}''(1) = 18$ so $6a + 6b = 18 \quad (2)$

$(2) - 2 \times (1) : 10b = -18$

$b = -1.8$

$a = 4.8$

19 $\mathrm{f}(x) = \sqrt{x^3} + 15\sqrt{x} = x^{\frac{3}{2}} + 15x^{\frac{1}{2}}$

$\mathrm{f}'(x) = 9$ therefore $\dfrac{3}{2}x^{\frac{1}{2}} + \dfrac{15}{2}x^{-\frac{1}{2}} = 9$

$3x^{\frac{1}{2}} + 15x^{-\frac{1}{2}} = 18$

$3x^{\frac{1}{2}} + \dfrac{15}{x^{-\frac{1}{2}}} = 18$ so $3x + 15 = 18x^{\frac{1}{2}}$

$x - 6x^{\frac{1}{2}} + 5 = 0$ thus $(\sqrt{x} - 1)(\sqrt{x} - 5) = 0$

$\sqrt{x} = 1$ *or* 5 therefore $x = 1$ or 25

20 $y = x^4 - 2x^2 + 3$ therefore $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 4x^2$ then $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 4$

Gradient is decreasing where $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$:

So $12x^2 - 4 < 0$

$x^2 - 3 < 0$

Positive quadratic is less than zero between its roots. $\frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Chapter 14 worked solutions

14 Applications of differentiation

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 14A

2 $y = \dfrac{x^2+4}{\sqrt{x}} = x^{\frac{3}{2}} + 4x^{-\frac{1}{2}}$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3}{2}x^{\frac{1}{2}} - 2x^{-\frac{3}{2}}$

When $x = 4$: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3}{2}(4)^{\frac{1}{2}} - 2(4)^{-\frac{3}{2}} = \dfrac{11}{4}$

$y = \dfrac{4^2+4}{\sqrt{4}} = 10$

Tangent equation: $y - 10 = \dfrac{11}{4}(x-4)$

$4y - 40 = 11x - 44$ therefore $11x - 4y - 4 = 0$

3 $y = 2x - \dfrac{3}{\sqrt{x}} = 2x - 3x^{-\frac{1}{2}}$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2 + \dfrac{3}{2}x^{-\frac{3}{2}}$

When $x = 4$: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2 + \dfrac{3}{2}(4)^{-\frac{3}{2}} = \dfrac{35}{16}$

$y = 2(4) - \dfrac{3}{\sqrt{4}} = \dfrac{13}{2}$

Normal has gradient $-\dfrac{16}{35}$

Normal equation: $y - \dfrac{13}{2} = -\dfrac{16}{35}(x-4)$

$70y - 455 = -32x + 128$ therefore $32x + 70y - 583 = 0$

4 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 4$

When $x = 2$: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 24 - 4 = 20$

$y = 16 - 8 = 8$

Normal has gradient $-\dfrac{1}{20}$

Normal equation: $y - 8 = -\dfrac{1}{20}(x-2)$

$20y - 160 = -x + 2$ therefore $x = 162 - 20y$

Normal crosses the x-axis when $y = 0$: P is the point $(162, 0)$

5 $y = \dfrac{3}{x^2} = 3x^{-2}$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -6x^{-3}$

When $x = 3$: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -6(3)^{-3} = -\dfrac{2}{9}$

$y = \dfrac{3}{3^2} = \dfrac{1}{3}$

Tanget equation: $y - \dfrac{1}{3} = -\dfrac{2}{9}(x-3)$

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$9y - 3 = -2x + 6$ therefore $2x + 9y - 9 = 0$.

On the tangent:

When $x = 0$, $y = 1 : A(0, 1)$

When $y = 0, x = \frac{9}{2} : B\left(\frac{9}{2}, 0\right)$

AOB is a right-triangled with area $A = \frac{1}{2} \times 1 \times \frac{9}{2} = \frac{9}{4}$.

6 $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 18x + 12$

Tangents parallel to the x-axis have zero gradient:

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$6x^2 - 18x + 12 = 0$ so $6(x^2 - 3x + 2) = 0$

$6(x - 1)(x - 2) = 0$ therefore $x = 1$ or 2

When $x = 1 : y = 2 - 9 + 12 + 1 = 6$

When $x = 2 : y = 16 - 36 + 24 + 1 = 5$

The two tangents required are $y = 5$ and $y = 6$

7 $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4x$

When $x = 1 : \frac{\mathrm{d}y}{\mathrm{d}x} = 3 - 4 = -1$

Normal has gradient $-\frac{1}{-1} = 1$

Require point on the curve where $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 : 3x^2 - 4x = 1$ so $3x^2 - 4x - 1 = 0$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 3 \times (-1)}}{2 \times 3} = \frac{2 \pm \sqrt{7}}{3}$$
$$= 1.55 \text{ or } -0.215$$

8 Given line has equation:

$x - 2y + 14 = 0$

$\Leftrightarrow y = \frac{1}{2}(x + 14)$ so has gradient $\frac{1}{2}$

$\therefore$ normal to the curve at $(1, 2)$ has gradient $\frac{1}{2}$ so curve has gradient $-\frac{1}{\frac{1}{2}} = -2$ at that point.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2 + 2bx$

When $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -2 : -2 = 3a(1)^2 + 2b(1)$ so $-2 = 3a + 2b$

Also, as $(1, 2)$ lies on the curve: $2 = a(1)^3 + b(1)^2$ so $2 = a + b$

Solving simultaneously:

$$\left.\begin{aligned} -2 &= 3a + 2b \\ 2 &= a + b \end{aligned}\right\}$$

$a = -6$, $b = 8$

9 Given line has equation:

$2x + 3y - 12 = 0$

$\Leftrightarrow y = \frac{1}{3}(12 - 2x)$ so the gradient has $\frac{-2}{3}$

$\therefore$ tangent to the curve at $\left(1, -\frac{7}{6}\right)$ has gradient $-\frac{2}{3}$

$y = p\sqrt{x} + qx = px^{\frac{1}{2}} + qx$ therefore $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}px^{-\frac{1}{2}} + q$

When $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{3}$:

$-\frac{2}{3} = \frac{1}{2}p(1)^{-\frac{1}{2}} + q$ therefore $-\frac{2}{3} = \frac{1}{2}p + q$

Also, as $\left(1, -\frac{7}{6}\right)$ lies on the curve:

$-\frac{7}{6} = p\sqrt{1} + q(1)$ therefore $-\frac{7}{6} = p + q$

Solving simultaneously:

$$\left.\begin{aligned} -\frac{2}{3} &= \frac{1}{2}p + q \\ -\frac{7}{6} &= p + q \end{aligned}\right\}$$

$p = -1, q = -\frac{1}{6}$

10 $\mathrm{f}(x) = \frac{2}{x} = 2x^{-1}$ therefore $\mathrm{f}'(x) = -2x^{-2}$

$\mathrm{f}'(1) = -2(1)^{-2} = -2$

Tangent equation at $x = 1$:

$y - 2 = -2(x - 1)$ so $y = -2x + 4$

$\mathrm{f}'(-2) = -2(-2)^{-2} = -\frac{1}{2}$

Normal at $x = -2$ has gradient $-\frac{1}{-\frac{1}{2}} = 2$

Normal equation:

$y + 1 = 2(x + 2)$ therefore $y = 2x + 3$

These two lines intersect when:

$2x + 3 = -2x + 4$

$4x = 1$ therefore $x = \frac{1}{4}$

When $x = \frac{1}{4}$, $y = \frac{7}{2}$, intersection point is $\left(\frac{1}{4}, \frac{7}{2}\right)$.

11 $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x$

When $x = 2$: $\frac{\mathrm{d}y}{\mathrm{d}x} = 3(2)^2 - 6(2) = 0$

$y = 2^3 - 3(2)^2 = -4$

Tangent is $y = -4$

Intersection with original curve when:

$x^3 - 3x^2 = -4$ therefore $x^3 - 3x^2 + 4 = 0$

$(x - 2)^2(x + 1) = 0$ so $x = 2$ or -1

So the additional intersection occurs when $x = -1$.

$y = (-1)^3 - 3(-1)^2 = -4$

Point is $(-1, -4)$

> Note
>
> To factorise the cubic equation above, note that you already knew that $\mathrm{x} = 2$ was a double root to this cubic (since the line $\mathrm{y} = -4$ is a tangent to the curve, represented by a repeated intersection root), and so you immediately had $(x-2)^2$ as a factor.

12 $y = (x-1)^2 = x^2 - 2x + 1$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x - 2$.

At the point $(a,\, (a-1)^2)$, the gradient is $2a - 2$

The normal gradient at that point is $\dfrac{1}{2-2a}$

The normal equation at that point is $\left(y - (a-1)^2\right) = \dfrac{1}{2-2a}(x-a)$

Require that this normal passes through the origin: $x = 0$, $y = 0$

$-(a-1)^2 = -\dfrac{a}{2-2a}$ therefore $(a-1)^3 = -\dfrac{a}{2}$

Using a calculator to solve this: $a = 0.410$

So the point is $(0.410, 0.348)$.

13 The given normal has gradient $-\dfrac{1}{2}$ so the curve has gradient 2 at that point.

$y = x^2 + 5x + 4 \,\text{therefore}\, \dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x + 5$

$2x + 5 = 2 \,\text{so}\, x = -\dfrac{3}{2}$

$y\left(-\dfrac{3}{2}\right) = \dfrac{9}{4} - \dfrac{15}{2} + 4 = -\dfrac{5}{4}$

Substituting $x = -\dfrac{3}{2}$, $y = -\dfrac{5}{4}$ into the normal equation: $-\dfrac{5}{4} = \dfrac{3}{4} + c$ so $c = -2$.

14 $y = kx^{-1}$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -kx^{-2}$

Tangent at $\left(a,\, \dfrac{k}{a}\right)$ has gradient $-\dfrac{1}{a^2}$

Tangent has equation $y - \dfrac{k}{a} = -\dfrac{1}{a^2}(x-a)$

$y = -\dfrac{x}{a^2} + \dfrac{1+k}{a} = \dfrac{(1+k)a - x}{a^2}$

$x = 0 : P\left(0,\, \dfrac{1+k}{a}\right)$

$y = 0 : Q(a(1+k), 0)$

$\text{Area } OPQ = \dfrac{1}{2} \times a(1+k) \times \dfrac{1+k}{a} = \dfrac{(1+k)^2}{2}$

Area is indeed independent of a.

15 $y = x^3 - x$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 1$

Tangent at $(a, a^3 - a)$ has gradient $(3a^2 - 1)$

Tangent has equation $y - (a^3 - a) = (3a^2 - 1)(x - a)$ so $y = (3a^2 - 1)x - 2a^3$

Intersection with the original curve is known to have a double root at $x = a$, since this is the point at which the line is tangent to the curve.

$x^3 - x = (3a^2 - 1)x - 2a^3$ may be factorised accordingly: $x^3 - 3a^2x + 2a^3 = 0$

$(x-a)(x-a)(x+2a)=0$

The third root is seen to be $x=-2a$, so the tangent does indeed meet the curve again when $x=-2a$, unless $a=0$ when all three roots coincide.

EXERCISE 14B

2 $y=x^3+3x^2-24x+12$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=3x^2+6x-24=3(x^2+2x-8)=3(x+4)(x-2)$

Stationary points at $x=-4$ or 2

$y(-4)=-64+48+96+12=92$

$y(2)=8+12-48+12=-16$

Positive cubic so the stationary point with lower value x will be a local maximum, the other will be a local minimum.

Tip

A concrete assertion of this sort, together with an explanatory thumbnail, is sufficient justification for classifying a standard polynomial's stationary points. For any more complicated curves, use second derivative or sign-change analysis.

Stationary points: $(-4,92)$ is local maximum, $(2,-16)$ is local minimum.

3 $y=x^3-3x^2+4x-1$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x}=3x^2-6x+4$

For stationary point, require $3x^2-6x+4=0$

Discriminant $\Delta=(-6)^2-4\times3\times4=-12<0$

Negative discriminant, so no real roots to the quadratic.

4 **a** $\mathrm{f}(x)=9x+x^{-1}$ therfore $\mathrm{f}'(x)=9-x^{-2}$

$\mathrm{f}'(x)=0$ when $x=\pm\dfrac{1}{3}$

b $\mathrm{f}''(x)=2x^{-3}$

$\mathrm{f}''\left(\dfrac{1}{3}\right)>0$ so $\left(\dfrac{1}{3},6\right)$ is a local minimum.

$\mathrm{f}''\left(-\dfrac{1}{3}\right)<0$ so $\left(-\dfrac{1}{3},-6\right)$ is a local maximum.

5 $y=x-x^{\frac{1}{2}}$ therfore $\dfrac{\mathrm{d}y}{\mathrm{d}x}=1-\dfrac{1}{2}x^{-\frac{1}{2}}=0$ at stationary point.

$x^{-\frac{1}{2}}=2$ so $x=\dfrac{1}{4}$

$y\left(\dfrac{1}{4}\right)=\dfrac{1}{4}-\dfrac{1}{2}=-\dfrac{1}{4}$

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{1}{4}x^{-\frac{3}{2}}$

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(\dfrac{1}{4}\right)>0$ so $\left(\dfrac{1}{4},-\dfrac{1}{4}\right)$ is a local minimum.

6 **a** $y=6x^{\frac{7}{3}}-7x^2$ therfore $\dfrac{\mathrm{d}y}{\mathrm{d}x}=14x^{\frac{4}{3}}-14x=0$ at stationary point

$x^{\frac{4}{3}} - x = 0$ so $x\left(x^{\frac{1}{3}} - 1\right) = 0$

$x = 0$ or 1

Stationary points are $(0, 0)$ and $(1, -1)$

b $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 14\left(\dfrac{4}{3}x^{\frac{1}{3}} - 1\right)$

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}(0) = -14 < 0$ so $(0, 0)$ is a local maximum.

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}(1) = \dfrac{14}{3} > 0$ so $(1, -1)$ is a local minimum.

7 **a** $y = x^3 + 10x^2 + kx - 2$ therfore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 20x + k = 0$ when $x = -8$.

$3(64) + 20(-8) + k = 0 = 32 + k$ so $k = -32$

Factorising: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 20x - 32 = (x + 8)(3x - 4)$

The other root is $x = \dfrac{4}{3}$.

b $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x + 20$

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}(-8) = -28 < 0$ so there is a local maximum at $x = -8$.

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(\dfrac{4}{3}\right) = 28 > 0$ so there is a local minimum at $x = \dfrac{4}{3}$.

8 $y = ax^2 + bx - 2$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b = 0$ when $x = -2$

$-4a + b = 0$ so $b = 4a$

$y(1) = 13 = a + b - 2$

Substituting: $5a - 2 = 13$

$a = 3$, $b = 12$

9 $\mathrm{f}(x) = 4 - 9x + 6x^2 - x^3$

$\mathrm{f}'(x) = -9 + 12x - 3x^2 = -3(x^2 - 4x + 3) = -3(x - 1)(x - 3)$

Stationary points at $x = 1$, $x = 3$

Interval of x is $-2 \leqslant x \leqslant 4$

$\mathrm{f}(-2) = 4 + 18 + 24 + 8 = 54$, $\mathrm{f}(1) = 4 - 9 + 6 - 1 = 0$, $\mathrm{f}(3) = 4 - 27 + 54 - 27 = 4$

$\mathrm{f}(4) = 4 - 36 + 96 - 64 = 0$

Greatest value of $\mathrm{f}(x)$ is 54

Least value of $\mathrm{f}(x)$ is 0

$0 \leqslant \mathrm{f}(x) \leqslant 54$ for $-2 \leqslant x \leqslant 4$

10 $\mathrm{f}(x) = -\dfrac{1}{4}x^4 - x^3 + 2x^2 - 10$

$\mathrm{f}'(x) = -x^3 - 3x^2 + 4x = -x(x^2 + 3x - 4) = -x(x + 4)(x - 1)$

Stationary points at $x = -4, 0$ and 1.

$\mathrm{f}(x)$ is a negative quartic so it will have maxima at the first and third of these stationary points.

$\mathrm{f}(-4) = -64 + 64 + 32 - 10 = 22$ and $\mathrm{f}(1) = -\dfrac{1}{4} - 1 + 2 - 10 = -\dfrac{37}{4}$

So the upper bound for $\mathrm{f}(x)$ is $\mathrm{f}(-4) = 22$

So $f(x) \leqslant 22$ for all x.

11 a $y = 3x^4 - 16x^3 + 18x^2 + 6$

$\frac{dy}{dx} = 12x^3 - 48x^2 + 36x = 12x(x^2 - 4x + 3) = 12x(x-1)(x-3)$

Stationary points where $\frac{dy}{dx} = 0 : x = 0, 1$ or 3

$y(0) = 6$, $y(1) = 3 - 16 + 18 + 6 = 11$ and $y(3) = 243 - 432 + 162 + 6 = -21$

Stationary points are $(0, 6)$, $(1, 11)$ and $(3, -21)$

b A horizontal line will cross the quartic in four different points as long as it is lower than the maximum at $(1, 11)$ and greater than the upper local minimum at $(0, 6) : 6 < c < 11$

12 $y = kx^3 + 6x^2$ therfore $\frac{dy}{dx} = 3kx^2 + 12x$

Stationary points where $\frac{dy}{dx} = 0$

$\Rightarrow kx^2 + 4x = 0 \Rightarrow x(kx + 4) = 0 \Rightarrow x = 0$ or $-\frac{4}{k}$

Stationary points are $(0, 0)$ and $\left(-\frac{4}{k}, \frac{32}{k^2}\right)$

$\frac{d^2y}{dx^2} = 6kx + 12$

$\frac{d^2y}{dx^2}(0) = 12 > 0 \Rightarrow (0, 0)$ is a local minimum

$\frac{d^2y}{dx^2}\left(-\frac{4}{k}\right) = -12 < 0 \Rightarrow \left(-\frac{4}{k}, \frac{32}{k^2}\right)$ is a local maximum

EXERCISE 14C

2 a $\text{Area} = x(30 - x) = 30x - x^2$ therefore $\frac{\text{d Area}}{dx} = 30 - 2x$

Stationary point when $\frac{\text{d Area}}{dx} = 0 \Rightarrow x = 15$

Since the extreme values $x = 0$ and $x = 30$ clearly give zero area while intermediate values give a positive area, the end points do not provide a maximum value, and the stationary point is a maximum.

$\text{Area}\ (15) = 15 \times 15 = 225\ \text{m}^2$

b $\text{Perimeter} = 2x + 2(30 - x) = 60\ \text{m}$

3 a If the angle of the sector is θ then $A = \pi r^2 \times \frac{\theta}{360^\circ}$

$\frac{\theta}{360^\circ} = \frac{A}{\pi r^2}$

Perimeter of the circle equals the straight edges plus the arc.

$40 = 2r + 2\pi r \times \frac{\theta}{360^\circ}$

Substituting from above: $40 = 2r + 2\pi r \times \frac{A}{\pi r^2} = 2r + \frac{2A}{r}$

$20 - r = \frac{A}{r}$ so $A = 20r - r^2$

b i $\frac{dA}{dr} = 20 - 2r$

Stationary point is at $r = 10$.

ii A is a negative quadratic in r so its stationary point must be the maximum.

4 **a** The base of the box is a square with side length $12 - 2x$, and the box sides have height x : $V = x(12 - 2x)^2$

b $V = 4x(6 - x)^2 = 4x(x^2 - 12x + 36) = 4(x^3 - 12x^2 + 36x)$

$\frac{\mathrm{d}V}{\mathrm{d}x} = 4(3x^2 - 24x + 36) = 12(x^2 - 8x + 12) = 12(x - 2)(x - 6)$

Stationary values for V occur when $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$.

$x = 6$ or 2

Clearly, $x = 6$ represents a minimum, with zero volume, so $x = 2$ represents the maximum.

Checking using the second derivative: $\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -12(6 - x) - 12(2 - x) = -12(8 - 2x)$

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2}(2) = -48 < 0$ so volume is indeed a maximal value at $x = 2$.

5 Let $y = x + x^{-1}$, where $x > 0$ therefore $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - x^{-2}$

Stationary point at $x = 1$ (not considering negative x)

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2x^{-3}$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(1) = 2 > 0$ so there is a minimum at $x = 1$

The minimum possible value of the sum of a positive real number and its reciprocal is $1 + 1 = 2$

6 **a** Volume $V = \pi r^2 h$

Surface area $S = 2\pi r^2 + 2\pi rh = 2\pi r(r + h) = 450$

$h = \frac{450}{2\pi r} - r$

$V = \pi r^2 \left(\frac{450}{2\pi r} - r\right) = 225r - \pi r^3$

b $\frac{\mathrm{d}V}{\mathrm{d}r} = 225 - 3\pi r^2$

Stationary values of V occur when $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$

$\Rightarrow 3\pi r^2 = 225 \Rightarrow r = \sqrt{\frac{75}{\pi}}$

$\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = -6\pi r \Rightarrow \frac{\mathrm{d}^2V}{\mathrm{d}r^2}\left(\sqrt{\frac{75}{\pi}}\right) < 0$

There is a local maximum in V at $r = \sqrt{\frac{75}{\pi}}$; $V\left(\sqrt{\frac{75}{\pi}}\right) = \sqrt{\frac{75}{\pi}}(225 - 75) = 733\,\mathrm{cm}^3$.

7 **a** Let the height be h. Then: volume $V = x^2h$

Surface area $S = 2x^2 + 4xh = 486$

$h = \frac{486 - 2x^2}{4x}$

Substituting into the formula for volume: $V = \frac{x^2(486 - 2x^2)}{4x} = \frac{(243 - x^2)x}{2}$

b $\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{243 - 3x^2}{2}$

Stationary values of V occur when $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$.

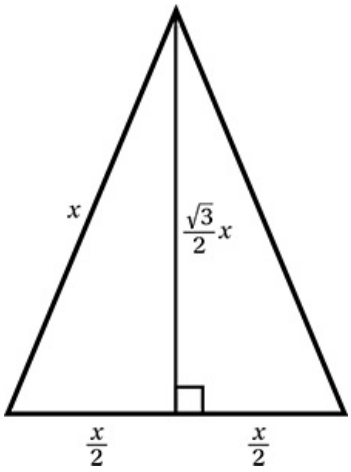
$3x^2 = 243$ therefore $x^2 = 81$ so $x = \pm 9$

Clearly only the positive value is relevant, and because V is a negative cubic, the greater stationary point is the maximum, so V is a maximum at $x = 9$.

$V(9) = 729$

The maximum volume is $729\,\mathrm{cm}^3$.

8 **a** Area of an equilateral triangle with side length x $\frac{1}{2} \times x \times \frac{\sqrt{3}x}{2} = \frac{\sqrt{3}x^2}{4}$



For the box: Volume $V = \frac{\sqrt{3}x^2y}{4} = 128$ so $y = \frac{512}{\sqrt{3}x^2}$

Surface area $A = 3xy + 2 \times \frac{\sqrt{3}x^2}{4}$

Substituting: $A = 3x \times \left(\frac{512}{\sqrt{3}x^2}\right) + \frac{\sqrt{3}x^2}{2}$ therefore $A = \sqrt{3}\left(\frac{512}{x} + \frac{x^2}{2}\right)$.

b $\frac{\mathrm{d}A}{\mathrm{d}x} = \sqrt{3}\left(-\frac{512}{x^2} + x\right)$

Minimum A when $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$.

$x = \frac{512}{x^2}$ therefore $x^3 = 512$ so $x = 8$

$A(8) = 96\sqrt{3} \approx 166\,\mathrm{cm}^2$

c $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = \sqrt{3}\left(\frac{1024}{x^3} + 1\right)$

$\frac{\mathrm{d}^2A}{\mathrm{d}x^2}(8) > 0$ so this is indeed a minimum.

9 **a** Volume of cone $V = \frac{1}{3}\pi r^2 h = 81\pi$ therfore $h = \frac{243}{r^2}$

Curved surface area $S = \pi r l$

Where $l = \sqrt{r^2 + h^2}$ is the angled length of the cone side.

$S = \pi r\sqrt{r^2 + \left(\frac{243}{r^2}\right)^2} = \pi\sqrt{r^4 + \frac{243^2}{r^2}}$

b $S^2 = \pi^2(r^4 + 243^2 r^{-2})$ therfore $\frac{\mathrm{d}}{\mathrm{d}r}(S^2) = \pi^2(4r^3 - 2 \times 243^2 r^{-3})$

Minimum S^2 when $\frac{\mathrm{d}}{\mathrm{d}r}\left(S^2\right) = 0$.

$2r^3 = 243^2 r^{-3}$ therefore $r = \left(\frac{243^2}{2}\right)^{\frac{1}{6}} = 5.56$

$h = \frac{243}{r^2} = 7.86$

Check that this is minimum: $\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(S^2\right) = \pi^2\left(12r^2 + 6 \times 243^2 r^{-4}\right) > 0$ for $r = 5.56$.

10 $y = 6 - x$

$S = x^2 + y^2 = x^2 + (6-x)^2 = 2x^2 - 12x + 36$

$\Rightarrow \frac{\mathrm{d}S}{\mathrm{d}x} = 4x - 12$

Stationary values of S when $\frac{\mathrm{d}S}{\mathrm{d}x} = 0$

$\Rightarrow x = 3$

$\frac{\mathrm{d}^2 S}{\mathrm{d}x^2} = 4 > 0$ so $x = 3$, $y = 3$ represents a minimum of the sum of squares.

a $x = y = 3$

b End point values of 0 and 6 produce the maximum value.

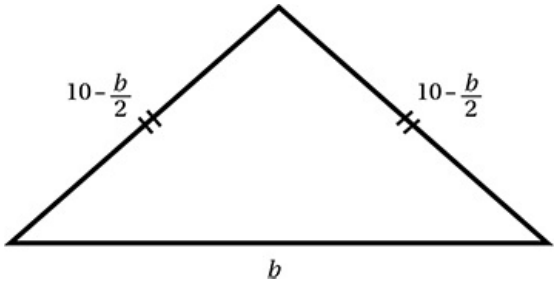
11 From context, $0 \leqslant p \leqslant 100$ (though a 0% or 100% saturated fat butter probably wouldn't be legitimately called 'butter').

$\frac{\mathrm{d}t}{\mathrm{d}p} = \frac{p}{5000} + \frac{1}{100}$

$\frac{\mathrm{d}t}{\mathrm{d}p} = 0$ when $p = -50$, which is not in the available interval of values for p.

Checking the extreme available values: $t(0) = 2$, $t(100) = 4$ so these must be the maximum and minimum values for t.

12 a The triangle has side lengths $b, 10 - \frac{b}{2}, 10 - \frac{b}{2}$.



Pythagoras' theorem gives the height h of the triangle are:

$h = \sqrt{\left(10 - \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2} = \sqrt{100 - 10b}$

Area of triangle $A = \frac{bh}{2} = \frac{b}{2}\sqrt{100 - 10b} = \frac{1}{2}\sqrt{100b^2 - 10b^3}$

b $A^2 = \frac{1}{4}(100b^2 - 10b^3)$ therefore $\frac{\mathrm{d}}{\mathrm{d}b}(A^2) = \frac{1}{4}(200b - 30b^2) = \frac{b}{2}(100 - 15b)$

Stationary value of A when $\frac{\mathrm{d}A}{\mathrm{d}b} = 0$.

$15b = 100$ or $b = 0$

Clearly $b = 0$ is a degenerate case and would produce a minimum area.

$b = \frac{100}{15} = \frac{20}{3}$

That is, the base length is one third of the perimeter of the isosceles triangle, so it is equilateral.

13 $x^2 + y^2 = a$ so $y = \sqrt{a - x^2}$

The product $P = xy = x\sqrt{a - x^2} = x\left(a - x^2\right)^{\frac{1}{2}}$

The square of the product will be easier to work with, and since the product is definitely positive, this is not a problem.

$P^2 = x^2(a^2 - x^2)$ therefore $\frac{\mathrm{d}}{\mathrm{d}x}(P^2) = 2a^2x - 4x^3 = 2x(a^2 - 2x^2)$

Stationary values of P when $\frac{\mathrm{d}P}{\mathrm{d}x} = 0$.

$2x^2 = a$ or $x = 0$, which makes for the minimum product $\sqrt{0}$.

$x^2 = \frac{a}{2} = y^2$ therefore $x = y$

$\frac{\mathrm{d}^2P^2}{\mathrm{d}x^2} = 2a^2 - 12x^2$

$\frac{\mathrm{d}^2P^2}{\mathrm{d}x^2}\left(\sqrt{\frac{a}{2}}\right) = 2a^2 - 6a^2 = -4a^2 < 0$

P is at a maximum when $x = y$

14 The distance d from point $(a,\ a^2)$ to $(0, 4)$ is given by:

$d^2 = a^2 + (4 - a^2)^2 = 16 - 7a^2 + a^4$ therefore $\frac{\mathrm{d}}{\mathrm{d}a}d^2 = -14a + 4a^3$

Stationary value of d^2 occur when $\frac{\mathrm{d}}{\mathrm{d}a}d^2 = 0$.

$4a^3 = 14a$ therefore $a^2 = \frac{7}{2}$ or $a = 0$

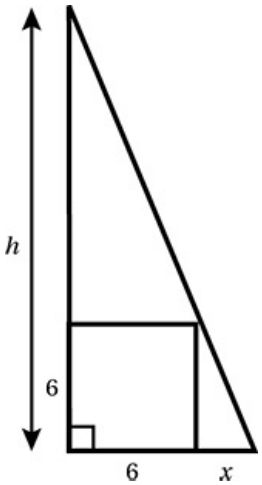
$\frac{\mathrm{d}^2}{\mathrm{d}a^2}d^2 = -14 + 12a^2$

$\frac{\mathrm{d}^2}{\mathrm{d}a^2}d^2(0) = -14 < 0 : a = 0$ represents a local maximum value for d^2.

$\frac{\mathrm{d}^2}{\mathrm{d}a^2}d^2\left(\sqrt{\frac{7}{2}}\right) = 28 > 0 : a = \pm\sqrt{\frac{7}{2}}$ represent local minimum values for d^2.

The point closest to the curve for $x \geqslant 0$ is $\left(\sqrt{\frac{7}{2}},\ \frac{7}{2}\right)$.

15 a Considering the cross-section triangle, the ratios of the sides of the large triangle must be the same as the ratio of the small triangle (they are similar):



$\frac{6 + x}{h} = \frac{x}{6}$

Rearranging: $h = \frac{6}{x}(6 + x) = \frac{36}{x} + 6$

b Volume of the cone is $V = \frac{1}{3}\pi r^2 h$.

$$V = \frac{1}{3}\pi(6+x)^2\left(\frac{36}{x}+6\right) = \pi(6+x)^2\left(\frac{12}{x}+2\right)$$
$$= 2\pi(36+12x+x^2)\left(\frac{6}{x}+1\right) = 2\pi\left(\frac{216}{x}+108+18x+x^2\right)$$

c $\frac{\mathrm{d}V}{\mathrm{d}x} = 2\pi\left(2x+18-\frac{216}{x^2}\right) = \frac{4\pi}{x^2}(x^3+9x^2-108)$

Either from a calculator or from trying values, you find $\frac{\mathrm{d}V}{\mathrm{d}x}(3) = 0$.

$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{4\pi}{x^2}(x-3)(x^2+12x+36) = \frac{4\pi}{x^2}(x-3)(x+6)^2$

So the only valid stationary point is at $x = 3$.

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = 4\pi\left(1+\frac{432}{x^3}\right)$

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2}(3) > 0$ so the volume is at minimum when $x = 3$.

Minimum cone volume is $V(3) = 2\pi(72+108+54+9) = 486\pi$.

MIXED PRACTICE 14

1 Stationary point has a horizontal tangent. The horizontal line passing $(2, \mathrm{f}(2))$ will have equation $y = \mathrm{f}(2)$.

2 $y = x^2 - 3x^{-1}$ so $y(2) = 4 - \frac{3}{2} = \frac{5}{2}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 3x^{-2}$ therefore $\frac{\mathrm{d}y}{\mathrm{d}x}(2) = 4 + \frac{3}{4} = \frac{19}{4}$

Tangent has equation $y - \frac{5}{2} = \frac{19}{4}(x-2)$

$4y - 10 = 19x - 38$ therefore $19x - 4y = 28$.

3 You need to expand the brackets before differentiating.

Tip

In Student Book 2 you will learn how to differentiate this expression without expanding the brackets.

$y = x^3 - 6x^2 + 12x - 8$ therefore $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12x + 12$

At $x = 2$, $y = \frac{\mathrm{d}y}{\mathrm{d}x} = 0$ so the normal is vertical through $(2, 0)$: $x = 2$

4 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{2} - 2x + 1 = \frac{1}{2}(x^2 - 4x + 2)$

Stationary point where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$x = 2 \pm \sqrt{2}$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x - 2$

At $x = 2 + \sqrt{2}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ and there is a minimum.

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At $x = 2 - \sqrt{2}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ and there is a maximum.

5 $y = \dfrac{1}{4}x^4 - 2x^3 + 4x^2 - 1$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^3 - 6x^2 + 8x = x(x-2)(x-4)$

a Stationary points where $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$x = 0, 2$ or 4

$y(0) = -1$, $y(2) = 4 - 16 + 16 - 1 = 3$ and $y(4) = 64 - 128 + 64 - 1 = -1$

$(0, -1)(2,\ 3)(4, -1)$

b Positive quartic with three stationary points so the first and third stationary pointsare minima and the second is a local maximum.

$(0, -1)$ and $(4, -1)$ are local minima.

$(2, 3)$ is a local maximum.

6 Let A have coordinates $(x, 0)$ where $x > 0$.

Then $\text{Area} = 2x(4 - x^2) = 8x - 2x^3$ therefore $\dfrac{\mathrm{d}\ \text{Area}}{\mathrm{d}x} = 8 - 6x^2$

Stationary value for area when $\dfrac{\mathrm{d}\ \text{Area}}{\mathrm{d}x} = 0$.

$x^2 = \dfrac{8}{6} = \dfrac{4}{3}$ so $x = \dfrac{2}{\sqrt{3}}$

It is clear that this is a maximum rather than a minimum from the context. Showing this rigorously:

$\dfrac{\mathrm{d}^2\ \text{Area}}{\mathrm{d}x^2} = -12x \Rightarrow \dfrac{\mathrm{d}^2\ \text{Area}}{\mathrm{d}x^2}\left(\dfrac{2}{\sqrt{3}}\right) < 0$

$y\left(\dfrac{2}{\sqrt{3}}\right) = 4 - \dfrac{4}{3} = \dfrac{8}{3}$

A has coordinates $\left(\dfrac{2\sqrt{3}}{3}, 0\right)$.

7 **a** The length of the enclosure is $20 - 2x$, so the area is:

$A = x(20 - 2x) = 20x - 2x^2$

b The maximum point has $\dfrac{\mathrm{d}A}{\mathrm{d}x} = 0$.

$\dfrac{\mathrm{d}A}{\mathrm{d}x} = 20 - 4x = 0$ when $x = 5$

To check that this is a maximum (rather than a minimum) point:

$\dfrac{\mathrm{d}^2x}{\mathrm{d}x^2} = -4 < 0$,so its is minimum.

The maximum value of A is $5(20 - 2 \times 5) = 50\,\mathrm{m}^2$.

8 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2 - 2ax + 3a$

Stationary point where $x^2 - 2ax + 3a = 0$.

Only one root, so discriminant $\Delta = 0$.

$\Delta = (-2a)^2 - 4(3a) = 0$

$4a^2 - 12a = 0$

$a = 0$ (specifically excluded) or 3.

9 $y = ax^3 + bx^2 + 8x - 1$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2 + 2bx + 8$

$\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1}{3}\right) = 0 : \frac{a}{3} + \frac{2b}{3} + 8 = 0$

$a + 2b = -24 \qquad (1)$

$\frac{\mathrm{d}y}{\mathrm{d}x}(4) = 0 : 48a + 8b + 8 = 0$

$6a + b = -1 \qquad (2)$

$(1) - 2(2) : -11a = -22$ therefore $a = 2$, $b = -13$

10 $\mathrm{f}(x) = x^2 - 4x^{\frac{3}{2}} + 4x - 3$ therefore $\mathrm{f}'(x) = 2x - 6x^{\frac{1}{2}} + 4$

Stationary points where $\mathrm{f}'(x) = 0$.

Let $u = x^{\frac{1}{2}}$

$2u^2 - 6u + 4 = 0$ therefore $2(u-1)(u-2) = 0$ so $u = 1$ or 2.

$x = 1^2 = 1$ or $2^2 = 4$

Two stationary points, $x = 1$ or 4.

$\mathrm{f}''(x) = 2 - 3x^{-\frac{1}{2}}$

$\mathrm{f}''(1) = 2 - 3 = -1 < 0$: local maximum at $x = 1$.

$\mathrm{f}''(4) = 2 - \frac{3}{2} = \frac{1}{2} > 0$: local minimum at $x = 4$.

11 $\mathrm{f}(x) = x^3 - 9x$

a $\mathrm{f}'(x) = 3x^2 - 9 = 3\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$

Stationary points where $\mathrm{f}'(x) = 0$.

$x = \pm\sqrt{3}$

$\mathrm{f}\left(\pm\sqrt{3}\right) = \pm 6\sqrt{3}$

Stationary points are $\pm\left(\sqrt{3}, -6\sqrt{3}\right)$.

b Test values at stationary points and limits of interval $-2 \leqslant x \leqslant 5$.

$\mathrm{f}(-2) = 10$ and $\mathrm{f}(5) = 80$

Minimum value of $\mathrm{f}(x)$ over $-2 \leqslant x \leqslant 5$ is $-6\sqrt{3}$ at the curve minimum.

Maximum value of $\mathrm{f}(x)$ over $-2 \leqslant x \leqslant 5$ is 80 at $x = 5$.

12 $y = x^{-2}$ so $y(3) = \frac{1}{9}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x^{-3}$ so $\frac{\mathrm{d}y}{\mathrm{d}x}(3) = -\frac{2}{27}$

Tangent is $\left(y - \frac{1}{9}\right) = -\frac{2}{27}(x - 3)$ therefore $27y - 3 = -2x + 6$ so $2x + 27y = 9$

$x = 0 : M\left(0,\ \frac{1}{3}\right)$

$y = 0 : N\left(\frac{9}{2}, 0\right)$

Area of triangle MON is $\frac{1}{2} \times \frac{9}{2} \times \frac{1}{3} = \frac{3}{4}$.

13 $V = 300(t^2 - t^3) + 4$

a $V(0) = 4$

Initially 4 litres in the tank.

b $V(0.5) = 300(0.25 - 0.125) + 4 = 41.5$

The maximum capacity of the tank is 41.5 litres.

c $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 300(2t - 3t^2)$ is the flow rate.

$\dfrac{\mathrm{d}^2V}{\mathrm{d}t^2} = 600(1 - 3t)$ is the rate of change of flow rate.

Flow rate is greatest when $\dfrac{\mathrm{d}^2V}{\mathrm{d}t^2} = 0$.

Greatest flow rate at $t = \dfrac{1}{3}$, so after 20 seconds.

> Note
>
> This is clearly a maximum, since flow rate is a negative quadratic and in any case minimum flow rate is zero, at $t = 0$.

14 $A = \dfrac{1}{3}\pi r^2 + rx = 200$

a Perimeter runs over one radius r, two short sides of the rectangle x, the base of the rectangle r and the arc length $\dfrac{2\pi r}{3}$.

$P = 2r + 2x + \dfrac{2\pi r}{3}$

From above, $x = \dfrac{200}{r} - \dfrac{1}{3}\pi r$

$P = 2r + 2\left(\dfrac{200}{r} - \dfrac{1}{3}\pi r\right) + \dfrac{2\pi r}{3} = 2r + \dfrac{400}{r}$

b $\dfrac{\mathrm{d}P}{\mathrm{d}r} = 2 - \dfrac{400}{r^2}$

Stationary point at $\dfrac{\mathrm{d}P}{\mathrm{d}r} = 0 : r^2 = 200$ so $r = 10\sqrt{2}$

$\dfrac{\mathrm{d}^2P}{\mathrm{d}r^2} = 800r^{-3} > 0$ so this stationary point is a minimum.

$P\left(10\sqrt{2}\right) = 40\sqrt{2}$

15 a $y = 3x^2 - 6x^{-1} - 2$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6x + 6x^{-2} = 0$

$\Leftrightarrow 6x + \dfrac{6}{x^2} = 0 \Leftrightarrow x^3 = -1$

$\therefore x = -1, y = 3 + 6 - 2 = 7$

The coordinates are $(-1, 7)$.

b $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6 - 12x^{-3} = 6 - \dfrac{12}{-1} = 18 > 0$

The stationary point is a minimum.

16 $y = ax^3 + bx^2 + 4$

a Intersection of curve and line at $(2, 24)$

$24 = 8a + 4b + 4$ therefore $2a + b = 5 \quad (1)$

b Gradient of tangent line is 24

For curve: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2 + 2bx$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}(2) = 12a + 4b = 24$ therefore $3a + b = 6 \quad (2)$

c $(2) - (1) : a = 1$ Solution is $a = 1$, $b = 3$

d $24x - 24 = x^3 + 3x^2 + 4$

$x^3 + 3x^2 - 24x + 28 = 0$

You know there is a double factor of $(x - 2)$ because this is the point of contact between tangent and curve and must produce a double root to the intersection cubic. Factorising:
$(x - 2)^2(x + 7) = 0$

Final solution is $x = -7$, which represents the point at which the tangent cuts the curve again.

$y(-7) = 24 \times -8 = -192$

Line meets curve again at $(-7, -192)$.

17 $y = x^3$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$

Tangent at $x = a$ has gradient $3a^2$.

Tangent has equation $(y - a^3) = 3a^2(x - a)$ therefore $y = 3a^2x - 2a^3$

y-intercept of the tangent is $(0, -2a^3)$

Normal at $x = -a$ has gradient $-\dfrac{1}{3a^2}$

Normal has equation $(y + a^3) = -\dfrac{1}{3a^2}(x + a)$ therefore $y = -\dfrac{1}{3a^2}x - \dfrac{1}{3a} - a^3$

y-intercept of the normal is $\left(0, -\dfrac{1}{3a} - a^3\right)$

If these intercepts are co-located then $-2a^3 = -\dfrac{1}{3a} - a^3$

$\dfrac{1}{3a} = a^3$ therefore $a = 3^{-\frac{1}{4}}$

18 $y = ax^2 + 24x^{-1}$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2ax - 24x^{-2}$

Stationary point when $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$ax - 12x^{-2} = 0$ therefore $x^3 = \dfrac{12}{a}$

Stationary point is at $y = \dfrac{ax^3 + 24}{x} = 18$

Substituting $x^3 = \dfrac{12}{a}$: $\dfrac{12 + 24}{x} = \dfrac{36}{x} = 18$ therefore $x = 2$

$a = \dfrac{12}{x^3} = \dfrac{12}{8} = 1.5$

Chapter 15 worked solutions

15 Integration

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 15A

4 $\mathrm{f}(x) = \int \frac{x^3}{2} - 6x^{-\frac{5}{3}} + 2\,\mathrm{d}x = \frac{1}{2}\left(\frac{1}{3+1}x^{3+1}\right) - 6\left(\frac{1}{-\frac{5}{3}+1}x^{-\frac{5}{3}+1}\right) + 2x + c$

$= \frac{1}{2}\left(\frac{1}{4}x^4\right) - 6\left(-\frac{3}{2}x^{-\frac{2}{3}}\right) + 2x + c = \frac{1}{8}x^4 + 9x^{-\frac{2}{3}} + 2x + c$

5 $\int x^{\frac{3}{2}} - x^{-\frac{3}{2}}\,\mathrm{d}x = \frac{1}{\frac{3}{2}+1}x^{\frac{3}{2}+1} - \frac{1}{-\frac{3}{2}+1}x^{-\frac{3}{2}+1} + c = \frac{1}{\frac{5}{2}}x^{\frac{5}{2}} - \frac{1}{-\frac{1}{2}}x^{-\frac{1}{2}} + c$

$= \frac{2}{5}x^{\frac{5}{2}} + 2x^{-\frac{1}{2}} + c$

6 $\int \frac{x^5 - 3x^{-3} + x}{2}\mathrm{d}x = \frac{1}{2}\int x^5 - 3x^{-3} + x\mathrm{d}x$

$= \frac{1}{2}\left\{\frac{1}{5+1}x^{5+1} - 3\left(\frac{1}{-3+1}x^{-3+1}\right) + \frac{1}{2}x^{1+1}\right\} + c$

$= \frac{1}{2}\left\{\frac{1}{6}x^6 + \frac{3}{2}x^{-2} + \frac{1}{2}x^2\right\} + c = \frac{1}{12}x^6 + \frac{3}{4}x^{-2} + \frac{1}{4}x^2 + c$

EXERCISE 15B

4 $\int 10x\sqrt{x}\,\mathrm{d}x = \int 10x^{\frac{3}{2}}\,\mathrm{d}x = 10\left(\frac{2}{5}x^{\frac{5}{2}}\right) + c = 4x^{\frac{5}{2}} + c$

5 $\int x^2\left(\sqrt[3]{x} - 12\right)\,\mathrm{d}x = \int x^{\frac{7}{3}} - 12x^2\mathrm{d}x = \frac{3}{10}x^{\frac{10}{3}} - 4x^3 + c$

6 $\int 9\sqrt[4]{x^5}\,\mathrm{d}x = \int 9x^{\frac{5}{4}}\mathrm{d}x = 9\left(\frac{4}{9}x^{\frac{9}{4}}\right) + c = 4x^{\frac{9}{4}} + c$

7 $\int \frac{1}{3\sqrt[3]{x^2}}\mathrm{d}x = \int \frac{1}{3}x^{-\frac{2}{3}}\mathrm{d}x = \frac{1}{3}\left(3x^{\frac{1}{3}}\right) + c = x^{\frac{1}{3}} + c$

8 $\int \left(\frac{1}{\sqrt{x}} - 1\right)\mathrm{d}x = \int \left(x^{-\frac{1}{2}} - 1\right)\mathrm{d}x = 2x^{\frac{1}{2}} + c$

9 $\int \frac{1}{3x^3} + \frac{1}{4x^4}\mathrm{d}x = \int \frac{1}{3}x^{-3} + \frac{1}{4}x^{-4}\mathrm{d}x$

$= \frac{1}{3}\left(-\frac{1}{2}x^{-2}\right) + \frac{1}{4}\left(-\frac{1}{3}x^{-3}\right) + c = -\frac{1}{6}x^{-2} - \frac{1}{12}x^{-3} + c$

10 a $\mathrm{f}(x) = \frac{6+x}{\sqrt[3]{x}} = \frac{6}{x^{\frac{1}{3}}} + \frac{x}{x^{\frac{1}{3}}} = 6x^{-\frac{1}{3}} + x^{\frac{2}{3}}$

b $\int \mathrm{f}(x)\mathrm{d}x = 6\left(\frac{3}{2}x^{\frac{2}{3}}\right) + \frac{3}{5}x^{\frac{5}{3}} + c = 9x^{\frac{2}{3}} + \frac{3}{5}x^{\frac{5}{3}} + c$

11 a $y = \frac{(x+2)(x-2)}{2\sqrt{x}} = \frac{x^2-4}{2x^{\frac{1}{2}}} = \frac{x^2}{2x^{\frac{1}{2}}} - \frac{4}{2x^{\frac{1}{2}}} = \frac{1}{2}x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}$

$a = \frac{1}{2}, b = -2$

b $\int y\mathrm{d}x = \frac{1}{2}\left(\frac{2}{5}x^{\frac{5}{2}}\right) - 2\left(2x^{\frac{1}{2}}\right) + c = \frac{1}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + c$

12 $\int 12\sqrt{x} - \frac{4}{\sqrt{x}}\mathrm{d}x = \int 12x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}\,\mathrm{d}x = 12\left(\frac{2}{3}x^{\frac{3}{2}}\right) - 4\left(2x^{\frac{1}{2}}\right) + c$

$= 8x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + c = 8x^{\frac{1}{2}}(x-1) + c = 8\sqrt{x}(x-1) + c$

13 $\int \frac{(x+4)^2}{8x\sqrt{x}}\mathrm{d}x = \int \frac{x^2+8x+16}{8x^{\frac{3}{2}}}\,\mathrm{d}x = \int \frac{1}{8}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}}\,\mathrm{d}x$

$= \frac{1}{8}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + 2x^{\frac{1}{2}} + 2\left(-2x^{-\frac{1}{2}}\right) + c$

$= \frac{1}{12}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} + c$

EXERCISE 15C

3 $y = \int x - x^{-2}\mathrm{d}x = \frac{1}{2}x^2 + x^{-1} + c$

Passes through $(1, 3)$: $3 = \frac{1}{2} + 1 + c$ so $c = \frac{3}{2}$.

$\therefore y = \frac{1}{2}x^2 + \frac{1}{x} + \frac{3}{2}$

4 $\mathrm{f}'(x) = \frac{2x-1}{\sqrt{x}} = 2x^{\frac{1}{2}} - x^{-\frac{1}{2}}$

$\mathrm{f}(x) = \int 2x^{\frac{1}{2}} - x^{-\frac{1}{2}}\,\mathrm{d}x$

$= \frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$

$\mathrm{f}(4) = 2$

$\frac{4}{3}(4)^{\frac{3}{2}} - 2(4)^{\frac{1}{2}} + c = 2$ therefore $\frac{32}{3} - 4 + c = 2$ so $c = -\frac{14}{3}$

$\therefore \mathrm{f}(x) = \frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - \frac{14}{3}$

5 $\mathrm{f}'(x) = \sqrt{x}(5x - 4) = 5x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$

$\mathrm{f}(x) = \int 5x^{\frac{3}{2}} - 4x^{\frac{1}{2}}\,\mathrm{d}x = 2x^{\frac{5}{2}} - \frac{8}{3}x^{\frac{3}{2}} + c$

$\mathrm{f}(1) = \frac{19}{3}$

$2 - \frac{8}{3} + c = \frac{19}{3}$ therefore $-\frac{2}{3} + c = \frac{19}{3}$ so $c = 7$

$\therefore \mathrm{f}(x) = 2x^{\frac{5}{2}} - \frac{8}{3}x^{\frac{3}{2}} + 7$

$\mathrm{f}(4) = 2(4)^{\frac{5}{2}} - \frac{8}{3}(4)^{\frac{3}{2}} + 7 = \frac{149}{3}$

6 $\frac{dy}{dx} = kx$

When $x = 3$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 12 : 12 = 3k$ so $k = 4$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = 4x$

$y = \int 4x\,\mathrm{d}x = 2x^2 + c$

Passes through $(3,\, 2)$:

$2 = 2(3)^2 + c$ therefore $2 = 18 + c$ so $c = -16$

$\therefore y = 2x^2 - 16$

7 **a** Stationary points when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0 : x^2 - 4 = 0$ so $x = \pm 2$

Nature of stationary points: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2x$

When $x = 2 : \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4 > 0 \therefore x = 2$ is a minimum.

When $x = -2 : \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4 < 0 \therefore x = -2$ is a maximum.

b $y = \int x^2 - 4\mathrm{d}x = \frac{1}{3}x^3 - 4x + c$

Passes through $(0,\, 2) : 2 = 0 - 0 + c$ so $c = 2$

$\therefore y = \frac{1}{3}x^3 - 4x + 2$

When $x = -2 : y = \frac{1}{3}(-2)^3 - 4(-2) + 2 = 7\frac{1}{3}$

8 $\frac{\mathrm{d}y}{\mathrm{d}x} = \int 2 - x^2\mathrm{d}x = 2x - \frac{1}{3}x^3 + c$

When $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 5 : 5 = 2 - \frac{1}{3} + c$ so $c = \frac{10}{3}$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \frac{1}{3}x^3 + \frac{10}{3}$

$y = \int 2x - \frac{1}{3}x^3 + \frac{10}{3}\,\mathrm{d}x = x^2 - \frac{1}{12}x^4 + \frac{10}{3}x + d$

Passes through $(1, 3)$:

$3 = 1 - \frac{1}{12} + \frac{10}{3} + d$ so $d = -\frac{5}{4}$

$\therefore y = x^2 - \frac{1}{12}x^4 + \frac{10}{3}x - \frac{5}{4}$

9 $\mathrm{f}'(x) = 12x - \frac{5}{\sqrt[3]{x}} = 12x - 5x^{-\frac{1}{3}}$

$\mathrm{f}'(x) = \int 12x - 5x^{-\frac{1}{3}}\,\mathrm{d}x = 6x^2 - \frac{15}{2}x^{\frac{2}{3}} + c$

$\mathrm{f}'(1) = -\frac{9}{2}$

$6 - \frac{15}{2} + c = -\frac{9}{2}$ so $c = -3$

$\therefore \mathrm{f}'(x) = 6x^2 - \frac{15}{2}x^{\frac{2}{3}} - 3$

$\mathrm{f}(x) = \int 6x^2 - \frac{15}{2}x^{\frac{2}{3}} - 3\,\mathrm{d}x = 2x^3 - \frac{9}{2}x^{\frac{5}{3}} - 3x + d$

$\mathrm{f}(0) = 2$

$0 - 0 - 0 + d = 2$ so $d = 2$

$\therefore \mathrm{f}(x) = 2x^3 - \frac{9}{2}x^{\frac{5}{3}} - 3x + 2$

$\mathrm{f}(8) = 2(8)^3 - \frac{9}{2}(8)^{\frac{5}{3}} - 3(8) + 2 = 858$

10 Gradient of normal is x^2

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}$ so $y = \int -\frac{1}{x^2}\mathrm{d}x = \frac{1}{x} + c$

Passes through $(2, 3)$: $3 = \frac{1}{2} + c$ so $c = \frac{5}{2}$

$\therefore y = \frac{1}{x} + \frac{5}{2}$

EXERCISE 15D

2 $\int_2^8 3\sqrt{x} - 2x\,\mathrm{d}x = \int_2^8 3x^{\frac{1}{2}} - 2x\,\mathrm{d}x$

$= \left[2x^{\frac{3}{2}} - x^2\right]_2^8 = \left(2 \times 8^{\frac{3}{2}} - 64\right) - \left(2 \times 2^{\frac{3}{2}} - 4\right)$

$= 2 \times \left(2^3\right)^{\frac{3}{2}} - 2 \times 2^{\frac{3}{2}} - 60$

$= 2^{\frac{11}{2}} - 2^{\frac{5}{2}} - 60 = 2^{\frac{1}{2}}\left(2^5 - 2^2\right) - 60$

$= 28\sqrt{2} - 60$

3 $\int_1^3 9\sqrt{x} - \frac{4}{\sqrt{x}}\,\mathrm{d}x = \int_1^3 9x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}\,\mathrm{d}x = \left[9\left(\frac{2}{3}x^{\frac{3}{2}}\right) - 4\left(2x^{\frac{1}{2}}\right)\right]_1^3$

$= \left(6 \times 3^{\frac{3}{2}} - 8 \times 3^{\frac{1}{2}}\right) - (6 - 8) = 18 \times 3^{\frac{1}{2}} - 8 \times 3^{\frac{1}{2}} + 2$

$= 2 + 10\sqrt{3}$

$a = 2,\ b = 10$

4 $\int_1^k 2 - \frac{1}{x^2}\mathrm{d}x = \int_1^k 2 - x^{-2}dx = \left[2x + x^{-1}\right]_1^k = \left(2k + \frac{1}{k}\right) - (2 + 1)$

$= 2k + \frac{1}{k} - 3$

5 $\displaystyle\int_{-a}^{2} x^2(4x-3)\,\mathrm{d}x = \int_{-a}^{2} 4x^3 - 3x^2\,\mathrm{d}x = \left[x^4 - x^3\right]_{-a}^{2}$

$= (16-8) - \left(a^4 + a^3\right) = 8 - a^4 - a^3$

6 $\displaystyle\int_1^a \sqrt{t}\,\mathrm{d}t = 42$ therefore $\displaystyle\int_1^a t^{\frac{1}{2}}\,\mathrm{d}t = 42$

$\left[\frac{2}{3}t^{\frac{3}{2}}\right]_1^a = 42$ so $\frac{2}{3}a^{\frac{3}{2}} - \frac{2}{3} = 42$

$a^{\frac{3}{2}} = 64$ therefore $a = 16$

7 $\displaystyle\int_{-2}^{p} 1 - \frac{10}{3x^2}\,\mathrm{d}x = -\frac{2}{3}$ therefore $\displaystyle\int_{-2}^{p} 1 - \frac{10}{3}x^{-2}\,\mathrm{d}x = -\frac{2}{3}$

$\left[x + \frac{10}{3}\left(x^{-1}\right)\right]_{-2}^{p} = -\frac{2}{3}$ so $\left(p + \frac{10}{3}p^{-1}\right) - \left(-2 - \frac{5}{3}\right) = -\frac{2}{3}$

$p + \frac{10}{3}p^{-1} + \frac{11}{3} = -\frac{2}{3}$ so $3p + 10p^{-1} + 13 = 0$

$3p^2 + 13p + 10 = 0$ therefore $(3p+10)(p+1) = 0$ so $p = -\frac{10}{3}$ or -1

8 $\displaystyle\int_3^9 \mathrm{f}(x)\,\mathrm{d}x = 7$

$\displaystyle\int_3^9 2\mathrm{f}(x) + 1\,\mathrm{d}x = 2\int_3^9 \mathrm{f}(x)\,\mathrm{d}x + \int_3^9 1\,\mathrm{d}x = 2\times 7 + [x]_3^9$

$= 14 + (9-3) = 20$

9 $\displaystyle\int_{-1}^{4} \mathrm{f}(x)\,\mathrm{d}x = 10$

$\displaystyle\int_{-1}^{4} \frac{\mathrm{f}(x)}{2} + x - 1\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{4} \mathrm{f}(x)\,\mathrm{d}x + \int_{-1}^{4} x - 1\,\mathrm{d}x = \frac{1}{2}\times 10 + \left[\frac{1}{2}x^2 - x\right]_{-1}^{4}$

$= 5 + (8-4) - \left(-\frac{1}{2} + 1\right) = \frac{15}{2}$

10 a i $\displaystyle\int_3^5 2x\,\mathrm{d}x = \left[x^2\right]_3^5 = 5^2 - 3^2$

$= 16$

ii $\displaystyle\int_5^3 2x\,\mathrm{d}x = \left[x^2\right]_5^3 = 3^2 - 5^2$

$= -16$

b i $\displaystyle\int_b^a 2x\mathrm{d}x = [x^2]_b^a = a^2 - b^2$

ii $\displaystyle\int_a^b 2x\mathrm{d}x = [x^2]_a^b = b^2 - a^2$

c $\displaystyle\int_b^a \mathrm{f}(x)\mathrm{d}x = -\int_a^b \mathrm{f}(x)\mathrm{d}x$

To show this, suppose that the integral function is $\mathrm{F}(x)$, so that $\mathrm{F}'(x) = \mathrm{f}(x)$.

Then $\displaystyle\int_b^a \mathrm{f}(x)\mathrm{d}x = \mathrm{F}(a) - \mathrm{F}(b) = -\int_a^b \mathrm{f}(x)\mathrm{d}x$

11 a $\displaystyle\int_1^a x^2\mathrm{d}x = \left[\frac{1}{3}x^3\right]_1^a = \frac{1}{3}(a^3 - 1)$

b From question 11a: $\mathrm{f}(a) = \frac{1}{3}(a^3 - 1)$ so $\mathrm{f}'(a) = a^2$

c i $\mathrm{f}(a) = \displaystyle\int_2^a x^2\mathrm{d}x = \left[\frac{1}{3}x^3\right]_2^a = \frac{1}{3}(a^3 - 8)$

$\therefore \mathrm{f}'(a) = a^2$

ii $\mathrm{f}(a) = \int_{-1}^{a} x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_{-1}^{a} = \frac{1}{3}(a^3 + 1)$

$\therefore \mathrm{f}'(a) = a^2$ $\mathrm{f}'(a)$ is the same in both cases.

For a constant c,

$\mathrm{f}(a) = \int_{c}^{a} x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_{c}^{a} = \frac{1}{3}(a^3 - c^3)$

Taking the derivative with respect to a must give the same result since the derivative of the constant term will cancel.

In general, if $\mathrm{f}(a) = \int_{c}^{a} \mathrm{g}(x)\mathrm{d}x$ for a constant c then $\mathrm{f}'(a) = \mathrm{g}(a)$.

> Note
>
> This result assumes that the function $\mathrm{g}(x)$ is defined throughout the interval $[c, a]$. Where this is not the case, some additional care will be needed.

d Using the result in question 11c:

i $\mathrm{f}'(a) = 4a^3 - 2a$

ii $\mathrm{f}'(a) = \frac{3}{a^2}$

iii $\mathrm{f}'(a) = 1 - 3a^2$

EXERCISE 15E

3 $A = \int_0^1 x^2 - 4x + 3\,\mathrm{d}x - \int_1^2 x^2 - 4x + 3\,\mathrm{d}x$

$= \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_0^1 - \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_1^2$

$= 2\left(\frac{1}{3} - 2 + 3\right) - \left(\frac{8}{3} - 8 + 6\right) = 2$

4 $\int_0^k x^{\frac{1}{2}}\,\mathrm{d}x = 18$ therefore $\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^k = 18$ so $\frac{2}{3}k^{\frac{3}{2}} = 18$

$k^{\frac{3}{2}} = 27$ therefore $k = 9$

5 **a** $\int_0^3 x^2 - 1\mathrm{d}x = \left[\frac{1}{3}x^3 - x\right]_0^3 = 6$

b $A = 6 - 2\int_0^1 x^2 - 1dx = 6 - 2\left[\frac{1}{3}x^3 - x\right]_0^1 = 6 + \frac{4}{3} = \frac{22}{3}$

6 **a** $x^2 - 7x + 10 = (x - 2)(x - 5)$

b $A = \int_2^5 -x^2 + 7x - 10\,\mathrm{d}x = \left[-\frac{1}{3}x^3 + \frac{7}{2}x^2 - 10x\right]_2^5$

$= \left(-\frac{125}{3} + \frac{175}{2} - 50\right) - \left(-\frac{8}{3} + 14 - 20\right) = \frac{9}{2}$

7 **a** $x^2 - kx = 0$ therefore $x(x - k) = 0$ so $x = 0$ or k

Curve crosses the x-axis at $(0, 0)$ and $(k, 0)$

b $\int_k^3 x^2 - kx\,\mathrm{d}x = -\int_0^k x^2 - kx\,\mathrm{d}x$

$\left[\frac{1}{3}x^3 - \frac{k}{2}x^2\right]_k^3 = -\left[\frac{1}{3}x^3 - \frac{k}{2}x^2\right]_0^k$ so $\left(9 - \frac{9k}{2}\right) - \left(\frac{1}{3}k^3 - \frac{1}{2}k^3\right) = -\left(\frac{1}{3}k^3 - \frac{1}{2}k^3\right)$

$9 - \frac{9k}{2} = 0$ therefore $k = 2$

Alternative method:

The integral over $[0, 3]$ will have value zero, since the area below will cancel the area above the axis.

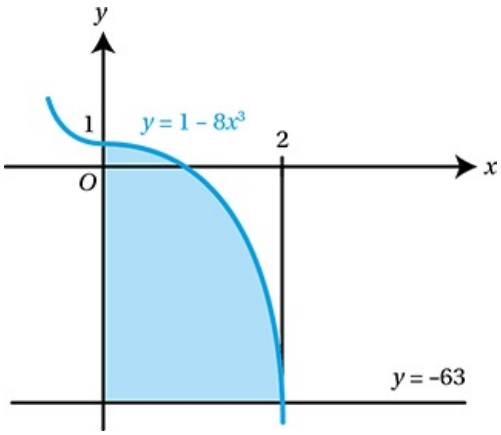
$\int_0^3 x^2 - kx \, \mathrm{d}x = 0$ so $\left[\frac{1}{3}x^3 - \frac{k}{2}x^2\right]_0^3 = 0$

$9 - \frac{9k}{2} = 0$ therefore $k = 2$

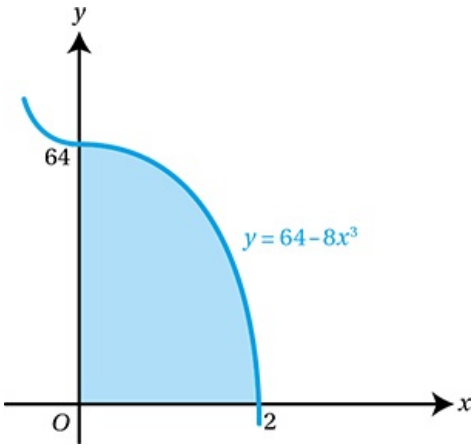
8 **a** $\int_0^3 x^2 + 2\mathrm{d}x = \left[\frac{1}{3}x^3 + 2x\right]_0^3 = 15$

b $A =$ area of rectangle $-$ area under curve $= 33 - 15 = 18$

9 Intersection of the curve and the line: $1 - 8x^3 = -63$ so $x^3 = 8$ therefore $x = 2$



This area is the same as the area between the curve and the x-axis for $y = 64 - 8x^3$ in the region $0 \leqslant x \leqslant 2$.



$A = \int_0^2 64 - x^3 \mathrm{d}x = \left[64x - \frac{1}{4}x^4\right]_0^2 = 128 - 4 = 124$

10 Intersection of the two curves: $36 - x^2 = 18x - x^2$ so $36 = 18x$ therefore $x = 2$.

Shaded area is the area under the first curve in $[0, 2]$ and the area under the second curve in $[2, 6]$.

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$$A = \int_0^2 18x - x^2\,\mathrm{d}x + \int_2^6 36 - x^2\,\mathrm{d}x = \left[9x^2 - \frac{1}{3}x^3\right]_0^2 + \left[36x - \frac{1}{3}x^3\right]_2^6$$
$$= \left(36 - \frac{8}{3}\right) + (216 - 72) - \left(72 - \frac{8}{3}\right) = 108$$

11 Shaded area equals area under curve less the area under the line $y = \frac{1}{\sqrt{p}}x$.

$$A = \int_0^p x^{\frac{1}{2}} - \frac{1}{\sqrt{p}}x\,\mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2\sqrt{p}}x^2\right]_0^p$$
$$= \frac{2}{3}p^{\frac{3}{2}} - \frac{1}{2}p^{\frac{3}{2}} = \frac{1}{6}p^{\frac{3}{2}}$$

12 Point P has coordinates $\left(\frac{a}{2}, \frac{3a^2}{4}\right)$.

Shaded area equals area under curve less the area of the triangle with vertices P, $\left(\frac{a}{2}, 0\right)$ and $(a, 0)$.

Triangle has height $\frac{3a^2}{4}$ and base length $\frac{a}{2}$ so has area $\frac{1}{2} \times \frac{a}{2} \times \frac{3a^2}{4} = \frac{3a^3}{16}$

$$A = \int_{\frac{a}{2}}^a a^2 - x^2\,\mathrm{d}x - \frac{3}{16}a^3 = \left[a^2x - \frac{1}{3}x^3\right]_{\frac{a}{2}}^a - \frac{3}{16}a^3$$
$$= \left(a^3 - \frac{1}{3}a^3\right) - \left(\frac{1}{2}a^3 - \frac{1}{24}a^3\right) - \frac{3}{16}a^3 = \frac{1}{48}a^3$$

MIXED PRACTICE 15

1 $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x - \sqrt{x} = 3x - x^{\frac{1}{2}}$ therefore $y = \int 3x - x^{\frac{1}{2}}\,\mathrm{d}x = \frac{3}{2}x^2 - \frac{2}{3}x^{\frac{3}{2}} + c$

Passes through $(4, -1)$: $-1 = \frac{3}{2} \times 16 - \frac{2}{3} \times 8 + c$ so $c = -\frac{59}{3}$

$\therefore y = \frac{3}{2}x^2 - \frac{2}{3}x^{\frac{3}{2}} - \frac{59}{3}$

2 $\int \frac{1 + x\sqrt{x}}{x^2}\mathrm{d}x = \int x^{-2} + x^{-\frac{1}{2}}\mathrm{d}x = -x^{-1} + 2x^{\frac{1}{2}} + c = \frac{1}{x} + 2\sqrt{x} + c$

3 $\mathrm{f}'(x) = (1 - x)(\sqrt{x} + 2) = -x^{\frac{3}{2}} - 2x + x^{\frac{1}{2}} + 2$

$\mathrm{f}(x) = \int -x^{\frac{3}{2}} - 2x + x^{\frac{1}{2}} + 2\,\mathrm{d}x = -\frac{2}{5}x^{\frac{5}{2}} - x^2 + \frac{2}{3}x^{\frac{3}{2}} + 2x + c$

$\mathrm{f}(1) = 3$

$-\frac{2}{5} - 1 + \frac{2}{3} + 2 + c = 3$ so $c = \frac{26}{15}$

$\therefore \mathrm{f}(x) = -\frac{2}{5}x^{\frac{5}{2}} - x^2 + \frac{2}{3}x^{\frac{3}{2}} + 2x + \frac{26}{15}$

4 **a** $\int_2^{2\sqrt{3}} 3 - \frac{12}{x^2}\,\mathrm{d}x = \int_2^{2\sqrt{3}} 3 - 12x^{-2}\,\mathrm{d}x$

$= \left[3x + 12x^{-1}\right]_2^{\sqrt{12}} = \left(3\sqrt{12} + \sqrt{12}\right) - (6 + 6)$

$= 4\sqrt{12} - 12 = -12 + 8\sqrt{3}$

b Rectangle area is $2 \times 2\sqrt{3} = 4\sqrt{3}$

$$\text{Shaded area} = \text{area of rectangle} - \int_2^{2\sqrt{3}} 3 - \frac{12}{x^2}\mathrm{d}x = 4\sqrt{3} - \left(-12 + 8\sqrt{3}\right)$$
$$= 12 - 4\sqrt{3}$$

5 **a** $\int x^2 - 2x + 5 \,\mathrm{d}x = \frac{1}{3}x^3 - x^2 + 5x + c$

b $11 = \left(\frac{1}{3}x^3 - x^2 + 5x + c\right)\Big|_{x=3} = 9 - 9 + 15 + c$ so $c = -4$

$y = \frac{1}{3}x^3 - x^2 + 5x - 4$

6 Derivative of the integral is the function being integrated. $\mathrm{f}'(x) = 4x + 5$

7 **a** $\int (x^3 - 6x^2 + 4x - 24)\mathrm{d}x = \frac{1}{4}x^4 - 2x^3 + 2x^2 - 24x + c$

b Intersection of the curve $y = 8x^{-2} - 2$ with the x-axis:

$8x^{-2} = 2$

$x^{-2} = \frac{1}{4} = 2^{-2}$ therefore $x = 2$

$$\text{Shaded area} = \int_0^1 6x^{\frac{2}{3}} \,\mathrm{d}x + \int_1^2 8x^{-2} - 2 \,\mathrm{d}x = \left[\frac{18}{5}x^{\frac{5}{3}}\right]_0^1 + \left[-8x^{-1} - 2x\right]_1^2$$
$$= \left(\frac{18}{5} - 0\right) + (-4 - 4 + 8 + 2) = \frac{28}{5}$$

8 $\mathrm{f}'(x) = \frac{4x^2 - 3\sqrt{x}}{x} = 4x - 3x^{-\frac{1}{2}}$

$\mathrm{f}(x) = \int 4x - 3x^{-\frac{1}{2}} \,\mathrm{d}x = 2x^2 - 6x^{\frac{1}{2}} + c$

$\mathrm{f}(1) = 2$ so $2 - 6 + c = 2$ therefore $c = 6$

$\therefore \mathrm{f}(x) = 2x^2 - 6x^{\frac{1}{2}} + 6$

$\mathrm{f}(4) = 2 \times 4^2 - 6 \times 4^{\frac{1}{2}} + 6 = 26$

9 $\int_1^a 2x - 3 \,\mathrm{d}x = 6$ therefore $\left[x^2 - 3x\right]_1^a = 6$ so $a^2 - 3a - (1 - 3) = 6$

$a^2 - 3a - 4 = 0$ therefore $(a - 4)(a + 1) = 0$ so $a = 4$ (since $a > 0$)

10 **a** $\int_0^a x^3 - x \,\mathrm{d}x = 0$ therefore $\left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^a = 0$ so $\frac{a^2}{4}\left(a^2 - 2\right) = 0$

$a = 0, \pm\sqrt{2} \therefore a = \sqrt{2}$ (since $a > 0$)

b Curve intersects the x-axis when:

$x^3 - x = 0$ therefore $x\left(x^2 - 1\right) = 0$ so $x = 0, \pm 1$

Since $\int_0^a x^3 - x \,\mathrm{d}x = 0$ (defined in question 10a), $-\int_0^1 x^3 - x \,\mathrm{d}x = \int_1^a x^3 - x \,\mathrm{d}x$

> Note
>
> The negative sign since the curve is below the x-axis for $0 < x < 1$).

$$\text{Total area} = -\int_0^1 x^3 - x \,\mathrm{d}x + \int_1^a x^3 - x \,\mathrm{d}x = -2\int_0^1 x^3 - x \,\mathrm{d}x$$
$$= -2\left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

11 Crosses x-axis where:

$k^2 - x^2 = 0$ so $(k - x)(k + x) = 0$ therefore $x = \pm k$

i.e. at $(\pm k,\ 0)$

$$A = \int_{-k}^{k} k^2 - x^2 \mathrm{d}x = \left[k^2 x - \frac{x^3}{3}\right]_{-k}^{k} = \frac{4k^3}{3}$$

12 a i $\mathrm{f}(2) = 2(2)^3 - 3(2)^2 - 3(2) + 2 = 2 \times 8 - 3 \times 4 - 3 \times 2 + 2 = 0$

Therefore, by the factor theorem $(x - 2)$ is a factor of $\mathrm{f}(x)$.

$\mathrm{f}(x)$ is a cubic function so its general form must be $\mathrm{f}(x) = (x - 2)(ax^2 + bx + c)$

Expanding: $ax^3 + x^2(b - 2a) + x(c - 2b) - 2c = 2x^3 - 3x^2 - 3x + 2$

Comparing coefficients:

$x^3 : a = 2$
$x^2 : b - 2a = -3 \Rightarrow b = 1$
$x^1 :\ c - 2b = -3 \Rightarrow c = -1$
$x^0 : -2c = 2$ is consistent with $c = -1$

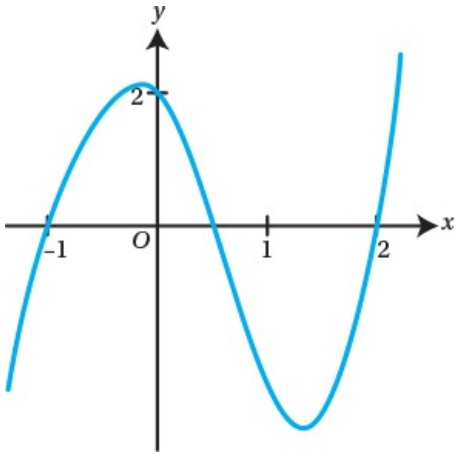
Tip

It is always wise to compare all coefficients as a quick check.)

$\mathrm{f}(x) = (x - 2)(2x^2 + x - 1) = (x - 2)(2x - 1)(x + 1)$

ii Roots at $x = 2$, $\frac{1}{2}$ and -1, y-intercept at $(0,\ 2)$

Positive cubic shape



b $$A = \int_{-1}^{\frac{1}{2}} 2x^3 - 3x^2 - 3x + 2 \,\mathrm{d}x - \int_{\frac{1}{2}}^{2} 2x^3 - 3x^2 - 3x + 2 \,\mathrm{d}x$$

$$= \left[\frac{1}{2}x^4 - x^3 - \frac{3}{2}x^2 + 2x\right]_{-1}^{\frac{1}{2}} - \left[\frac{1}{2}x^4 - x^3 - \frac{3}{2}x^2 + 2x\right]_{\frac{1}{2}}^{2}$$

$$= \left(\frac{1}{32} - \frac{1}{8} - \frac{3}{8} + 1\right) - \left(\frac{1}{2} + 1 - \frac{3}{2} - 2\right) - (8 - 8 - 6 + 4) + \left(\frac{1}{32} - \frac{1}{8} - \frac{3}{8} + 1\right)$$

$$= \frac{81}{16}$$

13 a i Upper right vertex is $(a,\ a^n)$ so

$\text{Area} = a \times a^n = a^{n+1}$

ii $B = \int_a^b x^n \mathrm{d}x = \left[\frac{1}{n+1}x^{n+1}\right]_a^b = \frac{1}{n+1}(b^{n+1} - a^{n+1})$

b Pink area $P = b^{n+1} - a^{n+1} - B$

But also, $P = 3B$

$\therefore b^{n+1} - a^{n+1} - B = 3B$

$4B = b^{n+1} - a^{n+1}$

$\frac{4}{n+1}\left(b^{n+1} - a^{n+1}\right) = b^{n+1} - a^{n+1}$ therefore $\frac{4}{n+1} = 1$

$n + 1 = 4$ so $n = 3$

14 a **i** Crosses x-axis where:

$a^2 - x^2 = 0$ therefore $(a - x)(a + x) = 0$ so $x = \pm a$

Points are $A : (-a,\, 0)$, $B : (a,\, 0)$ and $C : (0,\, a^2)$.

ii Shaded area is the area under the curve minus the area of triangle ABC.

Triangle area is $\frac{1}{2} \times 2a \times a^2 = a^3$

$$\text{Shaded area} = \int_{-a}^{a} a^2 - x^2 \mathrm{d}x - a^3 = \left[a^2x - \frac{1}{3}x^3\right]_{-a}^{a} - a^3$$
$$= 2\left(a^3 - \frac{1}{3}a^3\right) - a^3 = \frac{1}{3}a^3$$

b Triangle ABP has area $\frac{1}{2} \times 2a \times (a^2 - p^2) = a^3 - ap^2$

Since the triangle area is reduced by ap^2, the shaded area is greater by that amount:

Shaded area $= \frac{1}{3}a^3 + ap^2$

To minimise the shaded area, require ap^2 to be minimum, which occurs at $p = 0$.

Minimal shaded area is found in question 14a ii: $\frac{1}{3}a^3$.

15 If R is the point $(p,\, 0)$ then shaded area is the area under the curve minus the area of the triangle PQR.

$$\text{Area of } PQR = \frac{1}{2} \times (2 - p) \times \left(4 - p^2\right) = \frac{1}{2}(2 - p)^2\,(2 + p)$$

$$\text{Shaded area} = \int_p^2 4 - x^2 \mathrm{d}x - \frac{1}{2}(2 - p)^2\,(2 + p)$$
$$= \left[4x - \frac{1}{3}x^3\right]_p^2 - \frac{1}{2}(2 - p)^2\,(2 + p)$$
$$= \left(8 - \frac{8}{3}\right) - \left(4p - \frac{1}{3}p^3\right) - \frac{1}{2}(2 - p)^2\,(2 + p)$$
$$= \frac{16}{3} - 4p + \frac{1}{3}p^3 - \frac{1}{2}(2 - p)^2\,(2 + p)$$
$$= \frac{1}{6}\left[32 - 24p + 2p^3 - 3(2 - p)^2\,(2 + p)\right]$$
$$= \frac{1}{6}\left[(2 - p)\left(16 - 4p - 2p^2\right) - 3(2 - p)^2\,(2 + p)\right]$$
$$= \frac{1}{6}(2 - p)\left[(2 - p)\,(8 + 2p) - (2 - p)\,(6 + 3p)\right]$$
$$= \frac{1}{6}(2 - p)^2\,(2 - p) = \frac{1}{6}(2 - p)^3$$

Tip

Rather than expanding all brackets, given you know that factors of $(2-p)$ are required, it can be easier to keep any such factors and collect what remains.

16 a $\mathrm{f}''(3) = 6(3) + 6 = 24 > 0$

so $(3, 19)$ is a local minimum.

b $\mathrm{f}'(x) = \int 6x + 6\mathrm{d}x = 3x^2 + 6x + c$

Since $x = 3$ is a stationary point, $\mathrm{f}'(3) = 0 : 27 + 18 + c = 0 \text{ so } c = -45$

$\therefore \mathrm{f}'(x) = 3x^2 + 6x - 45$

$\mathrm{f}(x) = \int 3x^2 + 6x - 45\,\mathrm{d}x = x^3 + 3x^2 - 45x + d$

$\mathrm{f}(3) = 19 \text{ so } 27 + 27 - 135 + d = 19$ therefore $d = 100$

$\therefore \mathrm{f}(x) = x^3 + 3x^2 - 45x + 100$

17 $y = 6\sqrt{x} = 6x^{\frac{1}{2}}$ therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^{-\frac{1}{2}}$

Gradient at $(9, 18)$ is $\dfrac{3}{\sqrt{9}} = 1$

Tangent at $(9, 18)$ given by: $y - 18 = 1(x - 9)$ therefore $y = x + 9$

Area required is area under the tangent minus area under the curve:

$$A = \int_0^9 x + 9\,\mathrm{d}x - \int_0^9 6x^{\frac{1}{2}}\,\mathrm{d}x = \int_0^9 x + 9 - 6x^{\frac{1}{2}}\,\mathrm{d}x$$

$$= \left[\frac{1}{2}x^2 + 9x - 4x^{\frac{3}{2}}\right]_0^9$$

$$= \frac{81}{2} + 81 - 4 \times 27 = \frac{27}{2}$$

Chapter 16 worked solutions

16 Working with data

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 16A

3 Median 12

$\text{IQR} = 18 - 9 = 9$

4 **a** Median will be at cumulative frequency 40, which corresponds to 197 seconds.

b Q_1 will be at cumulative frequency 20, which corresponds to $c = 191$ seconds.

Q_3 will be at cumulative frequency 60, which corresponds to $d = 203$ seconds.

c $\text{IQR} = Q_3 - Q_1 = 12$ seconds

5 **a** 30 minutes corresponds to cumulative frequency 56.

50 minutes corresponds to cumulative frequency 162.

Approximately 106 students take between 30 and 50 minutes to travel to school, which is 53% of the population.

b 20% will be at cumulative frequency 40, which corresponds to 28 minutes.

80% of students take more than $x = 28$ minutes to travel to school.

6 **a** Results in English were less spread out than in History, and the centre of the spread was lower than in History.

b The 50 mark is at the lower quartile for History, so the probability is 75%.

c To make this comparison (which is subjective and open to interpretation) you would at least need a direct comparison, with the same cohort of students taking both tests.

d The plot gives no indication of the number of students sitting each test.

7 **a** $\text{IQR} = Q_3 - Q_1 = 25 - 10 = 15$ minutes

b According to the data, 25% of calls to Alpha Commerce are answered within 5 minutes, but 0% of calls to Beta Bank

Choose Alpha Commerce for calls answered within 5 minutes, with a probability 25%.

c On the other hand, 50% of Beta Bank calls are answered within 15 minutes, which is a greater proportion than seen in Alpha Commerce.

Choose Beta Bank for calls answered within 15 minutes, with a probability 50%.

8 **a** From the histogram:

Earning	Cumulative frequency density
$< £10\,000$	1.5
$< £20\,000$	4.0
$< £30\,000$	6.0
$< £40\,000$	7.2
$< £50\,000$	8.2

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$< £60\,000$	9.0
$< £70\,000$	9.5
$< £80\,000$	9.8
$< £90\,000$	9.9
$< £100\,000$	10.0

So the conversion from frequency density to probability is a division by 10.

Probability of an employee earning between $£20\,000$ and $£30\,000$ is 0.2.

Lacking further information, estimate that half that interval has half that probability.

P(earn between $£20\,000$ and $£25\,000$) is approximately 0.1.

b Q_1 lies within $£10\,000$ to $£20\,000$

Q_2 lies within $£20\,000$ to $£30\,000$

Q_3 lies within $£40\,000$ to $£50\,000$

This corresponds to box-and-whisker A.

9 **a** $n = 20$ so median is the mean of the 10 th and 11 th scores.

$\text{median} = \dfrac{23 + 24}{2} = 23.5$ seconds

Q_1 is the mean of the 5 th and 6 th scores: $Q_1 = 19$

Q_3 is the mean of the 15 th and 16 th scores: $Q_1 = 31.5$

$\text{IQR} = Q_3 - Q_1 = 12.5$

b $\bar{x} = \dfrac{\sum x}{n} = \dfrac{504}{20} = 25.2$ seconds

c Group 2 has a symmetrical distribution ($6 - 8 - 6$ in the three groups) whereas Group 1 is skewed to the upper values ($3 - 7 - 10$ in the three groups).

The range of values is the same in both groups.

10 The gradient of the cumulative frequency curve corresponds to the frequency value in that interval.

A is steep in the centre and less steep at either end, so corresponds to frequency graph 3.

B is steepest at the low data values and gradually becomes less steep, so corresponds to frequency graph 2.

C is shallow then steep then less steep, then steep again and finally becomes shallow, so corresponds to frequency graph 1, which is bimodal.

11 Frequency density $h = \dfrac{a}{2}$.

Frequency density for the second bar is $\dfrac{ab}{3} = \dfrac{2bh}{3}$.

EXERCISE 16B

3 **a** Median is 8

Lower data set 5, 5, 7 has median 5, so $Q_1 = 5$

Upper data set 9, x, 13 has median x, so $Q_3 = x$

$\text{IQR} = 7 = Q_3 - Q_1$

$x - 5 = 7$ therefore $x = 12$

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b From calculator: $\sigma = 2.92$

4 **a** $\bar{x} = \dfrac{\sum x}{n} = \dfrac{468}{10} = 46.8$

b $\sigma^2 = \dfrac{\sum x^2}{n} - (\bar{x})^2 = \dfrac{27\,172}{10} - 46.8^2 = 527$

5 **a** $\bar{x} = \dfrac{\sum x}{n} = \dfrac{1245}{10} = 124.5\,\text{mph}$

b $\sigma = \sqrt{\dfrac{\sum x^2}{n} - (\bar{x})^2} = \sqrt{\dfrac{156\,403}{10} - 124.5^2} = 11.8\,\text{mph}$

c Standard deviation for Andy is $\sigma = \sqrt{89.6} = 9.47\,\text{mph}$.

Andy has the more consistent serve speed, since he has the lower standard deviation.

6 **a** From calculator: $\sigma = 15.2$

b It is not stated that the scores are percentages, and without a scale the standard deviations cannot be compared.

Additionally, it appears that one of the Physics scores (36) is an outlier, and is responsible for the extremely high standard deviation; without this score, $\sigma = 2.9$. Without some explanation of the score (which might be an anomaly – misrecorded, for example) it would be unwise to draw general conclusions based on a comparison.

7 $\bar{x} = \dfrac{2 + 5 + 9 + x + y}{5} = 5$

$x + y = 9$

$\sigma^2 = \dfrac{2^2 + 5^2 + 9^2 + x^2 + y^2}{5} - 25 = 6$

$x^2 + y^2 = 155 - 4 - 25 - 81 = 45$

$(x + y)^2 = 81 = x^2 + y^2 + 2xy = 45 + 2xy$

$xy = 18$

8 The total IQ score before the new student is $\displaystyle\sum_{k=1}^{9} x_k = 9\bar{x}_9 = 1089$

The total sum of square IQ scores before the new student is
$\displaystyle\sum_{k=1}^{9} x_k^2 = 9\left(\sigma_9^2 + (\bar{x}_9)^2\right) = 9(226 + 121^2) = 133\,803$

The new student has score x_{10} where $\sigma_{10}^2 = \dfrac{133\,803 + x_{10}^2}{10} - \left(\dfrac{1089 + x_{10}}{10}\right)^2 = 239.4$

$1\,338\,030 + 10x_{10}^2 - 1089^2 - 2 \times 1089x_{10} - x_{10}^2 = 23\,940$ so $9x_{10}^2 - 2178x_{10} + 128\,169 = 0$

$x_{10}^2 - 242x_{10} + 14\,241 = 0$ therefore $(x_{10} - 101)(x_{10} - 141) = 0$ therefore $x_{10} = 101$ or 141

9 **a** $\text{Range} = x_{\max} - x_{\min}$

$x_{\max} \geqslant x_i$ for any i, by definition of maximum

$x_{\min} \leqslant \bar{x}$, since $\bar{x} = \dfrac{1}{n}\sum x_i \geqslant \dfrac{1}{n}\sum x_{\min} = x_{\min}$

Since $x_{\min} \leqslant \bar{x}$, it follows that $-x_{\min} \geqslant -\bar{x}$

So $x_{\max} - x_{\min} \geqslant x_i + (-\bar{x}) = x_i - \bar{x}$ for any i

$\text{Range} \geqslant x_i - \bar{x}$ for any i.

b Both range and standard deviation σ are non-negative, so squaring and taking square roots through the inequalities is valid in the following working:

$\sigma^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$

Using the result from question 9a: $\sigma^2 \leqslant \frac{1}{n}\sum \text{range}^2$

$\sigma^2 \leqslant \frac{1}{n}(n \times \text{range}^2)$ so $\sigma^2 \leqslant \text{range}^2$

$\sigma \leqslant \text{range}$

EXERCISE 16C

3 **a** The youngest could be 17 years exactly (i.e. 17 th birthday on the day the data is measured), the oldest could be one day short of 21 years old (i.e. 21 st birthday the day after the data is measured).

b The least is 17 pencils, the most is 20 pencils.

c The lower bound is $16.5\,\text{cm}$, the upper bound is $20.5\,\text{cm}$.

d The lower bound is $£16.01$ per hour, the upper bound is $£20.00$ per hour.

4 **a**

Number of cars	0	1	2	3	4
Number of households	10	52	64	15	5

b Approximations for mean and standard deviation:

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{52 + 128 + 45 + 20}{146} = 1.68$$

$$\sigma^2 = \frac{\sum f_i x_{i2}}{\sum f_i} - \bar{x}^2 = \frac{52 + 256 + 135 + 80}{146} - 1.68^2 = 0.766$$

$\sigma = 0.875$

c Households in Mediton tend to have more cars (higher mean number of cars) but there is a greater variation.

5 **a** Approximations for mean and standard deviation, using the midpoints of the given intervals:

Mass (kg)	0.5 to 2.5	2.5 to 4.5	4.5 to 6.5	6.5 to 8.5
Midpoint x	1.5	3.5	5.5	7. 5
Frequency	12	20	16	2

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{12 \times 1.5 + 20 \times 3.5 + 16 \times 5.5 + 2 \times 7.5}{50} = 3.82$$

$$\sigma = \sqrt{\frac{\sum f_i x_{i2}}{\sum f_i} - \bar{x}^2} = 1.67$$

b A more accurate estimate would be achieved by splitting the categories (e.g. $0\,\text{kg}$, $1\,\text{kg}$, $2\,\text{kg}$, etc).

Measuring more accurately (for example correct to the nearest $100\,\text{g}$) and keeping the raw data would also allow for a more accurate calculation.

6 **a** 50 boxes: Median will be the mean of the 25th and 26th box

$\text{median} = \frac{1+2}{2} = 1.5$ broken eggs

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b $\bar{x} = \dfrac{(17 \times 0) + (1 \times 8) + (2 \times 7) + (3 \times 7) + (4 \times 6) + (5 \times 5) + (6 \times 0)}{50} = 1.84$ broken eggs

c $\sigma^2 = \dfrac{(0^2 \times 17) + (1^2 \times 8) + (2^2 \times 7) + (3^2 \times 7) + (4^2 \times 6) + (5^2 \times 5) + (6^2 \times 0)}{50} - 1.5^2$

$= 4.15$

$\sigma = \sqrt{4.15} = 2.04$

d **i** The new process has a lower variance, suggesting more consistency in the process.

ii The new process has a higher mean breakage, suggesting a less effective packaging method.

iii 50 boxes, given the small difference detected, is too small a sample to reach a reliable conclusion.

7

Percentage of teachers	**Frequency density (fd)**	**Frequency $= fd \times$ width**	**Midpoint** x
0 to 5	5.6	28	2.5
5 to 10	4.4	22	7.5
10 to 15	3.0	15	12.5
15 to 20	1.4	7	17.5
20 to 30	0.6	6	25
TOTAL		78	

a Estimates of mean and standard deviation:

$\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i}$

$= \dfrac{28 \times 2.5 + 22 \times 7.5 + 15 \times 12.5 + 7 \times 17.5 + 6 \times 25}{78} = 8.91$

$\sigma = \sqrt{\dfrac{\sum {f_i}^2 x_i}{\sum f_i} - \bar{x}^2} = 6.65$

b The estimate assumes that all data values in a group are located at the centre of the interval.

c The data is skewed to the left, so the median (halfway point of the data) will be lower than the mean, which will be raised by the small number of high values.

8 **a** The spread of the data will be the same for

$p = 1 + a$ and $p = 1 - a$, since the distribution will be equivalent, just reflected through $x = 1$.

b $\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{8 + 2p}{10} = 0.8 + 0.2p$

$\sigma = \sqrt{\dfrac{\sum f_i x_{i2}}{\sum f_i} - \bar{x}^2}$

$= \sqrt{\dfrac{8 + 2p^2}{10} - (0.8 + 0.2p)^2} = 0.8k$

$0.8 + 0.2p^2 - 0.64 - 0.32p - 0.04p^2 = 0.64k^2$

$0.16p^2 - 0.32p + 0.16 - 0.64k^2 = 0$ so $p^2 - 2p + 1 - 4k^2 = 0$

$p = \dfrac{2 \pm \sqrt{4 - 4(1 - 4k^2)}}{2} = 1 \pm 2k$

c $0.8 + 0.2p = 2.2$ therefore $p = 7$

$7 = 1 \pm 2k$ so $k = 3$, since $k > 0$.

9 $\text{Mean } 32 = \dfrac{(20 \times 12 + 40q + 8p)}{20 + q}$ so $640 + 32q = 240 + 40q + 8p$

$80 + 4q = 30 + 5q + p$ therefore $p + q = 50$

Variance $136 = \dfrac{(20^2 \times 12 + 40^2 \times q + p^2 \times 8)}{20 + q} - 32^2$ so $1160(20 + q) = 20^2 \times 12 + 40^2 \times q + p^2 \times 8$

$23\,200 + 1160q = 4800 + 1600q + 8p^2$ therefore $440q + 8p^2 = 18\,400$

$$p^2 + 55(50 - q) - 2300 = 0$$
$$p^2 - 55p + 450 = 0$$
$$(p - 45)(p - 10) = 0$$
$$If\ p = 45\ q = 5,\ or\ p = 10\ q = 40$$

10 If 'average' can shift its meaning, there are numerous ways this can be true. However, if 'average' represents 'mean' throughout then it can still be true, if the number of attempts varies.

For example:

Amy plays level 1 twice and scores 90 and 100 (mean 95)

Bob plays level 1 four times and scores 90, 91, 93 and 94 (mean 92)

Amy plays level 2 twice and scores 50 and 60 (mean 55)

Bob plays level 2 twice and scores 42 and 52 (mean 45)

Clearly Amy's average is higher than Bob's in both level 1 and level 2

However, their overall averages are:

Amy: $\dfrac{90 + 100 + 50 + 60}{4} = 75$

Bob: $\dfrac{90 + 91 + 93 + 94 + 42 + 52}{6} = 77$

So Bob has a higher overall average.

EXERCISE 16D

3 **a** $r = 0.984$: Very strong positive correlation shows a reliable linear relationship, where any increase in processor speed is proportional to the corresponding increase in life expectancy.

b Correlation is not causation – the strong correlation gives no insight into whether one factor drives the other or whether both are resultant from an external factor, such as overall technological development, new materials, design or production improvements.

4 **a** $r = 0.92$: Strong positive correlation shows a reliable linear relationship, where increase in car age is proportional to the corresponding increase in braking distance.

b Positive correlation between age and distance supports Nicole's assertion.

Tip

Importantly, although there is a strong correlation, you have no information of the specific quantities involved. For example, from the limited information, it may be that every 10 years of age increases braking distance by an average $10\,\text{mm}$ (irrelevant to safety concerns) or an average $30\,\text{m}$ (extremely relevant for safety). Either could produce $r = 0.92$, since the correlation coefficient does not inform us as to the gradient value of the best fit line, only whether it is positive or negative. Avoid making interpretations which would relate to actual values when answering this sort of question unless you are given concrete information on which to base your assertions.

5 **a** $r = 0.88$: strong positive correlation shows a reliable linear relationship, where increase in mass is proportional to the corresponding increase in age.

b **i** 0.5 is the typical mass gain (in kg) of a baby each month

ii 3.2 is the typical mass of a baby at birth

c The model is for 0 to 18-month-old babies, so a 14-year-old would be far beyond the tested limits of the linear relationship (extrapolation) and so cannot be reliably estimated from the model.

The model does not split the subjects by gender, and even within the 0 to 18-month interval, there might be a significantly different model for girls and for boys, if the data were disaggregated and separate best fit lines derived.

6 **a** **i** The predicted $100\,\text{m}$ time for a child starting primary school is 24.8 seconds.

ii The typical time improvement a child achieves for each year at school is 0.609 seconds.

b Even within the data set (primary school children), the correlation coefficient – and so the reliability of the linear model – is very low.

A 60-year-old runner is also well beyond the studied data, so any prediction would be extrapolation.

7 **a** $=\text{A}$: Strong positive correlation – a close fit to a positive gradient straight line.

b $=\text{D}$: Weak negative correlation – a poor linear fit with a negative gradient trend.

c $=\text{B}$: Strong negative correlation – a close fit to a negative gradient straight line.

d $=\text{C}$: Weak positive correlation – a poor linear fit with a positive gradient trend.

8 **a** Reading from the best-fit line on the graph:

i $1.5\,\text{m}$

ii $3.0\,\text{m}$

b The correlation coefficient is high, indicating that within the interval of the data, the linear model is very reliable.

The age value in question 8ai is within the mass of data, so produces a valid estimate.

The age value in question 8aii is well beyond the mass of data, so the estimate is an extrapolation and not reliable.

9 **a** **i** $3.5\,\text{kW}$ intersects only one of the path lines at $5\,\text{Hz}$.

ii $2\,\text{kW}$ intersects three of the path lines, so it is impossible to say by this method.

b There is no ordering of these points, so the path provided is not useful for this interpretation.

c If the points are ordered, and especially if the horizontal axis is an independent variable (for example, time or age) then the pattern generated by a connected graph of this sort could be informative and useful for prediction. For example, if the data represented the height of an object falling over time, a connected graph of this sort (with the data ordered by time) might be useful for predicting the height at any given time within the interval measured.

10 **a** False: there is no *linear* relationship between the data (but there may be, for example, a quadratic relationship).

b False: if $k = 0$ then $r = 0$ and if $k < 0$ then $r = -1$.

c True: the sign of r is the same as the sign of the gradient of the line of best fit.

d False: r^2 increases as the closeness of fit increases, it has no relation to the value of the gradient.

EXERCISE 16E

3 It appears that there are two separate groups within the ant data, each with its own strong positive correlation. The biologist should investigate whether there is any characteristic which allows consistent separation of the ants into one or other population, and if so to calculate the

correlation coefficient for each group separately.

Tip

This appears to be a clear-cut case, from the scatter plot, but it is important to note that if no basis for splitting the population can be found other than 'it makes the data work better' then a separation of this kind is statistically dubious. After all, any population will have a strong correlation if you exclude the data points which don't fall into the pattern you want to show.

4 **a** $Q_1 = 7$, $Q_3 = 12$. $\text{IQR} = Q_3 - Q_1 = 5$ attempts

b $1.5 \times \text{IQR} + Q_3 = 19.5$, but clearly there was an individual who took 25 attempts, and is therefore an outlier. (There may be other outliers, but you know that there was at least one person who took 25 attempts).

5 $Q_1 = 40$ marks, $Q_3 = 65$ marks, $\text{IQR} = 25$ marks.

$Q_1 - 1.5 \times \text{IQR} = 2.5$ marks which is below the recorded minimum.

$Q_3 + 1.5 \times \text{IQR} = 102.5$ marks which is above the recorded maximum.

By this definition of outlier, there were no outliers in this data set.

6 **a** Underestimate: The basketball team are likely to be taller pupils.

b Close: Pupils from all year groups could be ill.

c Overestimate: Year 7 pupils will be among the shortest in the school.

7 Total for the 542 students $= 542 \times 48 = 26016$

If all the missing students scored zero, then minimum possible mean

$= 26016 \div 560 = 46.5$

If all the missing students scored 70, then maximum possible mean

$= (26016 + 18 \times 7) \div 560 = 48.7$

8 **a** This refers to the point lying outside the general trend:

$T = 21$.

b **i** The mean number of matches will decrease, since the point removed had a higher than average value.

ii The standard deviation will decrease, since the point removed was far from the centre of the data.

9 **a** $\bar{t} = \dfrac{\sum t}{n} = \dfrac{440}{7} = 62.9$ minutes

$\sigma = \sqrt{\dfrac{\sum t^2}{n} - (\bar{t})^2} = \sqrt{\dfrac{28040}{7} - 62.9^2} = 7.40$ minutes

b Thursday seems to lie outside the general trend, with a relatively long time for a relatively short distance. This would be likely the day of adverse weather conditions.

c Without Thursday: $\bar{t} = \dfrac{\sum t}{n} = \dfrac{440 - 62}{6} = 63$ minutes

$\sigma = \sqrt{\dfrac{\sum t^2}{n} - (\bar{t})^2} = \sqrt{\dfrac{28040 - 62^2}{6} - 63^2} = 7.98$ minutes

d The gradient gives the runner's general speed: $0.314\,\text{km}$ per minute.

e The data covers distances less than $24\,\text{km}$, so using the regression line to predict the time taken for a $40\,\text{km}$ run would be extrapolation, and the estimate would be unreliable.

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10 a $\text{Mean} + 2\sigma = 18 + 2\sqrt{\dfrac{631}{3}} = 47$

The value 50 is greater than this boundary, and so is classified as an outlier, under the given definition.

b If the data value 50 is removed, then for the remaining data.

$\sum x = 18 \times 6 - 50 = 58$

$\sum x^2 = 6\left(\dfrac{631}{3} + 18^2\right) - 50^2 = 706$

$\bar{x} = \dfrac{\sum x}{n} = \dfrac{58}{5} = 11.6$

$\sigma^2 = \dfrac{\sum x^2}{n} - (\bar{x})^2 = \dfrac{706}{5} - 11.6^2 = 6.64$

$\sigma = \sqrt{6.64} = 2.58$

11 a Let $\mathrm{f}(x) = 8x^2 - 576x + 11826$ for $0 \leqslant x \leqslant 50$

Then $\mathrm{f}(0) = 11826$ and $\mathrm{f}(50) = 3026$ so the maximum value of $\mathrm{f}(x)$ is 11826

The line of symmetry of the quadratic $\mathrm{f}(x)$ is at $x = \dfrac{576}{2 \times 8} = 36$ so the minimum of $\mathrm{f}(x)$ is $\mathrm{f}(36) = 1458$

b Let the students' marks be given by x_i where x_9 is the unrecorded mark of the ninth student.

$\bar{x}_8 = 36$ so $\displaystyle\sum_{i=1}^{8} x_i = 36 \times 8 = 288$

$\mathrm{sd}_8 = 4.5$ so $\dfrac{1}{8}\displaystyle\sum_{i=1}^{8} x_i^2 - 36^2 = 4.5^2$ and therefore $\displaystyle\sum_{i=1}^{8} x_i^2 = 8 \times (4.5^2 + 36^2) = 10530$

The variance of all nine students can then be given as a function of x_9:

$$\begin{aligned}\mathrm{var}_9 &= \frac{1}{9}\sum_{i=1}^{9} x_i^2 - \left(\frac{1}{9}\sum_{i=1}^{9} x_i\right)^2 \\ &= \frac{1}{9}\left(10530 + x_9^2\right) - \frac{1}{81}\left(288 + x_9\right)^2 \\ &= \frac{1}{81}\left[94770 + 9x_9^2 - 82944 - 576x_9 - x_9^2\right] \\ &= \frac{1}{81}\left[8x_9^2 - 576x_9 + 11826\right] = \frac{1}{81}\mathrm{f}(x_9)\end{aligned}$$

The standard deviation is therefore $\mathrm{sd}_9 = \dfrac{1}{9}\sqrt{\mathrm{f}(x_9)}$ and so from part **a:**

The minimum standard deviation is $\dfrac{1}{9}\sqrt{1458} = 4.24$

The maximum standard deviation is $\dfrac{1}{9}\sqrt{11826} = 12.1$

Tip

In fact, with a little consideration, it should be clear that the minimum standard deviation will occur when the ninth value exactly matches the mean of the other eight ($x_9 = 36$) and the maximum standard deviation occurs when the ninth value is as far as possible from the mean of the other eight (that is, $x_9 = 0$).

12 a The range is more than 4 standard deviations. No matter where the mean lies within the range, either the maximum value or the minimum value (or both) must be more than two standard deviations from the mean.

Suppose the minimum $x_{\min}$ is *not* an outlier.

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Then $\bar{x} < x_{\min} + 18$.

But then $x_{\max} = x_{\min} + 50 > \bar{x} + 32 > \bar{x} + 2\sigma$

So $x_{\max}$ must be an outlier if $x_{\min}$ is not (and vice-versa, by a similar argument.)

b $\bar{x} = \dfrac{30q}{p+q}$

$\sigma^2 = \dfrac{900q}{p+q} - \left(\dfrac{30q}{p+q}\right)^2 = \dfrac{900q(p+q) - 900q^2}{(p+q)^2} = \dfrac{900pq}{(p+q)^2}$

$\sigma = \dfrac{30\sqrt{pq}}{p+q}$

30 is an outlier if $30 > \bar{x} + 2\sigma$

$\dfrac{30q}{p+q} + \dfrac{60\sqrt{pq}}{p+q} < 30$ so $30q + 60\sqrt{pq} < 30(p+q)$

$60\sqrt{pq} < 30p$ so $2\sqrt{q} < \sqrt{p}$ therefore $p > 4q$

So 30 is an outlier if $p > 4q$.

c With $q = 3$, require $\sigma = 9$.

$\dfrac{30\sqrt{3p}}{p+3} = 9$ so $10\sqrt{3p} = 3(p+3)$ therefore $300p = 9(p+3)^2 = 9p^2 + 54p + 81$

$9p^2 - 246p + 81 = 0$ so $3p^2 - 82p + 27 = 0$ therefore $(3p-1)(p-27) = 0$

$p = 27$ (reject the fraction solution, since p must be an integer)

For $q = 3$, $p = 27$, the distribution shown has a range of 30 and a standard deviation of 9, but has three outlier values at $x = 30$.

d Mean of the suggested (symmetrical) data set is 15.

$\sigma^2 = \dfrac{\sum f_i(\bar{x} - x_i)^2}{\sum f_i} = \dfrac{p \times 15^2 + 0 + p \times 15^2}{2p+q} = \dfrac{450p}{2p+q}$

Setting $\sigma^2 = 81$:

$\dfrac{450p}{2p+q} = 81$ so $50p = 9(2p+q)$ therefore $32p = 9q$

So any values $p = 9n$ and $q = 32n$ for some positive integer n will satisfy the condition, and for this distribution, range is 30, standard deviation is 9 but no values are 2 standard deviations from the mean (all values are above -3 and below 33).

MIXED PRACTICE 16

1 **a** Ordering the data: 108, 111, 115, 122, 124, 127, 135, 139, 140

$n = 9$

Median value is the 5 th, which is 124

Q_1 is the mean of the 2 nd and 3 rd: $Q_1 = 113$

Q_3 is the mean of the 7 th and 8 th: $Q_3 = 137$

$\text{IQR} = Q_3 - Q_1 = 24$

b $\bar{x} = \dfrac{\sum x}{n} = 125$

c $\sigma^2 = \dfrac{\sum x^2}{n} - (\bar{x})^2 = 124$

2 $\sigma^2 = \dfrac{\sum (x - \bar{x})^2}{n} = \dfrac{12}{3} = 4$

$\sigma = 2$

3 **a** $\bar{x} = \dfrac{\sum x}{n} = 4.08$ minutes

b $\sigma^2 = \dfrac{\sum x^2}{n} - (\bar{x})^2 = 8.80$ minutes^2

4 From the diagram, $Q_1 = 40\,\text{cm}$, $Q_3 = 80\,\text{cm}$ so $\text{IQR} = 40\text{cm}$.

The minimum is $30\text{cm} > Q_1 - 1.5 \times \text{IQR} = -20\text{cm}$.

The maximum is $130\,\text{cm} < Q_3 + 1.5 \times \text{IQR} = 140\,\text{cm}$.

By this definition of outlier, none of the data values are outliers.

5 **a** For two values a and b,

$\bar{x} = \dfrac{a+b}{2}$ so $\bar{x}^2 = \dfrac{(a+b)^2}{4}$

$\overline{x^2} = \dfrac{a^2+b^2}{2}$

If $\bar{x}^2 = \overline{x^2}$ then $a^2 + 2ab + b^2 = 2a^2 + 2b^2$

$a^2 - 2ab + b^2 = 0$ therefore $(a-b)^2 = 0$

So any data set of two unequal items must have $\bar{x}^2 \neq \overline{x^2}$, for example the data set $\{0, 1\}$.

b By the above working, if $\bar{x}^2 = \overline{x^2}$ then $a = b$, so the standard deviation must be zero, as for example in the data set $\{1, 1\}$.

6 **a**

l	$10-12$	$13-15$	$16-20$	$21-30$
Midpoint x	11	14	18	25.5
Frequency f	1	13	20	6

i Estimates of mean and standard deviation:

$\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{1 \times 11 + 13 \times 14 + 20 \times 18 + 6 \times 25.5}{40} = 17.65\,\text{km}$

$\sigma^2 = \dfrac{\sum f_i x_{i2}}{\sum f_i} - \bar{x}^2 = \dfrac{1 \times 11^2 + 13 \times 14^2 + 20 \times 18^2 + 6 \times 25.5^2}{40} - 17.65^2 = 14.74$

$\sigma = \sqrt{14.74} = 3.84\,\text{km}$

ii The data values have been grouped, and the raw data values lost, so the above calculation has to make the assumption that all values in a group are equal to the midpoint value. The statistics are therefore estimates of the mean and standard deviation, not exact calculations of them.

b $\text{Frequency density} = \text{frequency} \div \text{class width}$

The '$16-20$' class represents lengths from $15.5\,\text{km}$ to $20.5\,\text{km}$, since the values are 'to the nearest kilometre', so the class width is $5\,\text{km}$.

$\text{Frequency density} = 4\,\text{km}^{-1}$

c $n = 40$ so the median is the mean of the 20th and 21st values when listed in order. Since both these values are in the $16-20$ class, the median also lies in this class.

d **i** The mean will increase, since a value in the data set has increased. The estimate of the mean will also increase, since the value has moved up a class.

ii The standard deviation will decrease, because a value distant from the estimated mean has

moved closer to the estimated mean.

7 **a** The data table is as follows:

Value	$0 \leqslant x \leqslant 5$	$5 \leqslant x \leqslant 10$	$10 \leqslant x \leqslant 15$	$15 \leqslant x \leqslant 20$	$20 \leqslant x \leqslant 25$
Midpoint x	2.5	7.5	12.5	17.5	22.5
Frequency f	20	15	7	16	4

Estimates of mean and standard deviation:

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{20 \times 2.5 + 15 \times 7.5 + 7 \times 12.5 + 16 \times 17.5 + 4 \times 22.5}{20 + 15 + 7 + 16 + 4} = 10$$

$$\sigma = \sqrt{\frac{\sum f_i x_i^2}{\sum f_i} - (\bar{x})^2}$$

$$= \sqrt{\frac{20 \times 2.5^2 + 15 \times 7.5^2 + 7 \times 12.5^2 + 16 \times 17.5^2 + 4 \times 22.5^2}{20 + 15 + 7 + 16 + 4} - 10^2} = 6.71$$

b $\mathrm{P}(15 \leqslant X \leqslant 25) = \dfrac{16 + 4}{20 + 15 + 7 + 16 + 4} = 0.323$

8 **a** Require that $\dfrac{3 \times 0.72 + x}{4} \geqslant 0.75$

$x \geqslant 3 - 2.16$ therefore $x \geqslant 0.84$

She requires at least 84% in the fourth paper.

b $\dfrac{3 \times 0.72 + 1}{4} = 0.79$.At most she can score 79% overall.

c Data: 0.72, 0.72, 0.72, 1.00

$$\sigma = \sqrt{\frac{\sum x_i^2}{n} - (\bar{x})^2} = 0.121$$

Her new standard deviation is 12.1%.

9 **a** Median speed (read from $\mathrm{CF} = 25$) is approximately $66\,\mathrm{mph}$.

b Speed $75\,\mathrm{mph}$ corresponds approximately with $\mathrm{CF} = 39$, so 11 of the 50 cars will be stopped.

c Q_1 (read from $\mathrm{CF} = 12.5$) is approximately $a = 61\,\mathrm{mph}$.

Q_3 (read from $\mathrm{CF} = 37.5$) is approximately $b = 73\,\mathrm{mph}$.

d

Speed v	**Frequency** f	**Midpoint** x
$40 < v \leqslant 50$	2	45
$50 < v \leqslant 60$	8	55
$60 < v \leqslant 70$	26	65
$70 < v \leqslant 80$	8	75
$80 < v \leqslant 90$	4	85
$90 < v \leqslant 100$	2	95

e Estimates of mean and standard deviation:

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$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$
$$= \frac{2 \times 45 + 8 \times 55 + 26 \times 65 + 8 \times 75 + 4 \times 85 + 2 \times 95}{50} = 67 \,\text{mph}$$
$$\sigma = \sqrt{\frac{\sum {f_i}^2 x_i}{\sum f_i} - \bar{x}^2} = 10.6 \,\text{mph}$$

f Outlier boundaries are $67 \pm 2 \times 10.6 = 45.8$ and 88.2.

Although it is unknown whether any of the data are actually below $45.8\,\text{mph}$, it is certain that at least 2 data values are above $88.2\,\text{mph}$ (since they are between $90\,\text{mph}$ and $100\,\text{mph}$).

However, these values seem to be valid elements of the data set, albeit extreme, and there is no indication that they should be removed from consideration – they are simply part of an extended right tail of the distribution.

10 a The scatter diagram shows the relationship between the two variables, and also highlights any trends and points which deviate from the overall trend.

b The data values (all but one) lie closely on a straight line with positive gradient, so there is a strong, positive correlation.

c The point which breaks from the overall trend is $200\,\text{km}^2$, with a population of $900\,000$.

d The mean population will decrease from that stated, since the point removed had a population greater than that mean.

e For every additional km^2 in a city, there are typically an additional 3.05 thousand inhabitants.

3.05 is therefore the typical population density (thousand people per km2).

f The regression line is based on data of cities with areas up to $350\,\text{km}^2$, so the capital city has an area well outside the modelled straight line. As such, this extrapolation is unreliable, as there is no evidence that the linear relationship continues that far.

11 a $n = 14$ so Q_1 is the median of the lower 7 values, which is the 4 th value. $Q_1 = 23$

b Median is the mean of the 7 th and 8 th values: $30 + x$ and 34.

$\frac{(30 + x) + 34}{2} = 32$ therefore $x = 0$

Since the data values are ordered in the stem-and-leaf diagram, $w = x = 0$.

c Q_3 is the median of the upper 7 values, which is the 11 th value, $30 + z$

Since the data values are ordered in the stem-and-leaf diagram, $y \geqslant 8$ and $z \geqslant y$

So Q_3 could be 38 or 39.

d The stem-and-leaf diagram retains all the actual data values, for further calculation, whereas a box-and-whisker plot reduces the data to 5 values (min, Q_1, median, Q_3 and max).

e The box-and-whisker plot has distilled the data into a form which is easy to interpret and compare with other data sets, irrespective of the size of those data sets. Stem-and-leaf diagrams allow ready comparison of small, similarly sized data sets but become cumbersome for large data sets and do not allow ready comparison of data sets of disparate sizes.

12 a α: Most data at the right end of the range: $\mathbf{D}$

β: Data concentrated in the centre, symmetrical: $\mathbf{B}$

γ: Data concentrated away from the centre, symmetrical: $\mathbf{C}$

b Steep gradient corresponds to high frequency.

i Data concentrated in the centre, symmetrical: $\mathbf{B}$

ii Data evenly distributed for a constant gradient cumulative frequency: $\mathbf{A}$

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iii Data concentrated away from the centre, symmetrical: **C**

13 For the 10 data values: $\sum x = n \times \bar{x} = 10 \times 115 = 1150$

$\sum x^2 = n(\sigma^2 + \bar{x}^2) = 10(154 + 115^2) = 133\,790$

For the 11 data values: $\sum x = n \times \bar{x} = 11 \times 114 = 1254$

Therefore the additional item has value $1254 - 1150 = 104$

Then for the 11 data values: $\sigma^2 = \dfrac{\sum x^2}{n} - (\bar{x})^2 = \dfrac{133\,790 + 104^2}{11} - 114^2 = 150$

14 $\bar{x} = \dfrac{1542}{20} = 77.1$, $n = 20$

$\sum x^2 = n(\sigma^2 + \bar{x}^2)$

To minimise $\sum x^2$, require σ^2 to be as small as possible.

Least possible value for σ^2 is zero (if all the data values are equal)

Then the minimum value for $\sum x^2$ is $\sum x^2 = 20 \times \bar{x}^2 = 118\,888.2$

Chapter 17 worked solutions

17 Probability

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 17A

3 **a** **i** When A and B are independent, $\mathrm{P}(A \text{ and } B) = \mathrm{P}(A)\,\mathrm{P}(B) = pq$.

ii When A and B are mutually exclusive, they cannot both happen

$\mathrm{P}(A \text{ and } B) = 0$.

b $pq = 0$ only if either $p = 0$ or $q = 0$

Two events can only be both independent and mutually exclusive if the probability of at least one of them is zero.

For example, if a bag contains two balls, one blue and one red, and one ball is drawn.

A: ball is blue. B: ball is green

$\mathrm{P}(A) = \frac{1}{2}$, $\mathrm{P}(B) = 0$

$\mathrm{P}(A \text{ or } B) = \frac{1}{2} = \mathrm{P}(A) + \mathrm{P}(B)$: mutually exclusive

$\mathrm{P}(A \text{ and } B) = 0 = \mathrm{P}(A)\mathrm{P}(B)$: independent

4 The two throws are independent of each other.

a $\mathrm{P}(2 \text{ heads}) = \mathrm{P}(\text{heads})^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

b $\mathrm{P}(\text{At least one heads}) = 1 - \mathrm{P}(2 \text{ tails}) = 1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}$

5 Each ordered combination has an equal probability $\frac{1}{36}$.

There are four ordered pairs with product 6: $(1, 6)$, $(2, 3)$, $(3, 2)$, $(6, 1)$

Probability of the scores having product 6 is $\frac{4}{36} = \frac{1}{9}$.

6 **a** ACT, ATC, CAT, CTA, TAC, TCA

b $\mathrm{P}(\text{CAT or ACT}) = \frac{2}{6} = \frac{1}{3}$

7 **a** 2 is the only even prime.

$\mathrm{P}(A \text{ and } B) = \frac{1}{6}$

b $\mathrm{P}(A) = \frac{1}{2}$, $\mathrm{P}(B) = \frac{1}{2}$

$\mathrm{P}(A \text{ and } B) = \frac{1}{6} \neq \frac{1}{2} \times \frac{1}{2} = \mathrm{P}(A)\mathrm{P}(B)$

A and B are not independent.

8 **a** **i** $\mathrm{P}(\text{Chem}) = \frac{95}{300} = \frac{19}{60}$

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ii $\mathrm{P}(\text{Y11 not Bio}) = \dfrac{30+31}{98} = \dfrac{61}{98}$

b $\mathrm{P}(\text{Y9}) = \dfrac{97}{300}.\ \mathrm{P}(\text{Phy}) = \dfrac{92}{300}$

$\mathrm{P}(\text{Y9 and Phy}) = \dfrac{27}{300} = \dfrac{9}{100}$

$\mathrm{P}(\text{Y9 and Phy}) \neq \mathrm{P}(\text{Y9}) \times \mathrm{P}(\text{Phy})$ so the events are not independent.

9 Each spin is independent of the others.

$\mathrm{P}(3\text{ ones}) = \mathrm{P}(\text{one})^3 = \left(\dfrac{1}{4}\right)^3 = \dfrac{1}{64}$

$\mathrm{P}(3\text{ fours}) = \mathrm{P}(\text{one})^3 = \left(\dfrac{1}{4}\right)^3 = \dfrac{1}{64}$

These results are mutually exclusive so can be added to find the 'or' probability.

$\mathrm{P}(3\text{ ones or }3\text{ fours}) = \dfrac{2}{64} = \dfrac{1}{32}$

10 a Result of each throw is independent of the others.

$\mathrm{P}(3\text{ tails}) = \mathrm{P}(\text{tails})^3 = \left(\dfrac{1}{2}\right)^3 = \dfrac{1}{8}$

b 'At least one head' is the complement of '3 tails'.

$\mathrm{P}(\text{At least one head}) = 1 - \dfrac{1}{8} = \dfrac{7}{8}$

11 a $\mathrm{P}(\text{None late}) = 0.85^{12} = 0.142$

$\mathrm{P}(\text{At least one late}) = 1 - \mathrm{P}(\text{None late}) = 0.858$

b Independence is not reasonable; some students might habitually travel to lessons together (and so be late or on time together), and students might be jointly affected by external factors, such as school bus breakdowns, fire drills, illnesses or school events overrunning.

EXERCISE 17B

3 a Probabilities sum to 1: $\dfrac{1}{12} + \dfrac{1}{6} + \dfrac{1}{12} + \dfrac{1}{4} + 2k + 3k = 1$

$5k = \dfrac{5}{12}$ therefore $k = \dfrac{1}{12}$

b X is the score of the roll: $\mathrm{P}(X > 4) = 5k = \dfrac{5}{12}$

4 a Probabilities sum to 1: $0.15 + 0.21 + k + 0.26 + 0.12 + 0.07 = 1$

$k = 0.19$

b X is the number of absentees: $\mathrm{P}(X \leqslant 2) = 0.15 + 0.21 + k = 0.55$

5 a $\mathrm{P}(\text{A} \geqslant 1) = 1 - 0.2 = 0.8$

b Assuming independence between Ben and Anna's performances,

$\mathrm{P}(A = 3 \text{ and } B = 3) = \mathrm{P}(A = 3) \times \mathrm{P}(B = 3) = 0.1 \times 0.1 = 0.01$

6 a

Score = 1st + 2nd		**1st spin**			
		1	**2**	**3**	**4**
	1	2	3	4	5
	2	3	4	5	6

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2nd spin					
	3	4	5	6	7
	4	5	6	7	8

$\boldsymbol{x}$	2	3	4	5	6	7	8
$\mathbf{P}(\boldsymbol{X}=\boldsymbol{x})$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

b $\mathrm{P}(X \geqslant 6) = \frac{6}{16} = \frac{3}{8}$

7 Probabilities sum to 1: $a + b + 0.7 = 1$ therefore $a + b = 0.3$

$\mathrm{P}(X < 4) = 0.2 = a + 0.1$

$a = 0.1$, $b = 0.2$

8 a

$\boldsymbol{x}$	2	3	4	5	6
$\mathbf{P}(\boldsymbol{X}=\boldsymbol{x}) = k(x+1)$	$3k$	$4k$	$5k$	$6k$	$7k$

Probabilities sum to 1: $25k = 1$ therefore $k = \frac{1}{25} = 0.04$

b $\mathrm{P}(X \geqslant 4) = 18k = 0.72$

9 a Probabilities sum to 1: $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + k = 1$ therefore $k = \frac{13}{60}$

b Total of four after two rolls can occur from three ordered rolls

$\mathrm{P}(1,\, 3) = \frac{1}{3} \times \frac{13}{60} = \frac{13}{180}$, $\mathrm{P}(2,\, 2) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$, $\mathrm{P}(3,\, 1) = \frac{13}{60} \times \frac{1}{3} = \frac{13}{180}$

$\mathrm{P}(\text{Total four}) = \frac{13}{180} + \frac{1}{16} + \frac{13}{180} = \frac{149}{720} = 0.207$

10 a Total of one ball potted can occur two ways:

$\mathrm{P}(R = 1,\, J = 0) = 0.2 \times 0.3 = 0.06$. $\mathrm{P}(R = 0,\, J = 1) = 0.1 \times 0.5 = 0.05$

$\mathrm{P}(\text{Total one ball potted}) = 0.06 + 0.05 = 0.11$

b $\mathrm{P}\left(\text{Pot at least one ball}\right) = 1 - \mathrm{P}(\text{Pot 0 balls}) = 1 - 0.1 \times 0.3 = 0.97$

11 a X only takes values 1, 2 or 3 so $\mathrm{P}(X \leqslant 3) = 1$

$9k = 1$ therefore $k = \frac{1}{9}$

b Using the formula given, and $\mathrm{P}(X = x) = \mathrm{P}(X \leqslant x) - \mathrm{P}(X \leqslant x - 1)$

x	1	2	3
$\mathrm{P}(X \leqslant x)$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{9}{9}$
$\mathrm{P}(X = x)$	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{5}{9}$

12 a Probabilities sum to 1 so there must be 9 integers in the range $[-k,\, k]$.

$k = 4$

b $Y = 0$ can only arise from $X = 0$, but each other $Y = n^2$ can arise from $X = \pm n$, so has twice the probability.

y	0	1	4	9

$\mathrm{P}(Y = y)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$

EXERCISE 17C

5 **a** Binomial assumes a constant probability, but although the school is described as 'large' it is still finite, so drawing without replacement causes the probabilities to change.

For example, suppose there are 500 students in the school: 75 travel by bus.

If you randomly draw 5 students who travel by bus, then the probability that the sixth student also uses the bus is $\frac{70}{495} = 14.1\%$, noticeably different from 15%.

Tip

It may appear that you could also cite a lack of independence, in that (for example) siblings would be expected to have the same transport behaviour, or bus use might differ according to student age. However, since the sample was drawn randomly from the whole school, neither of these is a valid argument against independence within the sample.

b Let X be the number of students in the sample who travel by bus.

$X \sim \mathrm{B}(20,\ 0.15)$ in approximation

$\mathrm{P}(X = 5) = 0.103$

6 Let X be the total number of questions Chen answers correctly.

$X \sim \mathrm{B}\left(8,\ \frac{1}{4}\right)$

a $\mathrm{P}(X = 5) = 0.0231$

b $\mathrm{P}(X \geqslant 5) = 0.0273$

7 Let X be the number of people the doctor sees who have the virus.

$X \sim \mathrm{B}(80,\ 0.8\%)$

a $\mathrm{P}(X = 2) = 0.108$

b $\mathrm{P}(X \geqslant 3) = 0.0267$

c It is assumed that the patients will or will not have the virus independently of each other. In reality, if the prevalence nationwide is 0.8%, it is likely that there will be geographical pockets which are higher and lower than this, since a cold virus is contagious. Therefore, if the doctor sees one patient with the virus, it may be supposed that the virus is prevalent in the locality, so the probability of seeing others will be higher than 0.8%.

Tip

It is also assumed that the country is large enough that the doctor will not be seeing a significant fraction of the whole population; an island nation of a few hundred, for example, would not allow for a binomial model to be used for a sample of 80, for the same reasons as you saw in question 5a. However, given the context of the question, this would not be the answer the examiner would be looking for.

8 Let X be the number of sixes in four throws: $X \sim \mathrm{B}\left(4,\ \frac{1}{6}\right)$

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$\mathrm{P}(X = 3) = 0.0154$

Let Y be the number of fives or sixes in six throws: $Y \sim \mathrm{B}\left(6, \frac{1}{3}\right)$

$\mathrm{P}(Y = 5) = 0.0165$

Rolling five fives or sixes in six throws is the more probable.

9 (A probability tree may be useful to visualise the working below.)

a **i** Let B_n be the event that the n th draw is blue, and R_n be the event that the n th draw is red.

Tip

With no information or conditional on the first ball, the probability that the second ball is red is equal to the probability of drawing a red ball first (i.e. $\frac{6}{10}$)... consider that if you drew the first two balls blindfold, the choice of which you look at first is not going to change anything about its colour! However, you may feel more comfortable working this value out longhand rather than making the argument when answering an examination question, so that working is given below.

$$\mathrm{P}(R_2) = \mathrm{P}(B_1R_2) + \mathrm{P}(R_1R_2) = \frac{4}{10} \times \frac{6}{9} + \frac{6}{10} \times \frac{5}{9} = \frac{24 + 30}{90}$$

$$= 60\%$$

ii $\mathrm{P}(R_3) = \mathrm{P}(B_1B_2R_3) + \mathrm{P}(B_1R_2R_3) + \mathrm{P}(R_1B_2R_3) + \mathrm{P}(R_1R_2R_3)$

$$= \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} + \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} + \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} + \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{72 + 120 + 120 + 120}{720}$$

$$= 60\%$$

Tip

As you saw in question 9ai, since the previous draws are not considered, the probability is still $\frac{6}{10}$.

iii $\mathrm{P}(R_1\, R_2) = \frac{6}{10} \times \frac{5}{9} = \frac{1}{3}$

But $\mathrm{P}(R_1) = 60\% = \mathrm{P}(R_2)$

So $\mathrm{P}(R_1\, R_2) \neq \mathrm{P}(R_1)\mathrm{P}(R_2)$

The two events R_1 and R_2 are not independent.

iv $\mathrm{P}(\text{One red}) = \mathrm{P}(R_1B_2B_3) + \mathrm{P}(B_1R_2B_3) + \mathrm{P}(B_1B_2R_3)$

$$= \frac{6}{10} \times \frac{4}{9} \times \frac{3}{8} + \frac{4}{10} \times \frac{6}{9} \times \frac{3}{8} + \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} = \frac{3}{10}$$

$X \sim \mathrm{B}(3,\, 0.6)$

$\mathrm{P}(X = 1) = 0.288$

There is a slightly higher probability of drawing exactly one red ball when the draws are made without replacement.

b **i** $\mathrm{P}(\text{Draw a red ball}) = 0.6$

ii $\mathrm{P}(\text{Draw a red ball}) = 0.6$

iii $\mathrm{P}(\text{Draw two red balls in two draws}) = 0.6^2 = 0.36$

iv With replacement, the number of red balls drawn is binomial $\mathrm{B}(3,\, 0.6)$ so $\mathrm{P}(\text{One red}) = 0.288$, taken from the binomial distribution answer in question 9aiv.

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10 a As in question 9, if no information is given about previous results, the probability of pulling out a red sock on the n th attempt is constant.

However, if information *is* given about previous results, the probabilities will change; therefore this scenario shows no independence but constant probability.

b No roll can affect another (rolls are independent) and the probability of success is constant.

c Each student's result is independent of the other students' results, but clearly the probabilities vary, since the probability of success p will variously equal 0, $\frac{1}{2}$ or 1, depending on the student's name.

11 Let X be the number of times out of the ten flips that the coin shows 'heads':

$X \sim \mathrm{B}(10,\, 0.5)$

$\mathrm{P}(X \geqslant 6) = 1 - \mathrm{P}(X \leqslant 5) = 0.377$

12 a Let X be the number of correct guesses out of 20:

$X \sim \mathrm{B}(20,\, 0.2)$

$\mathrm{P}(X < 5) = 0.630$

b To use the binomial distribution there must be a fixed number of trials which are independent and have a constant probability of success. The question says that the first two conditions are satisfied, so a further assumption needed is that the guesses are independent.

13 a Consider the conditions for the binomial distribution:

1. The number of trials is fixed.
2. The outcomes can be classified in two groups: a girl or not a girl.
3. The genders of different babies are independent.
4. The probability of a girl is constant.

The first two conditions are satisfied by the set-up of the question. The other two conditions are not necessarily satisfied, so we need to make these assumptions:

The genders of different babies are independent. The probability may be constant, but we also need to assume that it equals 0.5, so both genders are equally likely.

b $\mathrm{P}(\mathrm{G} > 15) = 1 - \mathrm{P}(\mathrm{G} \leqslant 15) = 0.00591$

14 a The probability of being late probably depends on the distance to school and the mode of transport, as well as students' individual characteristics. Hence the assumption is not reasonable.

b It is also necessary that one student being late is independent of all other students. This condition is unlikely to be satisfied, as several students may be travelling together (so if, for example, a bus is late, this will affect more than one student).

15 $X \sim \mathrm{B}(8,\, p)$

$$\mathrm{P}(X = 5) = \binom{8}{5} p^5 (1-p)^3 = 56p^5(1-p)^3$$

$$\mathrm{P}(X = 6) = \binom{8}{6} p^6 (1-p)^2 = 28p^6(1-p)^2$$

$\mathrm{P}(X = 5) = \mathrm{P}(X = 6) \Rightarrow 2(1-p) = p$ therefore $2 = 3p$ so $p = \frac{2}{3}$

16 $X \sim \mathrm{B}(4,\, p)$

$$\mathrm{P}(X = 2) = \binom{4}{2} p^2 (1-p)^2 = 0.3456$$

$$p^2(1-p)^2 = \frac{0.3456}{6} = 0.0576$$

$p(1-p) = 0.24$ so $p^2 - p + 0.24 = 0$ therefore $(p - 0.4)(p - 0.6) = 0$

$p = 0.4$ or 0.6

17 X-B$(n,\ 0.4)$

a $\mathrm{P}(X = 2) = \binom{n}{2} 0.4^2 0.6^{n-2} = \dfrac{n(n-1)}{2} \times 0.16 \times \dfrac{0.6^n}{0.36} = \dfrac{2n(n-1)}{9} \times 0.6^n$

b $\dfrac{2n(n-1)}{9} \times 0.6^n = 0.121$

From calculator, $n = 10$.

18 X-B$(n,\ 0.8)$

$\mathrm{P}(X = n) = 0.8^n$

$\mathrm{P}(X = 0) = 0.2^n$

$0.8^n = 64 \times 0.2^n$ so $4^n = 64$ therefore $n = 3$.

19 X-B$(n,\ p)$

$\mathrm{P}(X = 4) = \binom{n}{4} p^4 (1-p)^{n-4}$, $\mathrm{P}(X = 5) = \binom{n}{5} p^5 (1-p)^{n-5}$

$\mathrm{P}(X = 4) = \mathrm{P}(X = 5)$: $\dfrac{n!}{4! \times (n-4)!} p^4 (1-p)^{n-4} = \dfrac{n!}{5! \times (n-5)!} p^5 (1-p)^{n-5}$

Cancelling all common terms, brackets and factorials gives: $\dfrac{1-p}{n-4} = \dfrac{p}{5}$

$5 - 5p = np - 4p$ therefore $n = \dfrac{5-p}{p}$

MIXED PRACTICE 17

Tip

It is always wise to define your random variable clearly at the start of a question, especially on occasions where you will use one distribution result to inform a different distribution.

Defining your variable at the start of working makes the calculations clear to the reader, whether that is the examiner or yourself, when checking your answer.

1 Let X be the number of defective bottles in a sample of 20:

X-B$(20,\ 0.015)$

$\mathrm{P}(X \geqslant 1) = 1 - \mathrm{P}(X = 0) = 1 - 0.985^{20} = 26.1\%$

2 **a** Probabilities sum to 1: $0.05 + 0.23 + 0.35 + k + 0.21 = 0$ so $k = 0.16$

b $\mathrm{P}(\text{Score} \geqslant 3) = 0.35 + k + 0.21 = 0.72$

c Modal mark is 3.

3

$X = \lvert 1\text{st spin} - 2\text{nd spin} \rvert$		**1st spin**			
		1	2	5	7
2nd spin	1	0	1	4	6
	2	1	0	3	5
	5	4	3	0	2
	7	6	5	2	0

x	0	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

4 X~B(12, 0.4)

a $\mathrm{P}(X = 3) = 0.142$

b $\mathrm{P}(X > 5) = 1 - \mathrm{P}(X \leqslant 4) = 0.562$

5 **a** Probabilities sum to 1: $p + 0.2 + 0.3 + 0.3 + q = 1$ therefore $p + q = 0.2$

b $\mathrm{P}(X \geqslant 4) = 0.3 + 0.3 + q = 0.75$

$q = 0.15$, $p = 0.05$

6 Use calculator to find cumulative or individual probabilities:

a $\mathrm{P}(X < 6) = \mathrm{P}(X \leqslant 5) = 0.572$

b $\mathrm{P}(X = 8) = 0.0319$

c $\mathrm{P}(5 \leqslant X < 7) = \mathrm{P}(X = 5 \text{ or } 6) = 0.2786 + 0.2587 = 0.537$

7 Let X be the number of times Robyn hits the target in 8 attempts.

X~B(8, 0.6)

a $\mathrm{P}(X = 4) = 0.232$

b $\mathrm{P}(\text{fails to qualify}) = \mathrm{P}(X \leqslant 6) = 0.894$

c We need to assume that the shots are independent. (The other two conditions for binomial distribution – that there is a fixed number of trials and a constant probability of success – are given in the question.)

8 **a** $\mathrm{P}(S = 2) = \mathrm{P}(S \leqslant 2) - \mathrm{P}(S \leqslant 1) = 0.16$

16% of candidates received grade 2.

b $\mathrm{P}(3 \leqslant S \leqslant 5) = \mathrm{P}(S \leqslant 5) - \mathrm{P}(S \leqslant 2) = 0.52$

52% of candidates received grades between 3 and 5 inclusive.

9 Let X be the number of questions answered correctly out of 15: X~B(15, 0.7)

$\mathrm{P}(7 < X < 12) = \mathrm{P}(X \leqslant 11) - \mathrm{P}(X \leqslant 7) = 0.653$

10 Let X be the number of times the coin shows heads in 6 throws. X~B(6, p)

$\mathrm{P}(X = 0) = (1 - p)^6 = 0.072$ so $p = 1 - 0.072^{\frac{1}{6}} = 0.355 \approx 0.36$

11 **a**

$X = \text{Roll } S - \text{Roll } E$		**Roll S**					
		1	2	3	4	5	6
Roll E	1	0	1	2	3	4	5
	2	−1	0	1	2	3	4
	3	−2	−1	0	1	2	3
	4	−3	−2	−1	0	1	2
	5	−4	−3	−2	−1	0	1
	6	−5	−4	−3	−2	−1	0

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\mathrm{P}(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

b $\mathrm{P}(\text{Roll } E > \text{Roll } S) = \frac{15}{36} = \frac{5}{12}$

12 a Let p be the probability of heads. Then $p^2 = 0.04$, so $p = 0.2$

So $\mathrm{P}(\text{tails}) = 0.8$ and hence $\mathrm{P}(\text{tails twice}) = 0.8^2 = 0.64$

b $\mathrm{P}(\text{heads once in two throws}) = 2p(1-p)$

$2p - 2p^2 = 0.42 \Rightarrow 2p^2 - 2p + 0.42 = 0$, so $p = 0.3$ or 0.7

13 a $\mathrm{P}(\text{At least one defective}) = 1 - \mathrm{P}(0 \text{ defective}) = 1 - 0.995^6 = 0.0296$

b Let X be the number of packs out of 20 containing at least one defective bulb:

X~B$(20,\ 0.296)$

$\mathrm{P}(X > 4) = 1 - \mathrm{P}(X \leqslant 4) = 0.000\,244 = 0.0244\%$

14 a $\mathrm{P}(5 \text{ heads}) = \left(\frac{1}{2}\right)^5 = 0.03125$

b $\mathrm{P}(\text{HHHHHT}) = \left(\frac{1}{2}\right)^6 = 0.015\,625$

15 a $\mathrm{P}(\text{test}) = \mathrm{P}(\text{at least one forgets}) = 1 - \mathrm{P}(0 \text{ forget}) = 1 - 0.95^{12} = 0.460$

b $1 - 0.95^n$

c Require $1 - 0.95^n > 0.8$ so $0.95^n < 0.2$

$n \log 0.95 < \log 0.2$ therefore $n > \dfrac{\log 0.2}{\log 0.95}$

Tip

Inequality reverses because $\log 0.95 < 0$.

$n > 31.4$ so least such n is $n = 32$

16 a

$X = \lvert 1\text{st roll} - 2\text{nd roll}\rvert$		**2nd roll**					
		1	2	3	4	5	6
1st roll	1	0	1	2	3	4	5
	2	1	0	1	2	3	4
	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

x	0	1	2	3	4	5
$\mathrm{P}(X = x)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

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b Let X be the number of times a zero is recorded in 10 trials.

X~B $\left(10,\ \frac{1}{6}\right)$

$\mathrm{P}(X > 3) = 1 - \mathrm{P}(X \leqslant 3) = 0.0697$

17 a For X to take a value less than or equal to k, that means that no values greater than k were rolled

For $X = 1$, that means all rolls were a 1

For $X = 2$, all rolls must have been 1 or 2

Etc.

The probability that a single roll is k or less is $\frac{k}{6}$ for $k = 1$ to 6.

$\mathrm{P}(X \leqslant k) = \mathrm{P}(\text{all four rolls are less than or equal to } k) = \left(\frac{k}{6}\right)^4$

b $\mathrm{P}(X = 1) = \left(\frac{1}{6}\right)^4$ and $\mathrm{P}(X = k) = \mathrm{P}(X \leqslant k) - \mathrm{P}(X \leqslant k - 1)$ for $k > 1$

The probability distribution is therefore:

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{6^4}$	$\frac{2^4 - 1^4}{6^4}$	$\frac{3^4 - 2^4}{6^4}$	$\frac{4^4 - 3^4}{6^4}$	$\frac{5^4 - 4^4}{6^4}$	$\frac{6^4 - 5^4}{6^4}$
	$\frac{1}{1296}$	$\frac{15}{1296}$	$\frac{65}{1296}$	$\frac{175}{1296}$	$\frac{369}{1296}$	$\frac{671}{1296}$

18 a Let X be the number of times a 6 is rolled in the first 7 rolls.

X~B $\left(7,\ \frac{1}{6}\right)$

$\mathrm{P}(X = 3) = 0.0781$

b $\mathrm{P}(\text{fourth 6 on eighth roll}) = \mathrm{P}(X = 3 \text{ and then a 6 is rolled})$

$= \mathrm{P}(X = 3) \times \frac{1}{6} = 0.0130$

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Chapter 18 worked solutions

18 Statistical hypothesis testing

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 18A

3 **a** Mean equals the sum of data values divided by the number of data values:

$$\text{mean} = \frac{51}{20} = 2.55$$

Standard deviation of the sample:

$$\sigma = \sqrt{\frac{1}{n}\sum w^2 - \left(\frac{1}{n}\sum w\right)^2} = \sqrt{\frac{138.32}{20} - 2.55^2} = 0.643.$$

b The deviation within the sample is likely to be less than the deviation within the population, so the population standard deviation is probably greater than 0.643.

c The estimate of the mean taken from a larger sample is *likely* to be more accurate as a representation of the true population mean, but it is not necessarily so.

4 **a** To randomly sample a population requires the ability to select any individual with equal probability. However, this would require having a complete listing of every member of the population (every fish in the North Sea) and being able to measure whichever fish on that list was selected for the random sample. Since the sampling method is going to be limited by opportunity (which ones can be caught) and the whole population is not known, a random sample is not achievable.

b The sample has been balanced to be in proportion to the believed population, so this is quota sampling.

c Given many fish species travel in groups, a quota sampling system prevents the sample being distorted by this effect. The quota sample should be more representative than an opportunity sample of the first 20 fish observed.

5 Obtain a numbered list of all the students; use a random number generator to generate 70 different random numbers; select students with those numbers.

6 **a** Opportunity sampling (he is using individuals who are available and willing to take part).

b Students who eat burgers may not be interested in healthy eating.

7 A common way to select a sample using random numbers is to ignore repeats and numbers outside the range of data.

938 is not used because there is no student with that number. The second 016 is not used because it is a repeat. There is no student with number 886. So a possible sample consists of students with numbers: 16, 762, 537, 722, 152.

8 **a** The population is 40% female so the sample should reflect this, with a total of 32 females and 48 males.

The median age is 42 so the sample should contain as many people less than 42 years old as are at least 42 years old – so males and females should each be split equally between age groups.

	Female	**Male**	**TOTAL**

Under 42	16	24	40
42 or over	16	24	40
TOTAL	32	48	80

b Stratified sampling uses random selection of participants (albeit the numbers in each group are constrained) and is therefore more likely to give a result which is representative of the population than a quota sample, where participants enter the sample on the basis of availability, which may introduce bias.

c Quota sampling is simpler, and so in a practical sense is usually cheaper, and because it seeks available people to measure, it is also more likely to be achievable. To undertake stratified sampling would mean being able to compel individuals selected for the sample to allow measurement and would also require a complete list of all people in the city, and their ages.

9 a A: Picking samples at random from randomly selected subgroups of the population = cluster sampling.

B: Random selection from a full list of the population being studied = simple random sampling.

C: (Non-random) selection from every subgroup in proportion to its size = quota sampling.

D: Random selection from every subgroup in proportion to its size = stratified sampling.

b D is likely to give the most representative result. It is credible that different branches might exhibit different profiles so in this situation, simple random sampling might introduce bias because of the likelihood of over- or under-representing branches showing unusually high or low wellbeing.

c Method A will be simpler (and consequently faster and cheaper) to undertake than method B, since the owner need only travel to 4 branches to gather data.

10 If all participants answered the questionnaire, this would produce a random sample of voters in the borough. However, some recipients of the questionnaire will doubtless not provide a response, and so non-participants can never be part of a data sample. Since a simple random sample requires that every member of the population has an equal chance of being a part of the sample, this cannot be a simple random sample.

11 a $\frac{500}{1000} \times \frac{499}{999} \times \ldots \times \frac{491}{991} = 0.000\,933$

b i The number of ways of choosing k different names from n is $\binom{n}{k}$ so for ten different individuals from 1000 there are $\binom{1000}{10} = 2.63 \times 10^{23}$ ways. The number of ways of choosing k names (with replacement) from n is $\binom{n+k-1}{k}$ so for ten names from 1000 there are $\binom{1009}{10} = 2.88 \times 10^{23}$ ways.

Probability that the ten names are all different is $\frac{2.63}{2.88} = 0.914$

So the probability that at least one person is picked more than once is $1 - 0.914 = 0.0861$

ii Probability of a student not being selected is $0.999^{10} = 0.990\,04$

So the probability for any student of being picked (at least once) is $1 - 0.990\,04 = 0.009\,96$

c Ways of picking boys 10 times with replacement is $(0.5)^{10} = 0.000\,977$

Percentage difference between this value and the answer to question 11a is

$$\frac{(0.000\,977 - 0.000\,933)}{0.000\,933} \times 100\% = 4.65\%$$

d Taking a group of 10 students who have chosen to be in the same location will increase the probability of taking a cluster of the same gender.

EXERCISE 18B

Tip

Always lay out your answers formally. Define distributions, hypotheses (determine whether one or two tailed) and state the data you will use clearly. Ensure that your conclusion is in context.

3 Let X be the number of households in a sample which have at least one car.

X~B(n, p)

a $\mathrm{H}_0 : p = 0.63$

$\mathrm{H}_1 : p < 0.63$: one tailed test

b $n = 50$, $x = 29$, 10% significance test

$\mathrm{P}(X \leqslant 29 | \mathrm{H}_0) = 0.276 > 0.1$

There is insufficient evidence to reject H_0 at the 10% significance level

You cannot conclude that David's neighbourhood has a significantly lower car ownership level than the national average.

4 Let X be the number of 16- to 19-year-olds in a sample who attend sixth-form college.

X~B(n, p)

$\mathrm{H}_0 : p = 0.68$

$\mathrm{H}_1 : p > 0.68$: one-tailed test

$n = 60$, $x = 46, 5\%$ significance test.

$\mathrm{P}(X \geqslant 46 | \mathrm{H}_0) = 0.0943 > 0.05$

There is insufficient evidence to reject H_0 at the 5% significance level

You *cannot* conclude that the proportion of $16 - 19$-year-olds in the town attending sixth-form college has significantly increased.

5 Let X be the number of times a '3' is rolled in a series of trials.

X~B(n, p)

$\mathrm{H}_0 : p = \frac{1}{6}$

$\mathrm{H}_1 : p < \frac{1}{6}$: one-tailed test

$n = 30$, $x = 4$

Choose a 5% significance test.

Tip

If the question does not specify a significance level, choose 5% as your default, and state this clearly.

$\mathrm{P}(X \leqslant 4 | \mathrm{H}_0) = 0.424 > 0.05$

There is insufficient evidence to reject H_0 at the 5% significance level.

You cannot conclude that the a '3' is rolled with a significantly lower probability than $\frac{1}{6}$.

6 Let X be the number of teachers in the sample who drive to work.

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X~B(n, p)

$H_0 : p = 33\%$

$H_1 : p > 33\%$: one-tailed test

$n = 28$, $x = 12$, 5% significance test.

$P(X \geqslant 12|H_0) = 1 - P(X \leqslant 11|H_0) = 0.181 > 0.05$

There is insufficient evidence to reject H_0 at the 5% significance level.

You *cannot* conclude that the teachers at this college are more likely to drive to work than the average for the local authority.

7 Let X be the number of cases in a sample for which the new treatment is effective.

X~B(n, p)

$H_0 : p = 82\%$

$H_1 : p > 82\%$: one-tailed test

$n = 50$, $x = 43$, 2% significance test

$P(X \geqslant 43|H_0) = 1 - P(X \leqslant 42|H_0) = 0.30 > 0.02$

There is insufficient evidence to reject H_0 at the 2% significance level

You *cannot* conclude that the new treatment is significantly better.

8 Let X be the number of sixth-formers in a sample who have a younger sibling at the same school.

X~B(n, p)

$H_0 : p = \frac{1}{3}$

$H_1 : p \neq \frac{1}{3}$: two-tailed test

$n = 60$, $x = 27$, 5% significance test.

$P(X \geqslant 27|H_0) = 1 - P(X \leqslant 26|H_0) = 0.0397 > 0.025$

Tip

Remember that for a 2-tailed test, you must compare against half the significance level.

There is insufficient evidence to reject H_0 at the 5% significance level.

You *cannot* conclude that the number of sixth-formers with a younger sibling at the same school is different at the new school.

9 Let X be the number of times a '6' is rolled in a series of trials.

X~B(n, p)

$H_0 : p = \frac{1}{6}$

$H_1 : p < \frac{1}{6}$: one-tailed test

$n = 40$, $x = 3$, 10% significance test

$P(X \leqslant 3|H_0) = 0.081 < 0.1$

Reject H_0 at the 10% significance level.

You conclude that a '3' is rolled with a significantly lower probability than $\frac{1}{6}$.

10 Let X be the number of athletes in a sample who can achieve a 100-metre time less than 12 seconds:

X~B(n, p)

$H_0 : p = 28\%$

$H_1 : p \neq 28\%$: two-tailed test

Tip

Here you encounter a standard problem. Looking at the data it is tempting to use a one-tailed test in the direction the data suggests. Unfortunately, that is what is commonly called 'data snooping', and invalidates the result.

You have to interpret the question as someone asking 'has the success rate changed?', asking with no prior inclination one way or the other, or the question would instead refer to a hypothesis of worsening or improving. Be careful to read a question carefully for any clue as to a particular direction for H_1, and if there is no hint either directly or from context, you must use a two-tailed test.

$n = 75$, $x = 26$, 5% significance test.

$P(X \geqslant 26|H_0) = 1 - P(X \leqslant 25|H_0) = 0.125 > 0.025$

There is insufficient evidence to reject H_0 at the 5% significance level.

You cannot conclude that the proportion of athletes who can run 100 metres in under 12 seconds has changed.

11 Let X be the number of students in Year 12 in the sample who are in favour of the new uniform.

X~B(n, p)

$H_0 : p = 0.7$

$H_1 : p < 0.7$: one-tailed test

$H_2 : p \neq 0.7$: two-tailed test

$n = 25$, 10% significance test

Reject H_0 in the one-tailed test but not in the two-tailed test.

$0.05 < P(X \leqslant x|H_0) < 0.1$

Using the table function of the calculator or multiple calculations:

x	$P(X \leqslant x\|H_0)$
10	0.001778
12	0.01747
13	0.044246
14	0.0978
15	0.189436

The only value of x to satisfy the required inequality is $x = 14$.

12 Let X be the number of cats in a sample of the specified breed which are brown.

X~B(n, p)

$H_0 : p = 0.4$

$H_1 : p > 0.4$: one-tailed test

$n = 80, x = 40$

$\mathrm{P}(X \geqslant 40|\mathrm{H}_0) = 1 - \mathrm{P}(X \leqslant 39|\mathrm{H}_0) = 0.0445$

The conclusion was a rejection of H_0, so the significance level must be greater than 0.0445.

13 Let X be the number of patients in a sample who have the genetic condition.

$X \sim \mathrm{B}(n, p)$

$\mathrm{H}_0 : p = q$

$\mathrm{H}_1 : p < q$: one-tailed test

$n = 120$, $x = 0, 5\%$ significance test

$\mathrm{P}(X \leqslant 0|\mathrm{H}_0) = \mathrm{P}(X = 0|\mathrm{H}_0) = (1 - q)^{120}$

The conclusion, even with no patients showing the condition, is that H_0 *cannot* be rejected.

$(1 - q)^{120} > 0.05$

$(1 - q) > (0.05)^{\frac{1}{120}} = 0.9753$

$q < 0.0247$

EXERCISE 18C

3 Let X be the number of students in the sample who are girls:

$X \sim \mathrm{B}(n, p)$

$\mathrm{H}_0 : p = 0.5$

$\mathrm{H}_1 : p < 0.5$: one-tailed test, where p is the underlying proportion of A level students who are girls.

$n = 30$, 10% significance test

Find the greatest x such that $\mathrm{P}(X \leqslant x|\mathrm{H}_0) < 0.1$:

x	$\mathrm{P}(X \leqslant x\|\mathrm{H}_0)$
10	0.0494
11	0.1002

Critical region is $X \leqslant 10$.

4 **a** Let X be the number of patients out of a cohort of size n suffering the given disease who are cured by application of the new drug:

$X \sim \mathrm{B}(n, p)$

$\mathrm{H}_0 : p = 0.85$

$\mathrm{H}_1 : p > 0.85$: one-tailed test, where p is the underlying proportion of patients cured by the drug.

b $n = 180, 5\%$ significance test

Find the least x such that $\mathrm{P}(X \geqslant x|\mathrm{H}_0) < 0.05$

x	$\mathrm{P}(X \leqslant x\|\mathrm{H}_0)$
160	0.946
161	0.967

Critical region is $X \geqslant 162$.

5 **a** Let X be the number of people in a sample of size n who have purchased the manufacturer's

products.

X~B(n, p)

$\mathrm{H}_0 : p = 0.12$

$\mathrm{H}_1 : p > 0.12$: one-tailed test

b Critical region: $X \geqslant 32$

$n = 200$, $x = 30$

Value does not lie in the critical region so cannot reject H_0.

There is insufficient evidence to show the purchasing history in the population having increased from 12%.

c $\mathrm{P}(X \geqslant 32|\mathrm{H}_0) = 1 - \mathrm{P}(X \leqslant 31|\mathrm{H}_0) = 0.0556$

6 **a** $5 \leqslant X \leqslant 15$ is the acceptance region

b X~B$(80, p)$

$\mathrm{H}_0 : p = \dfrac{1}{8}$

$\mathrm{H}_1 : p \neq \dfrac{1}{8}$: two-tailed test

$\mathrm{P}(X < 5 \text{ or } X \rangle 15|\mathrm{H}_0) = \mathrm{P}(X \leqslant 4|\mathrm{H}_0) + 1 - \mathrm{P}(X \leqslant 15|\mathrm{H}_0) = 0.0600$

7 **a** Let X be the number of children in a sample of size n who walk to school.

X~B(n, p)

$\mathrm{H}_0 : p = 0.3$

$\mathrm{H}_1 : p > 0.3$: one-tailed test where p is the underlying proportion of children who walk to school.

b $n = 170$, 5% significance test.

Find the least x such that $\mathrm{P}(X \geqslant x|\mathrm{H}_0) < 0.05$

x	$\mathrm{P}(X \leqslant x\|\mathrm{H}_0)$
60	0.942
61	0.959

Critical region is $X \geqslant 62$

c Data lies in the critical region; reject H_0 at 5% significance level.

Conclude that the proportion of children who walk to school is greater than 30%.

8 **a** Let X be the number of students who pass their test on the first attempt:

X~B(n, p)

$\mathrm{H}_0 : p = 0.72$

$\mathrm{H}_1 : p \neq 0.72$: two-tailed test where p is the underlying proportion of students who pass on the first attempt.

b Acceptance region is $30 \leqslant X \leqslant 40$

So critical region is $X \leqslant 29$ or $X \geqslant 41$.

c $n = 50$

$\mathrm{P}\left(X \leqslant 29|\mathrm{H}_0\right) + \mathrm{P}\left(X \geqslant 41|\mathrm{H}_0\right) = \mathrm{P}\left(X \leqslant 29|\mathrm{H}_0\right) + \left(1 - \mathrm{P}\left(X \leqslant 40|\mathrm{H}_0\right)\right)$

$= 0.0232 + 0.0740 = 0.0972$

Since the probability of this error is by definition no greater than the significance level,

$\text{SL} \geqslant 0.0972$.

MIXED PRACTICE 18

1 Quota sampling (fixed numbers from different subgroups).

2 **a** Population has been divided into subsets (the tutor groups) and then from some of these subsets a random sample is taken. This is cluster sampling.

b There may not be independence within clusters (students in the same tutor group may choose to go together) and the clusters selected may affect the data values (students with a music teacher as a tutor may be more encouraged to attend).

c List all pupils, identifying each with an indexing number (1, 2, 3 etc.).

Use a random number generator to select 50 different integers less than or equal to the number of students and select the students with the corresponding indexing numbers.

3 **a** Let X be the number of times in n rolls that the die shows a 5.

$X \sim \text{B}(n, p)$

$\text{H}_0 : p = \dfrac{1}{6}$

$\text{H}_1 : p > \dfrac{1}{6}$: one-tailed test where p is the underlying probability of rolling a 5.

b $n = 75$, $x = 18$, 5% significance test.

$\text{P}(X \geqslant 18 | \text{H}_0) = 0.0654 > 0.05$

Cannot reject H_0.

There is insufficient evidence at the 5% significance level to conclude that the probability of rolling a 5 is greater than $\dfrac{1}{6}$.

4 Let X be the number of times in n days that Lisa is late for school:

$X \sim \text{B}(n, p)$

$\text{H}_0 : p = \dfrac{1}{8}$

$\text{H}_1 : p < \dfrac{1}{8}$: one-tailed test

where p is the underlying proportion of days that Lisa is late.

$n = 30$, $x = 2$, 10% significance test.

$\text{P}(X \leqslant 2 | \text{H}_0) = 0.258 > 0.1$

Cannot reject H_0.

There is insufficient evidence at the 10% significance level to conclude that Lisa's probability of being late has decreased from 1 in 8.

5 **a** $\dfrac{500}{600} = \dfrac{5}{6}$

b Not correct; this sample has the same probability as any other sample.

6 **a** Volunteers, being a self-selecting sample, are unlikely to represent the true opinion of all students – they are more likely to have strong political interests and therefore give more extreme results than the general population.

b List all n students in alphabetical order and number from 1 to n. Select a sample of the required size by random number selection from 1 to n, including the corresponding students in the sample.

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7 Let X be the number of people in the sample of 30 who suffer side effects:

X~B$(30, p)$

$\mathrm{H}_0 : p = 0.2$

$\mathrm{H}_1 : p < 0.2$ (one-tailed)

x	$\mathrm{P}(X \leqslant x)$
1	0.0105
2	0.0442
3	0.123

The rejection region is $X \leqslant 2$, so the largest number of sufferers for which H_0 would still be rejected is 2.

8 Let X be the number of families (in a survey of n families) that own their own home:

X~B(n, p)

$\mathrm{H}_0 : p = 0.64$

$\mathrm{H}_1 : p \neq 0.64$: two-tailed test, where p is the underlying proportion of families owning their own home

$n = 180$, $x = 98$, 5% significance test

$\mathrm{P}(X \leqslant 98 | \mathrm{H}_0) = 0.00523 < 0.025$

Reject H_0 at 5% confidence level.

Based on this sample, home ownership in Germany is significantly lower than 64%.

9 **a** A simple random sample requires that each possible group of a given size has an equal probability of being selected, and is carried out by a method equivalent to assigning a distinct index number (1, 2, 3, etc.) to each member of the population and using a random number generator to select a sufficient quantity of individuals (identified by their index numbers) to constitute the sample.

b **i** The population is classified into mutually exclusive and collectively exhaustive groups, each of which may be anticipated to have a behaviour distribution distinct from the others. The sample is drawn from each stratum in proportion with that stratum's size in the population.

This is stratified sampling.

ii The advantage over simple random sampling is that, if boys and girls have a different behaviour profile, then a random sample which (by chance) over-represented one group would give a distorted impression of the overall population.

c Let X be the number of students in a sample of size n who play for a school sports team.

X~B(n, p)

$\mathrm{H}_0 : p = 0.4$

$\mathrm{H}_1 : p < 0.4$: one-tailed test

$n = 70$, 10% significance test.

Find the least x such that $\mathrm{P}(X \geqslant x | \mathrm{H}_0) < 0.1$

x	$\mathrm{P}(X \leqslant x \mid \mathrm{H}_0)$
32	0.864
33	0.909

Critical region is $X \geqslant 34$.

d $\mathrm{P}(X \geqslant 34 | \mathrm{H}_0) = 0.0906$

e Data lies outside the critical region; cannot reject H_0.

There is insufficient evidence that the proportion of students who play in a school sports team is less than 40%.

10 a Let X be the number of occurrences of the letter e in a sample of n letters of text.

$X \sim \mathrm{B}(n, p)$

$\mathrm{H}_0 : p = 0.19$ $\mathrm{H}_1 : p < 0.19$ (one-tailed)

Test at 10% significance level for $n = 20$, $x = 1$

$\mathrm{P}(X \leqslant 1 | \mathrm{H}_0) = 0.0841 < 0.10$

Reject H_0 at the 10% significance level. There is evidence to conclude that the frequency of the letter e in the text from which the sample was drawn is less than 19%.

b Words are not random assortments of letters, and each letter will not occur independently (to take a more extreme example to illustrate this, the model would suggest that in a sample text 4 letters long, there would be a 0.1% chance that all the letters would be e).

11 $p_1 = \mathrm{P}(0, 1, 9, 10 | \mathrm{fair})$ must be less than the significance AND $p_2 = \mathrm{P}(0, 1, 2, 8, 9, 10 | \mathrm{fair})$ must be greater than the significance (or the criterion for concluding bias would have to be less stringent).

$p_1 = 0.02$ and $p_2 = 0.11$ so the only option given in the question which satisfies the conditions is 10%.

(20% would have rejection for $0, 1, 2, 8, 9, 10$)

Chapter 19 worked solutions

19 Introduction to kinematics

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 19A

1 **a** There is no need to worry about the size or shape of the car when considering the journey, simply its position – so using a particle model is fine.

b Size and shape of car do matter here as they will need to manoeuvre around each other. A particle model would not be wise.

c The box would turn or perhaps even spin, which would affect the application of the force. This aspect would be lost in a particle model.

d For the football to curve will need an appreciation of the ball spinning (and the friction with the ground or air that spin will produce) so a particle model will not provide the necessary detail

2 The lists below are not intended to be comprehensive, just offer some additional considerations to those offered in brief in the textbook answers.

3 **a** If the box is small, rigid and aerodynamically compact and is falling sufficiently far from the building not to interact with it, then the particle model will work well. If the building is not of an extreme height or in an extreme location other factors can also be ignored.

However, if the box is large or not compact then air resistance may be significant.

If it is not rigid then it may deform during the fall, again giving rise to air resistance issues and potential separation into multiple falling objects or interaction with the building.

If the box is tumbling close to the building it may touch it, leading to friction, tumbling or rolling which would interfere with the simple falling model of the particle.

If the building is very tall then the difference in g at the top and the bottom may be of small significance, though in a real-world simulation this may not be noticeable.

Climatic conditions (wind or precipitation) will have an effect on effective air resistance and may lead to a change in mass of the object as it falls.

b If the balls are perfectly balanced, equal size, lying on a perfectly horizontal smooth or consistent friction surface, and if the cushions have perfect elasticity while deforming exactly one ball radius on impact, and if no ball will have any spin affecting the trajectory then the particle model will be perfect.

However …

Balls have small flaws and certainly snooker table surfaces are not perfectly horizontal and flat, but are slightly erratic with areas of greater and lesser friction. Balls roll, they do not slide, and they can roll with spin, which will interact with the frictional surfaces and air to produce deviations to the straight line movement. Cushions are not perfectly elastic (balls will not come back with quite the same momentum as they approach a cushion) and they deform erratically on impact, so the centre of the ball does not reflect exactly from the point of first contact with the cushion at equal angles of approach and departure. The particle model also requires a perfect hit; the particle will be located at the centre of mass of each ball (or calculations are even more distorted) but the centres of two balls need not be aligned for them to strike one another. Indeed, if the balls each have a radius r, then any occurrence of the centres passing at less than $2r$ separation would be an effective impact.

IRANINOVIN

c If the bus has a constant mass and fuel consumption profile as a function of speed, and if the road is accurately modelled (as a straight line or otherwise, the question does not specify) then a particle model will be fine.

However ...

The bus may pick up or lose passengers.

It will definitely lose mass during the journey, from burning fuel (unless it is a sealed-unit electrically powered model) and from the passengers respiring.

Climatic conditions will play a part in varying the resistance to movement – the bus may have the same actual speed of progress with a head wind or a tail wind but would burn a different amount of fuel. Similarly, if driving up or down hill, or on a straight versus a winding road.

d The question is slightly contradictory. If you model the Earth as a sphere and require the plane to travel in a straight line, it needs both submarine and digging capabilities not normally within a plane's specification! If you interpret the 'straight line' as suggested meaning 'constant elevation above sea level' on a flat-Earth model, you then have a problem of determining which flat-Earth projection to choose that will be the shortest 'horizontal' distance but not encounter mountains. So you can elevate the plane sufficiently and choose the shortest route, from one point on a sphere to another, a great arc of the sphere. (You are also now ignoring the problem that the Earth is not actually perfectly spherical, even over its non-terrestrial surfaces.)

But elevating the plane (and lowering it at the other end) takes time, and there are limits to how high you can fly a passenger plane anyway. Take-off and landing times (and the additional distances involved in flying obliquely instead of 'horizontally') have not been included.

As was the case for the bus, climatic conditions (headwind versus tailwind for instance) will have a considerable impact, as these can effectively increase or decrease the journey distance from the perspective of the plane.

EXERCISE 19B

4 If you were told that the particle moves along a fixed linear path in the positive direction, then the calculation for mean acceleration would be:

$$\text{mean acceleration} = \frac{\text{change in speed}}{\text{time taken}} = \frac{(4.3 - 12.3)\,\text{m s}^{-1}}{5\,\text{s}} = -1.6\,\text{m s}^{-2}$$

However, acceleration is a vector quantity and speed a scalar, and the direction is not fixed. The particle could, for example, have been travelling west at $12.3\,\text{m s}^{-1}$ and by the end been travelling east at $4.3\,\text{ms}^{-1}$. The change in *velocity* is then $\pm(-12.3 - 4.3)\,\text{m s}^{-1}$ over five seconds, leading to an acceleration of $\pm\frac{(-12.3 - 4.3)\text{m s}^{-1}}{5\,\text{s}} = \pm 3.32\,\text{m s}^{-2}$, with the sign depending on the direction you determine to be positive. Likewise, the calculation above of $-1.6\,\text{m s}^{-2}$ assumes that the particle was travelling throughout in the direction specified as positive; if it was going 'backwards' at $12.3\,\text{m s}^{-1}$ and slowed to $4.3\,\text{m s}^{-1}$ then the acceleration in the positive direction would be $+1.6\,\text{m s}^{-2}$.

(In fact, if you free the particle to travel along a non-linear path (beyond the scope of this syllabus), the magnitude of the average acceleration can take any values in $-3.32 \leqslant a \leqslant -1.6$ and $1.6 \leqslant a \leqslant 3.32$.)

EXERCISE 19C

6 **a** $a = \frac{\text{d}v}{\text{d}t} = 2.4 - 3t\, a(2) = -3.6\,\text{m s}^{-2}$

b $a = 0 : t = 0.8$ therefore $v(0.8) = 0.96\,\text{m s}^{-1}$

IRANINOVIN

c $x(t) = \int v\mathrm{d}t = \int 2.4t - 1.5t^2\,\mathrm{d}t = 1.2t^2 - 0.5t^3 + c$

$x(0) = c = 0$

$x = 1.2t^2 - 0.5t^3$

7 **a** $v = \dfrac{\mathrm{d}x}{\mathrm{d}t} = 0.3t^2 - 2.4t + 3.5$ so $v(6) = -0.1$

Speed at 6 seconds is $0.1\,\mathrm{m\,s^{-1}}$.

$a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = 0.6t - 2.4$ so $a(6) = 1.2\,\mathrm{m\,s^{-2}}$

b $x = 0.1t(t^2 - 12t + 35) = -0.1t(t-5)(t-7)$

$x = 0$ at $t = 0$, 5 and 7

$v(5) = -1$

Speed when the object first returns to O is $1\,\mathrm{m\,s^{-1}}$.

c $v = 0.1(3t^2 - 24t + 35)$

$v = 0$ when $t = \dfrac{24 \pm \sqrt{24^2 - 4 \times 3 \times 35}}{6} = 4 \pm \dfrac{\sqrt{39}}{3}$

$v = 0$ when $t = 1.92$ or 6.08.

$x(1.92) = 3.00\,\mathrm{m}$

8 **a** $v(t) = \int a\mathrm{d}t = 0.04t^3 - 0.72t^2 + 4.32t + c_1$

$v(0) = 0 = c_1$

$v = 0.04t^3 - 0.72t^2 + 4.32t$

$x(t) = \displaystyle\int v\mathrm{d}t = 0.01t^4 - 0.24t^3 + 2.16t^2 + c_2$

$x(0) = 0 = c_2$

$x = 0.01t^4 - 0.24t^3 + 2.16t^2$

b $a = 0.12(t^2 - 12t + 36) = 0.12(t-6)^2$

$a = 0$ at $t = 6$

$v(6) = 8.64\,\mathrm{m\,s^{-1}}$

$x(6) = 38.9\,\mathrm{m}$

9 **a** $v(t) = \int a\mathrm{d}t = t - 0.3t^2 + c_1$

$v(0) = 7.5 = c_1$

$v = 7.5 + t - 0.3t^2$

b $x(t) = \int v\mathrm{d}t = 7.5t + 0.5t^2 - 0.1t^3 + c_2$

$x(0) = 25 = c_2$

$x = 25 + 7.5t + 0.5t^2 - 0.1t^3$

$x(10) = 50\,\mathrm{m}$

c $a = -2$ when $1 - 0.6t = -2$

$t = 5$

$x(5) = 62.5\,\mathrm{m}$

10 $v(t) = \int a\mathrm{d}t = 2t - 3t^2 + c$

$v(2) = -8 = 4 - 12 + c$

$c = 0$

$v(t) = 2t - 3t^2$

$x(t) = \displaystyle\int v\mathrm{d}t = t^2 - t^3 + d$

$x(8) = 64 - 512 + d = d - 448$

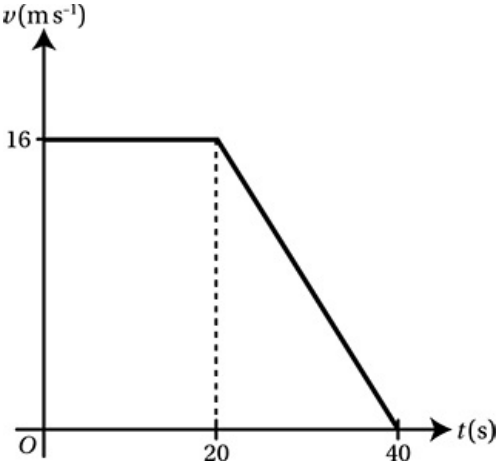
$x(5) = 25 - 125 + d = d - 100$

Total displacement between $t = 5$ and $t = 8$ is $x(8) - x(5) = -348$

$$\text{Average velocity} = \frac{\text{total displacement}}{\text{total time}} = \frac{-348}{8-5} = -116\,\text{m s}^{-1}$$

EXERCISE 19D

5 a



b Distance d = area under the graph, so $d = 16 \times 20 + \frac{1}{2} \times 16 \times 20 = 480\,\text{m}$

6 a $$\text{Acceleration} = \frac{(\text{change in velocity})}{\text{time}} = \frac{20}{10} = 2\,\text{m s}^{-2}$$

b Velocity changes sign (particle changes direction) at $t = 25\,\text{s}$.

c Total distance travelled equals the total area enclosed by the velocity–time graph.

If the velocity at $t = 32$ is $-v$ then: $325 = \frac{1}{2}25 \times 20 + \frac{1}{2}20 \times v$

$325 = 250 + 10v$ therefore $v = 7.5$

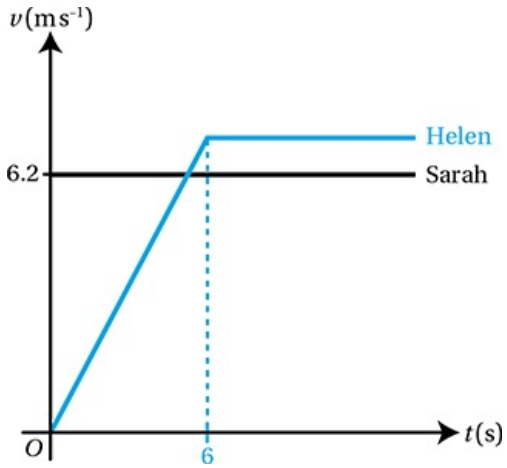
The velocity at $t = -32$ is $-7.5\,\text{m s}^{-1}$.

d Displacement equals the integral of the velocity–time graph, which is the difference between the triangle areas in this case.

$x = 250 - 75 = 175\,\text{m}$

The particle is $175\,\text{m}$ to the right of A at the end of the 45 seconds.

7 a



b Helen's end velocity is $1.3 \times 6 = 7.8\,\text{m s}^{-1}$.

Require the area under each graph to be equal. For $t > 6$:

$x_H(t) = \frac{1}{2} \times 6 \times 7.8 + 7.8(t - 6)$

$x_S(t) = 6.2t$

So when Helen catches up with Sarah, $x_H(t) = x_S(t)$

$7.8t - 23.4 = 6.2t$ therefore $1.6t = 23.4$ so $t = 14.6\,\text{s}$

8 a Total distance travelled equals area enclosed by the graph.

$275 = 20V + \frac{1}{2} \times V \times 10 + \frac{1}{2} \times 6 \times 25$

$25V + 75 = 275$ therefore $V = 8$

b Velocity–time graph has straight line section passing through $(20, V)$ and $(30, 0)$

Acceleration equals the gradient $\frac{0 - V}{30 - 20} = -0.8\,\text{m s}^{-2}$

There is a negative acceleration (deceleration) of $0.8\,\text{m s}^{-2}$.

c Three sections to graph:

For $0 \leqslant t \leqslant 20$ velocity is constant $8\,\text{m s}^{-1}$ so displacement is linear with slope 8.

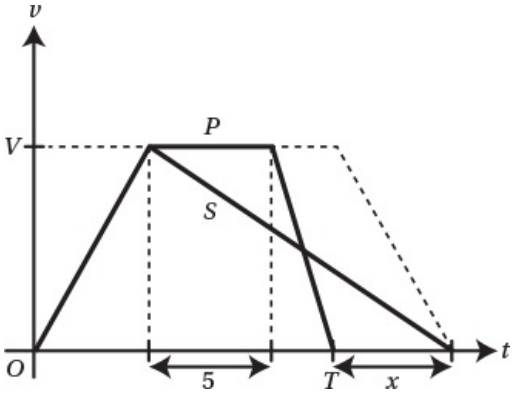
For $20 \leqslant t \leqslant 37.5$ velocity decreases through zero to $-6\,\text{m s}^{-1}$ so displacement describes a negative quadratic passing through its maximum at $t = 30$.

For $37.5 \leqslant t \leqslant 55$ velocity increases from -6 to zero so displacement describes a positive quadratic reaching its minimum at $t = 55$.

Graph C shows this behaviour.

9 Require the areas under the two velocity–time graphs to be equal.

Suppose Peter finishes the race in T seconds.



$\text{Area}_P = \frac{T + 5}{2} \times V$

$\text{Area}_S = \frac{1}{2} V \times (T + x)$

$V(T + 5) = V(T + x)$ therefore $x = 5$

10 a Require that the area above the horizontal is equal to the area below the horizontal for the end displacement to be zero.

$\frac{75 + T}{2} \times 6 = \frac{1}{2} \times 8 \times (165 - 75)$ so $225 + 3T = 360$

$3T = 135$ therefore $T = 45\,\text{s}$

b Maximum speed occurs when $v = -8$

Straight line passes through $(T, 6)$, $(75, 0)$ and $(t, -8)$

Gradient is $\frac{0 - 6}{75 - T} = -0.2\,\text{m s}^{-2}$

So $\dfrac{-8-0}{t-75} = -0.2$

$t - 75 = 40$ therefore $t = 115\,\text{s}$.

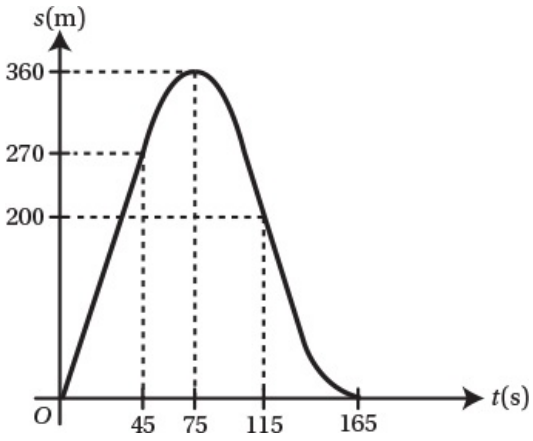
c 3 sections to the graph:

For $0 \leqslant t \leqslant 45$, straight line section with gradient 6, so connecting $(0, 0)$ to $(45, 270)$.

For $45 \leqslant t \leqslant 115$, negative quadratic with apex at $(75, 360)$.

For $115 \leqslant t \leqslant 165$, positive quadratic with apex at $(165, 0)$

Gradient (velocity) is continuous so there should be no corners, with each section flowing smoothly into the next. There should be an inflection at $t = 115$ where the second and third curves connect.



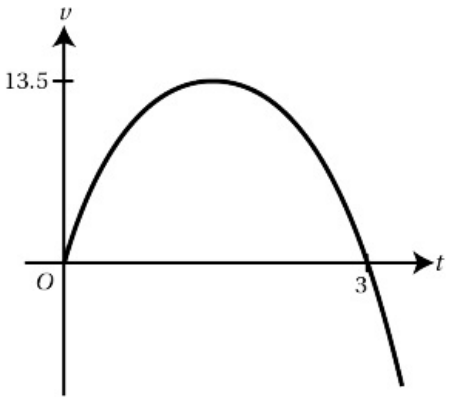
EXERCISE 19E

2 Total distance is given by the area under the graph: $\dfrac{25 + 10}{2} \times 3.5 = 61.25\,\text{m}$

$\text{Average speed} = \dfrac{\text{total distance}}{\text{total time}} = \dfrac{61.25}{25} = 2.45\,\text{m s}^{-1}$

3 **a** $v(t) = 6t(3 - t)$

Negative quadratic with roots at 0 and 3.



b Velocity over the first three seconds is positive, so distance travelled and displacement are the same in this period.

$\text{Total distance travelled} = \int_0^3 v\,\text{d}t = \left[9t^2 - 2t^3\right]_0^3 = 81 - 54 = 27$

$\text{Average speed} = \dfrac{\text{total distance}}{\text{total time}} = \dfrac{27}{3} = 9\,\text{m s}^{-1}$

4 **a** Particle is stationary at a distance of $400\,\text{m}$ from its start point in the period $12 \leqslant t \leqslant 19$.

b Point C is $200\,\text{m}$ in front of A.

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$$\text{Average velocity} = \frac{\text{change in displacement}}{\text{time taken}} = \frac{200}{30} = 6.67\,\text{m s}^{-1}.$$

c Particle travels out $200\,\text{m}$ and then back $200\,\text{m}$, for a total distance of $400\,\text{m}$.

$$\text{Average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{400}{30} = 13.3\,\text{m s}^{-1}$$

5 a Roots of the equation for velocity are at $t = 0$, 4 and 7.

Velocity changes from positive to negative at $t = 4$, so the particle ceases to move forward and starts to move backward at that time.

Velocity changes from negative to positive again at $t = 7$, so the particle stops its backward movement and begins to move forward again at that time.

b $v(t) = 3t^3 - 33t^2 + 84t$ so $x(t) = \int v\text{d}t = \frac{3}{4}t^4 - 11t^3 + 42t^2 + c$

Measuring displacement from position at $t = 0$ so $x(0) = 0 = c$

$x(t) = \frac{3}{4}t^4 - 11t^3 + 42t^2$ so $x(7) = 85.75$

c $\text{Average velocity} = \frac{\text{change in displacement}}{\text{change in time}} = \frac{85.75}{7} = 12.25\,\text{m s}^{-1}$

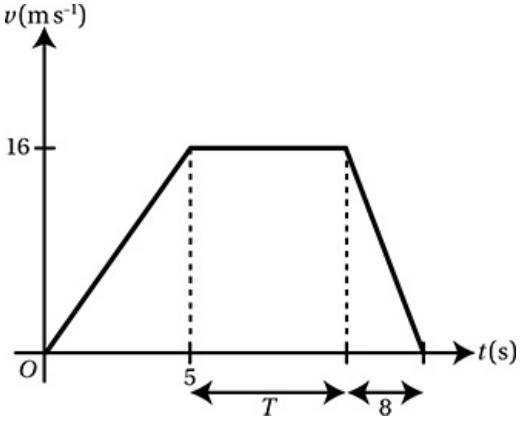
To calculate average speed, you need to know total distance travelled:

Distance travelled forwards is $x(4) = 160$.

You know that in the next 3 seconds the displacement reduces to 85.75, so the particle travels backwards $(160 - 85.75) = 74.25$.

$$\text{Average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{160 + 74.25}{7} = 33.5\,\text{m s}^{-1}.$$

6 a



b The decelerative part of the graph takes the velocity from $16\,\text{m s}^{-1}$ to $0\,\text{m s}^{-1}$ with an acceleration of $-2\,\text{m s}^{-2}$, and so lasts $\frac{16}{2} = 8$ seconds.

Total distance equals area under the graph; the graph consists of three parts:

Triangle, base $5\,\text{s}$, height $16\,\text{m s}^{-1}$: Area $\frac{1}{2} \times 5 \times 16 = 40\,\text{m}$

Rectangle, base $T\,\text{s}$ and height $16\,\text{m s}^{-1}$: Area $16T\,\text{m}$

Triangle, base $8\,\text{s}$, height $16\,\text{m s}^{-1}$: Area $\frac{1}{2} \times 8 \times 16 = 64\,\text{m}$

Total distance: $40 + 16T + 64 = 104 + 16T$

$$\text{Average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{104 + 16T}{5 + T + 8} = 12$$

$104 + 16T = 12(13 + T)$ so $4T = 52$ therefore $T = 13$

7 a $v(0) = 0$ so the particle is initially at rest.

$v(t) = 0.05t(t^2 - 20t + 96) = 0.05t(t - 8)(t - 12)$

So $v = 0$ when $t = 0$, 8 or $12\,\text{s}$.

b $v(6) = 3.6\,\text{m s}^{-1}$

$x\,(t) = \int v\,\text{d}t = \frac{1}{80}t^4 - \frac{1}{3}t^3 + \frac{12}{5}t^2 + c$

x measures displacement from the start point so $x(0) = c = 0$

$x(t) = \frac{1}{80}t^4 - \frac{1}{3}t^3 + \frac{12}{5}t^2$

$x(6) = 30.6\,\text{m}$

c The particle moves forward over $0 < t < 8$, backwards over $8 < t < 12$ and forwards for $t > 12$.

Total distance travelled in the first 8 seconds is $x(8) = 34.1\,\text{m}$.

Total distance travelled in $8 < t < 12$ is $|x(12)\ -\ x(8)| = |28.8\ -\ 34.13| = 5.33\,\text{m}$

Total distance travelled in the first 12 seconds is $34.13 + 5.33 = 39.5\,\text{m}$.

$\text{Average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{39.5}{12} = 3.29\,\text{ms}^{-1}$.

8 To calculate average speed, must determine total distance.

Particle travels forward for $0 < t < 2$ and backward for $2 < t < 7$.

$v(t) = t^3\ -\ 6t^2\ -\ 13t\ +\ 42$

$x(t) = \int v\text{d}t = \frac{1}{4}t^4 - 2t^3 - \frac{13}{2}t^2 +\ 42t + c$

x measures displacement from the start so $x(0) = 0 = c$

$x(t) = \frac{1}{4}t^4 - 2t^3 - \frac{13}{2}t^2 + 42t$

Total distance travelled in $0 < t < 2$ is $x(2) = 46\,\text{m}$.

Total distance travelled in $2 < t < 7$ is $x(2)\ -\ x(7) = 46 + 110.25 = 156.25$

Total distance travelled in $0 < t < 7$ is $46\ +\ 156.25 = 202.25$

$\text{Average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{202.25}{7} = 28.9\,\text{m s}^{-1}$.

9 Displacement of the particle for the first 5 seconds:

$x(t) = \int v\text{d}t = \frac{5}{2}t^2 - \frac{1}{6}t^3 + c$

$x(0) = 0 = c$

So $x(5) = 41.7$

Displacement of the particle for $5 < t \leqslant 15$:

$x(t) = \int v\text{d}t = \frac{15}{2}t - \frac{1}{4}t^2 + d$

Require that there is no jump in displacement, so this function evaluated at $t = 5$ must equal $x(5)$.

$x(5) = 41.7 = 31.25 + d$

$d = 10.4$

$$x(t) = \begin{cases} \frac{5}{2}t^2 - \frac{1}{6}t^3 & \text{for} \quad 0 \leqslant t \leqslant 5 \\ \frac{15}{2}t - \frac{1}{4}t^2\ + 10.4 & \text{for} \quad 5 \leqslant t \leqslant 15 \end{cases}$$

Since velocity is positive for $0 < t < 15$, distance travelled will equal displacement.

Require that $\frac{x(T)}{T} = 4$

$x(T) = 4T$

If $T < 5$: $\frac{5}{2}T^2 - \frac{1}{6}T^3 = 4T$

$T^3 - 15T^2 + 24T = 0$ so $T(T^2 - 15T + 24) = 0$

Roots of the quadratic are at $T = \frac{15 \pm \sqrt{15^2 - 4 \times 24}}{2} = \frac{15 \pm 11.4}{2}$

One root lies in the required interval: $T = 1.82\,\text{s}$

If $5 < T < 15$: $\frac{15}{2}T - \frac{1}{4}T^2 + 10.4 = 4T$

$T^2 - 14T + 41.7 = 0$ so $T = \frac{14 \pm \sqrt{14^2 - 4 \times 41.7}}{2} = \frac{14 \pm 5.4}{2}$

One root lies in the required interval: $T = 1.82\,\text{s}$.

So $T = 1.82\,\text{s}$ or $9.7\,\text{s}$.

10 $\text{Average velocity} = \frac{\text{change in displacement}}{\text{time taken}} = \frac{x(T)}{T} = 9$

So $24 - 3T = 9$ therefore $T = 5$

$v = \frac{\mathrm{d}x}{\mathrm{d}t} = 24 - 6t = 6(4 - t)$

The particle changes direction at $t = 4$

So total distance travelled in $0 < t < 4$ is $x(4) = 48\,\text{m}$.

And total distance travelled in $4 < t < 5$ is $x(4) - x(5) = 48 - 45 = 3\,\text{m}$.

So total distance travelled in $0 < t < 5$ is $48 + 3 = 51\,\text{m}$.

$\text{Average speed} = \frac{\text{total distance}}{\text{time taken}} = \frac{51}{5} = 10.2\,\text{m s}^{-1}$.

EXERCISE 19F

1 $x = 3.6t - 1.2t^2$

Tip

Always look for a quick way to answer a question.

The displacement here is a quadratic with roots at 0 and 3: $x = 1.2\,t(3 - t)$.

Since the vertex of a quadratic is midway between the roots, it is immediately known that the particle changes direction (is at its furthest point) at $t = 1.5$. You could use this argument to answer the question rather than making the algebraic calculations below.

$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = 3.6 - 2.4\,t$

Change of direction when $v = 0$

$3.6 = 2.4t$ therefore $t = 1.5$

The particle changes direction at 1.5 seconds after passing P for the first time.

2 $x(t) = 1.8t^2 - 0.2t^3 = 0.2t^2(9 - t)$

a $v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = 3.6t - 0.6t^2 = 0.6t(6 - t)$.

$v(0) = 0$ so at $t = 0$, Ellie is at rest

b v is a negative quadratic with roots at 0 and 6, so reaches its maximum midway between the roots at $t = 3$.

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$v(3) = 5.4\,\text{m s}^{-1}$

3 $v = -0.8t^2 + 1.44t + 9.72$

$= -0.04(20t^2 - 36t - 243)$

$v = 0$ when $t = \dfrac{36 \pm \sqrt{36^2 + 4 \times 20 \times 243}}{40} = \dfrac{36 \pm 144}{40} = -2.7$ or 4.5

Car comes to rest $4.5\,\text{s}$ after starting.

$x(t) = \displaystyle\int v dt = -\frac{8}{30}t^3 + 0.72t^2 + 9.72t + c$

$x(0) = c = 0$

$x = -\dfrac{8}{30}t^3 \ + \ 0.72t^2 \ + \ 9.72t$

$x(4.5) = 34.02\,\text{m}$

The car is $34.0\,\text{m}$ in front of its starting point when it comes to rest.

$a(t) = \dfrac{\text{d}v}{\text{d}t} = -1.6t + 1.44$

$a(4.5) = -5.76\,\text{m s}^{-2}$

The deceleration is $5.76\,\text{m s}^{-2}$ at the time the car comes to rest.

4 $v = 11.2t - 3t^2$

$x(t) = \displaystyle\int v\text{d}t = 5.6t^2 - t^3 + c$

$x(0) = 0 = c$ where x measures displacement from A

$x = 5.6t^2 - t^3 = t^2(5.6 - t)$

$x = 0$ at $t = 0$ or $t = 5.6$

Particle returns to A after 5.6 seconds.

5 **a** $a(t) = \dfrac{\text{d}v}{\text{d}t} = \begin{cases} 4 - t & \text{for} \quad 0 \leqslant t \leqslant 4 \\ 0.8t \ - 6.4 & \text{for} \quad t > 4 \end{cases}$

$a(5) = \ -2.4\,\text{m s}^{-2}$

b $x(t) = \displaystyle\int v\text{d}t = \begin{cases} 2t^2 - \dfrac{1}{6}t^3 \ + c_1 & \text{for} \quad 0 \leqslant t \leqslant 4 \\ \dfrac{2}{15}t^3 - \dfrac{16}{5}t^2 \ + \dfrac{136}{5}t \ + c_2 & \text{for} \quad t > 4 \end{cases}$

$x(0) = 0 = c_1$ where x measures displacement from the starting point.

$x(4) = 32 - \dfrac{64}{6} = \dfrac{64}{3}$

But by continuity of position, $x(4) = \dfrac{2}{15} \ \times \ 4^3 \ - \ \dfrac{16}{5} \ \times \ 4^2 \ + \ \dfrac{136}{5} \ \times \ 4 \ + \ c_2$

$\dfrac{128}{15} \ - \ \dfrac{256}{5} \ + \ \dfrac{544}{5} \ + \ c_2 = \dfrac{64}{3}$ therefore $c_2 = \ -\dfrac{224}{5} = \ -44.8$

$x(t) = \begin{cases} 2t^2 \ - \ \dfrac{1}{6}t^3 & \text{for} \quad 0 \leqslant t \leqslant 4 \\ \dfrac{2}{15}t^3 \ - \ \dfrac{16}{5}t^2 \ + \ \dfrac{136}{5}t \ - \ \dfrac{224}{5} & \text{for} \quad t > 4 \end{cases}$

i $x(3) = 13.5\,\text{m}$

ii $x(10) = 40.5\,\text{m}$

6 **a** $v(0) = 3.7 = u$

b $v(10) = 1.43\,\text{m s}^{-1}$

c Final displacement is the area under the velocity–time graph:

$$x(10) = \int_0^{10} v\mathrm{d}t = \int_0^5 3.7 + 0.4t\mathrm{d}t + \int_5^{10} 142.5t^{-2}\mathrm{d}t$$
$$= [3.7t + 0.2t^2]_0^5 + [-142.5t^{-1}]_5^{10} = (23.5 - 0) + (28.5 - 14.25)$$
$$= 37.75\,\mathrm{m}$$

d Require $x(t) = 32$

From question 6c, the dog travels $23.5\,\mathrm{m}$ in the first $5\,\mathrm{s}$, so you require a further $8.5\,\mathrm{m}$:

$$\int_5^T 142.5t^{-2}\,\mathrm{d}t = 8.5$$
$$\left[-142.5t^{-1}\right]_5^T = 8.5$$
$$28.5 - \frac{142.5}{T} = 8.5$$

$\frac{142.5}{T} = 20$ therefore $T = \frac{142.5}{20} = 7.125$ seconds.

7 $a(7) = -2.6$

$$a = \begin{cases} 6.5 - 1.3t & \text{for} \quad 0 \leqslant t \leqslant 7 \\ -2.6 & \text{for} \quad t > 7 \end{cases}$$

$$v(t) = \int a\mathrm{d}t = \begin{cases} 6.5t - 0.65t^2 + c_1 & \text{for} \quad 0 \leqslant t \leqslant 7 \\ -2.6t + c_2 & \text{for} \quad t > 7 \end{cases}$$

$v(0) = c_1 = 0$ since the particle starts from rest.

$v(7) = 6.5 \times 7 - 0.65 \times 7^2 = -2.6 \times 7 + c_2$ by continuity of velocity (assuming no impacts).

$c_2 = 9.1 \times 7 - 0.65 \times 7^2 = 31.85$

$$v = \begin{cases} 6.5t - 0.65t^2 & \text{for} \quad 0 \leqslant t \leqslant 7 \\ 31.85 - 2.6t & \text{for} \quad t > 7 \end{cases}$$

8 **a** For $0 \leqslant t \leqslant 5$ the acceleration is never negative, so the velocity can only increase in the first 5 seconds from its starting value of zero.

b $v(t) = \int a\mathrm{d}t = \frac{1}{30}(t-5)^3 + c_1$

$v(0) = -\frac{5^3}{30} + c_1 = 0$ since the particle starts from rest.

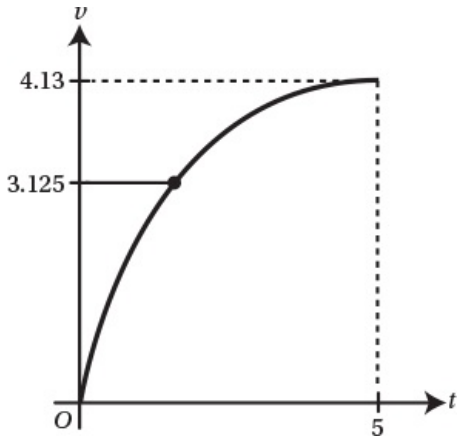
$$c_1 = \frac{125}{30}$$
$$v(t) = \frac{1}{30}(125 + (t-5)^3)$$
$$x(t) = \int v\mathrm{d}t = \frac{1}{120}(500t + (t-5)^4) + c_2$$

$x(0) = \frac{5^4}{120} + c_2 = 0$ since x measures displacement from the starting position.

$$c_2 = -\frac{625}{120}$$
$$x(t) = \frac{1}{120}(500t + (t-5)^4 - 625)$$
$$x(5) = \frac{1}{120}(2500 - 625) = 15.625$$

$$\text{Average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{15.625}{5} = 3.215\,\mathrm{m\,s^{-1}}$$

c $v(5) = \frac{125}{30} = 4.13\,\mathrm{m\,s^{-1}}$



Since the curve of v is increasing over $0 \leqslant t \leqslant 5$ from $v(0) = 0$ to $v(5) = 4.13$, it must pass through $v = 3.125$ in that interval of time.

9 a $v(t) = \int a\mathrm{d}t = t^3 - 7t^2 + 10t + c$

$v(0) = c = 0$ since the particle starts from rest

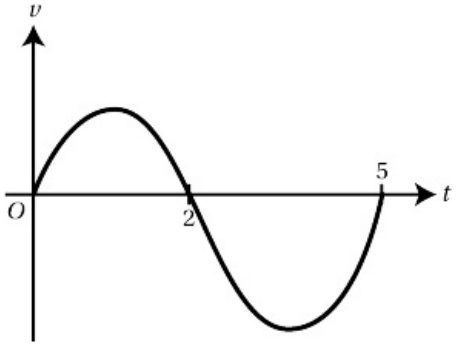
$v(t) = t^3 - 7t^2 + 10t$

b Maximum velocity occurs when acceleration is zero.

$a(t) = 3t^2 - 14t + 10$

$a = 0$ when $t = \dfrac{14 \pm \sqrt{14^2 - 4 \times 3 \times 10}}{2 \times 3} = 3.79\,\mathrm{s}$ or $0.880\,\mathrm{s}$

Velocity is a positive cubic, so the local maximum value is the first turning point, and since $v(0) = v(2) = v(5) = 0$, the local maximum and minimum are the two extreme values within the given time interval.



i $v(0.880) = 4.06\,\mathrm{m\,s^{-1}}$

ii $v(3.79) = -8.21\,\mathrm{m\,s^{-1}}$

So the maximum speed is $8.21\,\mathrm{m\,s^{-1}}$.

10 a $v = \begin{cases} 0.01t^2(12 - t) & \text{for} \quad 0 \leqslant t \leqslant 10 \\ 10 - 0.8t & \text{for} \quad 10 < t < 12.5 \end{cases}$

$v > 0$ during the first 10 seconds, so distance travelled is the same as displacement

$x(t) = \int v\mathrm{d}t = \begin{cases} 0.0025(16t^3 - t^4) + c_1 & \text{for} \quad 0 \leqslant t \leqslant 10 \\ 10t - 0.4t^2 + c_2 & \text{for} \quad 10 < t < 12.5 \end{cases}$

$x(0) = x_1 = 0$ since x measures displacement from the starting point

$x(10) = 15\,\mathrm{m}$

b Since the position must be continuous,
$x(10) = 15 = 10 \times 10 - 0.4 \times 10^2 + c_2$

$c_2 = 15 - 100 + 40 = -45$

Require $10t - 0.4t^2 - 45 = 17$

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$0.4t^2 - 10t + 62 = 0$

$t = \dfrac{10 \pm \sqrt{10^2 - 4 \times 0.4 \times 62}}{2 \times 0.4} = 12.5 \pm \dfrac{\sqrt{0.8}}{0.8} = 13.6 \text{ or } 11.4$

(can disregard 13.6 as it is out of the range).

The particle is at $17\,\text{m}$ from A when $t = 11.4\,\text{s}$.

11 $\text{Average velocity} = \dfrac{\text{end displacement}}{\text{total time}}$

$x(t) = \displaystyle\int v\,\text{d}t = \frac{20}{3}t^3 - \frac{1}{4}t^4 + c$

Set $x(0) = c = 0$ to measure displacement from the starting point.

$x(T) = \dfrac{20}{3}T^3 - \dfrac{1}{4}T^4$

Average velocity over the first T seconds is $\frac{x(T)}{T} = \frac{20}{3}T^2 - \frac{1}{4}T^3$

$v\left(\dfrac{T}{2}\right) = 20\left(\dfrac{T}{2}\right)^2 - \left(\dfrac{T}{2}\right)^3 = 5T^2 - \dfrac{1}{8}T^3$

If this equals the average velocity over the first T seconds then

$\dfrac{20}{3}T^2 - \dfrac{1}{4}T^3 = 5T^2 - \dfrac{1}{8}T^3$ so $\dfrac{5}{3}T^2 = \dfrac{1}{8}T^3$ therefore $T = \dfrac{40}{3} = 13.3$

12

Tip

Notice that this question uses two different units: m s^{-1} and also km h^{-1}. Choose which you will work in and convert all to that system.

$5\,\text{km h}^{-1}$ is equivalent to $5000\,\text{m h}^{-1} = \dfrac{5000}{3600}\text{m s}^{-1} = 1.39\,\text{m s}^{-1}$

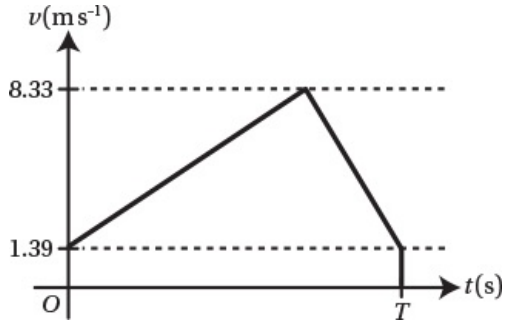
$30\,\text{km h}^{-1}$ is equivalent to $30\,000\,\text{m h}^{-1} = \dfrac{30000}{3600}\text{m s}^{-1} = 8.33\,\text{m s}^{-1}$

On a velocity–time graph of $v\,(\text{m s}^{-1})$ against $t(\text{s})$, if the first speed bump is encountered at $t = 0$, then with maximum acceleration, the upward slope from $(0,\ 1.39)$ has gradient 1.4.

The rise to $v = 8.33$ takes $\dfrac{8.33\ -\ 1.39}{1.4} = 4.96\,\text{s}$.

If the second speed bump is at $t = T$, the downward gradient of the line towards $(T,\ 1.39)$ representing deceleration is -3.2.

The deceleration from $v = 8.33$ takes $\dfrac{8.33\ -\ 1.39}{3.2} = 2.17$.



Hence $T = 4.96\ +\ 2.17 = 7.13\,\text{s}$, so that the speed of $30\,\text{km h}^{-1}$ is not exceeded.

Distance travelled equals the area under the velocity–time graph.

$\text{Distance} = \dfrac{1}{2}\ \times\ T\ \times\ (8.33\ -\ 1.39)\ +\ T\ \times\ 1.39 = 34.7\,\text{m}\,.$

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1 **a** $a(t) = \dfrac{\mathrm{d}v}{\mathrm{d}t} = 6t - 8$ therefore $a(2) = 4\,\mathrm{m\,s^{-2}}$.

b $x(t) = \int v\mathrm{d}t = t^3 - 4t^2 + c$

$x(0) = c = 0$ since x measures displacement from the initial position

$x(t) = t^3 - 4t^2$

2 **a** $v > 0$ for $t \geqslant 1$

Distance travelled equals the area under the velocity–time curve.

$\int_1^5 3 - t^{-2}\mathrm{d}t = [3t + t^{-1}]_1^5 = (15 - 0.2) - (3 + 1) = 10.8\,\mathrm{m}$

b $\text{Average speed} = \dfrac{\text{total distance}}{\text{total time}} = \dfrac{10.8}{4} = 2.7\,\mathrm{m\,s^{-1}}$

3 **a** Acceleration is the gradient of the velocity–time graph.

$a = \dfrac{7}{5} = 1.4\,\mathrm{m\,s^{-2}}$

b Particle is at rest when $v = 0$.

The particle is at rest at $0\,\mathrm{s}$, $12\,\mathrm{s}$ and $15\,\mathrm{s}$.

c Total distance travelled equals the area enclosed by the velocity–time graph and the horizontal axis.

$\text{Total distance} = \dfrac{1}{2} \times 12 \times 7 + \dfrac{1}{2} \times 3 \times 2 = 45\,\mathrm{m}$

$\text{Average speed} = \dfrac{\text{total distance}}{\text{total time}} = \dfrac{45}{15} = 3\,\mathrm{m\,s^{-1}}$

4 **a** Distance travelled equals area enclosed between the velocity–time graph and the horizontal axis.

Area of triangle: $\dfrac{1}{2} \times 3 \times 20 = 30\,\mathrm{m}$

b Displacement at the end of the graph is zero (the lorry is back on the weighbridge) so the area below must equal the area above.

Area of trapezium is $30\,\mathrm{m}$; trapezium has parallel lengths T and 4, and height 3, so has area $\dfrac{3(T+4)}{2}$.

$\dfrac{3(T+4)}{2} = 30$ so $T + 4 = 20$ therefore $T = 16$

End of the journey is $60 + T = 76\,\mathrm{s}$ after the start.

c The 16-second period depicted by the trapezium consists of:

Acceleration $0.4\,\mathrm{m\,s^{-2}}$ from 0 to $3\,\mathrm{m\,s^{-1}}$: $\dfrac{3}{0.4} = 7.5\,\mathrm{s}$

Constant velocity $3\,\mathrm{m\,s^{-1}}$: $4\,\mathrm{s}$

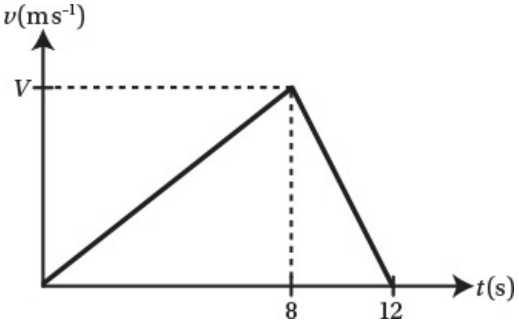
Deceleration a from 3 to $0\,\mathrm{m\,s^{-1}}$: $16 - 4 - 7.5 = 5.4\,\mathrm{s}$

$a = \dfrac{v-u}{t} = \dfrac{0-3}{5.4} = -0.667\,\mathrm{m\,s^{-2}}$

5 Total distance travelled equals the area enclosed by the velocity–time graph and the horizontal axis.

Let V be the maximum velocity reached.

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$$\text{Total distance} = \frac{1}{2} \times 12 \times V = 360\,\text{m}$$
$$V = 60\,\text{m s}^{-1}$$

Acceleration is the gradient of the velocity–time graph.

$$a = -\frac{60}{4} = -15\,\text{m s}^{-2}$$

6 **a** Conversion factor for km h^{-1} into m s^{-1} is $\frac{1000}{3600} = \frac{5}{18}$.

$$v = \frac{5}{18}\left(40 + 10t - \frac{1}{2}t^2\right) = \frac{100}{9} + \frac{25}{9}t - \frac{5}{36}t^2$$

b $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{25}{9} - \dfrac{5}{18}t = \dfrac{5}{18}(10 - t)$

Maximum velocity when $a = 0:\ t = 10\,\text{s}$.

c Distance $x(t) = \int v\,\mathrm{d}t = \dfrac{100}{9}t + \dfrac{25}{18}t^2 - \dfrac{5}{108}t^3 + c$

$$x(0) = 0 = c$$
$$x(t) = \frac{100}{9}t + \frac{25}{18}t^2 - \frac{5}{108}t^3$$
$$x(12) = 253\,\text{m}$$

d **i** Overtaking requires understanding of the dimensions of the vehicles involved and also the changing orientation of the car – simply considering the centre of each vehicle will not be sufficient, so a particle model is not suitable.

ii When considering travel time through a tunnel, the car can safely be modelled as a particle, since we are only interested in travel progress, not its orientation or dimensions (assuming it fits through the tunnel).

7 Suppose the bus takes T seconds to decelerate.

The velocity–time graph for the bus is a triangle, base length T and height 8.5.

Area equals total distance travelled $= \dfrac{1}{2} \times T \times 8.5 = \dfrac{17T}{4} = 44.2$

$$T = 44.2 \times \frac{4}{17} = 10.4\,\text{s}$$

The man runs $44.2\,\text{m}$ in $10.4\,\text{s}$ at a constant velocity, $V = \dfrac{44.2}{10.4} = 4.25\,\text{m s}^{-1}$

8 **a** $v = \dfrac{\mathrm{d}x}{\mathrm{d}t} = 0.04t^3 - 0.48t^2 + 1.44t$ so $v(2) = 0.32 - 1.92 + 2.88 = 1.28$

b $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = 0.12t^2 - 0.96t + 1.44$

$a = 0$ when $12t^2 - 96t + 144 = 0$

Dividing through by 12: $t^2 - 8t + 12 = 0$.

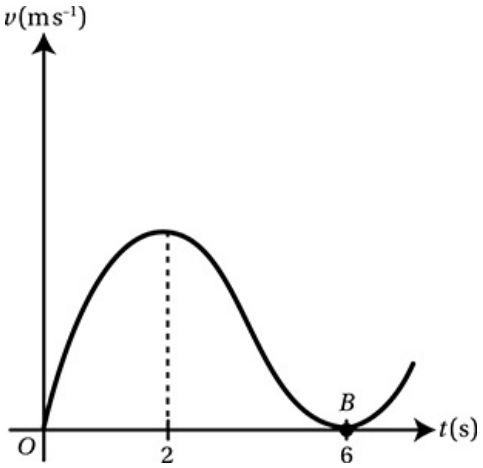
c Stationary value of v when $a = 0$

$(t - 2)(t - 6) = 0$ therefore $t = 2$ or $t = 6$.

v is a positive cubic, so the first stationary point is a local maximum and the second is a local minimum.

Minimum v when $t = 6$, $v = 0\,\mathrm{m\,s^{-1}}$.

d



The train has a positive velocity after leaving B so continues away from A.

e Distance AB is the displacement at $t = 6$: $x(6) = 4.32$ so AB is $4.32\,\mathrm{m}$.

9 $x(t) = \int v\mathrm{d}t = \begin{cases} 0.04t^4 - 0.06t^2 + 10.6t + c & \text{for} \quad 0 \leqslant t \leqslant 5 \\ 40t - t^2 + d & \text{for} \quad t > 5 \end{cases}$

Displacement is defined as distance from start point so $x(0) = 0 = c$

$x(5) = 0.04 \times 5^4 - 0.06 \times 5^2 + 10.6 \times 5 = 76.5$

Require continuity of position so $40 \times 5 - 5^2 + d = 76.5$

$d = -98.5$

$x(t) = \begin{cases} 0.04t^4 - 0.06t^2 + 10.6t & \text{for } 0 \leqslant t \leqslant 5 \\ 40t - t^2 - 98.5 & \text{for } t > 5 \end{cases}$

Require $x(t) = 200$

If $0 \leqslant t \leqslant 5$: $0.04t^4 - 0.06t^2 + 10.6t = 200$: no solutions in $0 \leqslant t \leqslant 5$

If $t > 5$: $40t - t^2 - 98.5 = 200$ so $t^2 - 40t + 298.5 = 0$

$t = \dfrac{40 \pm \sqrt{40^2 - 4 \times 298.5}}{2} = \dfrac{40 \pm 20.1}{2} = 9.93, 30.1$

10 a $v = \dfrac{1}{5}t^3 - 4t^2 + 20t$

Distance travelled in the first 10 seconds:

$\int_0^{10} v\mathrm{d}t = \left[\dfrac{1}{20}t^4 - \dfrac{4}{3}t^3 + 10t^2\right]_0^{10} = 167\,\mathrm{m}$

Remaining $133\,\mathrm{m}$ to be travelled after the car came to rest at $t = 10$.

Car travels for a further p seconds under acceleration $3.5\,\mathrm{m\,s^{-2}}$, so the area of the triangle up to the second traffic camera is $\dfrac{1}{2}p \times 3.5p = 133$

$\dfrac{7}{4}p^2 = \dfrac{400}{3}$

$p = \sqrt{\dfrac{1600}{21}} = 8.73\,\mathrm{s}$

The car takes $8.73\,\mathrm{s}$ to reach the second camera after starting from rest the second time.

b $90\,\mathrm{km\,h^{-1}} = 90 \times \dfrac{1000}{3600}\mathrm{m\,s^{-1}} = 25\,\mathrm{m\,s^{-1}}$

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When $0 \leqslant t \leqslant 10$, $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{3}{5}t^2 - 8t + 20$

Stationary point on velocity curve when $a = 0 : t = \dfrac{8 \pm \sqrt{8^2 - 4 \times \frac{3}{5} \times 20}}{2 \times \frac{3}{5}} = \dfrac{10}{3}, 10$

Maximum velocity when $0 \leqslant t \leqslant 10$ is $v\left(\dfrac{10}{3}\right) = 29.6\,\mathrm{m\,s^{-1}}$

The car clearly has speed exceeding the $90\,\mathrm{km\,h^{-1}}$ limit during the first 10 seconds.

After 8.73 seconds of $3.5\,\mathrm{m\,s^{-2}}$ acceleration, the car has velocity $8.73 \times 3.5 = 30.6\,\mathrm{m\,s^{-1}}$, which is even greater than the maximum in the first ten seconds, and so the car is again over the speed limit.

c Total distance travelled equals $300\,\mathrm{m}$.

For the cameras not to detect an infringement of the limit, the total time taken must be greater than $\dfrac{300}{25} = 12$ seconds.

The first period of travel took 10 seconds and the car then passed the second camera after $8.7\,\mathrm{s}$ of acceleration from rest, so T can equal zero, the car will still not be detected as having exceeded the limit.

11 $x = 0.8t^3 - 0.12t^4$ so $v(t) = \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2.4t^2 - 0.48t^3$

$a(t) = \dfrac{\mathrm{d}v}{\mathrm{d}t} = 4.8t - 1.44t^2 = 4.8t(1 - 3t)$

Stationary value of v occurs when $a = 0$; $a = 0$ when $t = 0, 3$.

Check stationary points and endpoints of the interval:

$v(0) = 0$, $v(3) = 8.64\,\mathrm{m\,s^{-1}}$, $v(6) = -17.3\,\mathrm{m\,s^{-1}}$

So although the maximum *velocity* is $8.64\,\mathrm{m\,s^{-1}}$, the maximum *speed* in the first 6 seconds is $17.3\,\mathrm{m\,s^{-1}}$.

Chapter 20 worked solutions

20 Motion with constant acceleration

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 20A

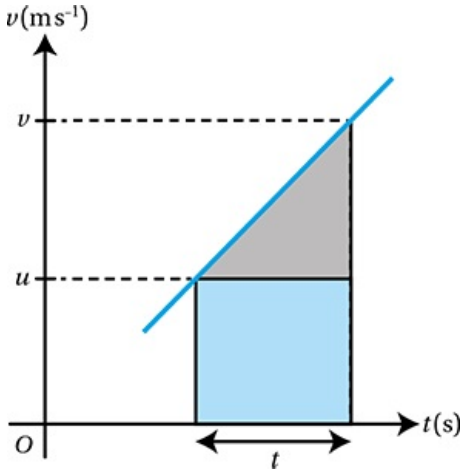
1 Substituting $v = u + at$:

$s = \frac{1}{2}(u + at + u)t = ut + \frac{1}{2}at^2$

2 **a** Constant acceleration is represented by the gradient of the line on the velocity–time graph

$a = \frac{\text{change in velocity}}{\text{time taken}} = \frac{v - u}{t}$

b Area under the graph represents change in displacement.



Area of the rectangle is ut

Area of the triangle is $\frac{1}{2}t(v - u) = \frac{1}{2}\frac{v - u}{t} \times t^2 = \frac{1}{2}at^2$

Total area, distance travelled during time t is $ut + \frac{1}{2}at^2$

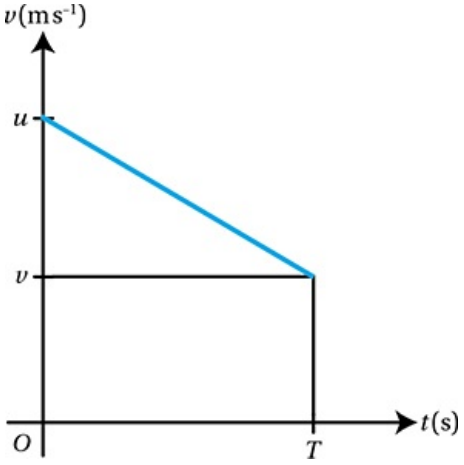
3 **a** Substituting $u = v - at$ into $s = ut + \frac{1}{2}at^2$:

$s = (v - at)t + \frac{1}{2}at^2 = vt - at^2 + \frac{1}{2}at^2 = vt - \frac{1}{2}at^2$

b $s = 300$, $t = 8$, $a = 3.1$, seeking v: use $s = vt - \frac{1}{2}at^2$

$300 = 8v - 32 \times 3.1$ therefore $v = \frac{300 + 99.2}{8} = 49.9\,\text{m s}^{-1}$

4 **a**



b Distance travelled equals area under the graph.

Area of a trapezium equals base length (T) multiplied by mean height $\frac{u+v}{2}$.

$\text{distance} = \frac{(u+v)T}{2}$

5 **a** $v = \int \text{adt} = \frac{1}{2}kt^2 + c$

When $\text{t} = 0$, $\text{v} = \text{u}$, so $\text{c} = \text{u}$. Hence, $v = \frac{1}{2}kt^2 + u$.

b **i** $s = \int v\,\text{dt} = \int u + \frac{1}{2}kt^2\,\text{dt} = ut + \frac{1}{6}kt^3 + c$

Since s is defined as displacement from initial position, $s(0) = 0 = c$

So $s = ut + \frac{1}{6}kt^3$

ii $v = u + \frac{1}{2}kt^2$ so $\frac{1}{6}kt^3 = \frac{1}{3}(v-u)t$

Substituting into the answer to part bi: $s = ut + \frac{1}{3}(v-u)t = \frac{1}{3}(2u+v)t$

iii From the working in part bii:

$kt^2 = 2(v-u)$ and $3s = (2u+v)t$

So $2(v-u)(2u+v)^2 = kt^2 \times \left(\frac{3s}{t}\right)^2 = 9ks^2$

6 Substitute $t = \frac{(v-u)}{a}$ into $s = ut + \frac{1}{2}at^2$:

$$s = \frac{u(v-u)}{a} + \frac{a}{2} \times \left(\frac{(v-u)}{a}\right)^2 = \frac{u(v-u)}{a} + \frac{(v-u)^2}{2a}$$

$$2as = 2u(v-u) + (v-u)^2 = (v-u)(2u+v-u) = (v-u)(v+u) = v^2 - u^2$$

So $v^2 = u^2 + 2as$

7 **a** $v_1(T) = u + aT$

b Require that $v_1(t) = v_2(t)$

$u + at = u + at^2$ therefore $t = 0$ or 1

The particles have the same velocity at 0 or 1 second.

c $s_1(t) = \int v_1\,\text{dt} = ut + \frac{1}{2}at^2$, $s_2(t) = \int v_2\,\text{dt} = ut + \frac{1}{3}at^3$

(The constant of integration is zero in both cases, since initial displacement is by definition zero.)

$uT + \frac{1}{2}aT^2 = uT + \frac{1}{3}aT^3$ so $3T^2 = 2T^3$ therefore $T = 0$ so $1.5\,\text{s}$

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8 **a** Displacement equals the area above the horizontal axis less the area below the axis.

The area above the axis is a triangle with sides z and u and the area below the axis is a triangle with sides $(t-z)$ and v so:

$s = \frac{1}{2}uz - \frac{1}{2}(t-z)v$

b Since the two triangles are similar,

$\frac{z}{u} = \frac{t-z}{v}$ and therefore $zv = ut - uz$

Rearranging: $z = \frac{ut}{u+v}$

c Substituting into the answer to part **a**:

$s = \frac{1}{2}(u+v)z - \frac{1}{2}vt = \frac{1}{2}ut - \frac{1}{2}vt = \frac{1}{2}(u-v)t$

EXERCISE 20B

2 $u = 0,\ v = 12.45,\ t = 6.5$

a $a = \frac{(v-u)}{t} = \frac{12.45}{6.5} = 1.92\,\text{m s}^{-2}$

b $s = \frac{(u+v)t}{2} = \frac{12.45 \times 6.5}{2} = 40.5\,\text{m}$

3 $u = 4.2,\ a = 0.3,\ t = 7$

a $v = u + at = 4.2 + 0.3 \times 7 = 6.3\,\text{m s}^{-1}$

b $s = ut + \frac{1}{2}at^2 = 4.2 \times 7 + \frac{1}{2}0.3 \times 7^2 = 36.75\,\text{m}$

4 $u = 16.3,\ v = 7.5,\ s = 120$

a $a = \frac{(v^2 - u^2)}{2s} = \frac{(7.5^2 - 16.3^2)}{2 \times 120} = -0.873\,\text{m s}^{-2}$

b $t = \frac{2s}{(u+v)} = \frac{2 \times 120}{16.3 + 7.5} = 10.08\,\text{s}$

Running for a further $5\,\text{s}$: $u = 16.3,\ a = -0.873,\ t = 15.08$

$s = ut + \frac{1}{2}at^2 = 16.3 \times 15.08 - \frac{1}{2} \times (-0.873) \times 15.08^2 = 146.6\,\text{m}$

The particle would travel a further $26.6\,\text{m}$.

5 $u = 0,\ a = 2.5,\ s = 250$

$s = ut + \frac{1}{2}at^2$ so $250 = 1.25t^2$ therefore $t = \sqrt{\frac{250}{1.25}} = 14.1\,\text{s}$

6 $u = 20,\ v = 8.2,\ s = 100$

$a = \frac{(v^2 - u^2)}{2s} = \frac{(8.2^2 - 20^2)}{2 \times 100} = -1.66\,\text{m s}^{-2}$

$t = \frac{2s}{(u+v)} = \frac{2 \times 100}{20 + 8.2} = 7.09\,\text{s}$

Running for a further $20\,\text{m}$: $u = 20,\ a = -1.66,\ s = 120$

$s = ut + \frac{1}{2}at^2$ so $120 = 20t + \frac{1}{2} \times (-1.66)t^2$ therefore $-0.832t^2 + 20t - 120 = 0$

$t = \frac{-20 \pm \sqrt{20^2 - 4 \times 0.832 \times 120}}{-2 \times 0.832} = 11.5\,\text{s}$ or $12.5\,\text{s}$

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The first value represents the time at which the particle first passes the additional $20\,\mathrm{m}$ mark, the second value is when the particle returns through the mark.

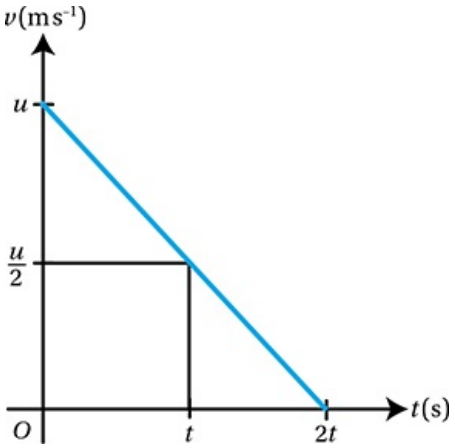
The additional $20\,\mathrm{m}$ takes $4.43\,\mathrm{s}$

7 $a = -3.6,\ s = 350,\ v = \dfrac{u}{2}$ so $t = \dfrac{(v-u)}{a} = \dfrac{u}{7.2}$ therefore $u = 7.2t,\ v = 3.6t$

$s = \dfrac{(u+v)t}{2}$ so $350 = \dfrac{10.8t^2}{2}$ therefore $t = \sqrt{\dfrac{700}{10.8}} = 8.05\,\mathrm{s}$

Tip

Perhaps simpler is to look at the shape of the velocity–time graph.



The time for the velocity to halve would be half the time for it to decrease to zero. However, the area enclosed is three-quarters of the area of the entire triangle, clear if one considers the similar shapes involved. So the particle would cover $\dfrac{4}{3} \times 350\,\mathrm{m}$ when coming to a stop, which would equal the area of the triangle with height u and base length $2t$: $ut = \dfrac{4}{3} \times 350$.

But the gradient of the slope is $-\dfrac{u}{2t} = -3.6$ so $u = 7.2t$

Combining gives $7.2t^2 = \dfrac{4}{3} \times 350$ so $t = \sqrt{\dfrac{4 \times 350}{3 \times 7.2}} = 8.05\,\mathrm{s}$

8 $u = 18,\ v = 9,\ s = 200$

$a = \dfrac{(v^2 - u^2)}{2s} = \dfrac{(9^2 - 18^2)}{400} = -0.6075$

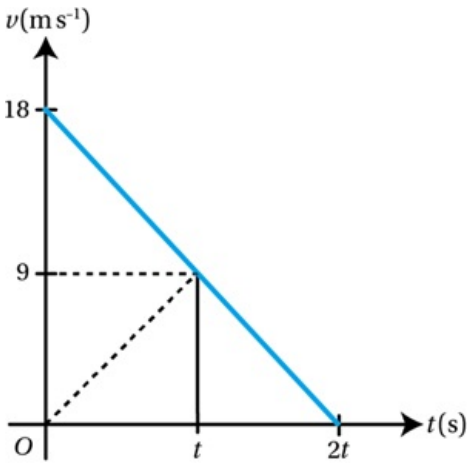
Calculating again, with the same start velocity and acceleration but end velocity zero:
$u = 18,\ v = 0,\ a = -0.6075$

$s = \dfrac{(v^2 - u^2)}{2a} = \dfrac{-18^2}{2 \times (-0.6075)} = 267\,\mathrm{m}$

The particle travels a further $66.7\,\mathrm{m}$.

Tip

Alternatively, much as for question 7, consider the velocity–time graph; the area under the line as it descends from velocity u to $\dfrac{u}{2}$ must be three time the area of the continuation to zero velocity.



The particle travels a further $66.7\,\mathrm{m}$

9 **a** Particle turns when $v = 0$.

Constant acceleration problem: $a = -3.4$, $v = 0$, $u = 6$. Find s.

$$s = \frac{v^2 - u^2}{2a} = \frac{-36}{-6.8} = 5.29\,\mathrm{m}$$

b Distance is the modulus of displacement.

After the particle turns it will move with increasing speed back towards the starting point and beyond it, so that the particle will eventually move further behind the start point than its maximum distance in front of the start point.

EXERCISE 20C

Tip

As advised in the textbook, it is generally accepted practice to take the dominant direction as being that of the initial velocity so that $u \geqslant 0$. However, it is always wise to explicitly state your dominant direction at the start of your working, so that both you and the reader can clearly interpret the working.

3 Taking the dominant direction as downward: $u = 0$, $a = 9.8$, $s = 25$. Find v.

$v^2 = u^2 + 2as = 2 \times 9.8 \times 25 = 490$ so $v = 22.1\,\mathrm{m\,s^{-1}}$

4 Taking the dominant direction as upward, $u = 18.5$, $a = -9.8$, $s = 15$. Find t and v.

a $s = ut + \frac{1}{2}at^2$ so $15 = 18.5t - 4.9t^2$ therefore $4.9t^2 - 18.5t + 15 = 0$

$$t = \frac{18.5 \pm \sqrt{18.5^2 - 4 \times 4.9 \times 15}}{2 \times 4.9} = 1.18 \text{ or } 2.60$$

It takes 1.18 seconds for the ball to reach a height of $15\,\mathrm{m}$.

Tip

The second time would represent the ball, having reached its maximum height, passing back through the height of $15\,\mathrm{m}$. The peak, should we need it, would occur midway between these times.

b $v = \sqrt{u^2 + 2as} = \sqrt{18.5^2 - 2 \times 9.8 \times 15} = 6.95\,\mathrm{m\,s^{-1}}$

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Tip

The negative root would of course be valid for the downward leg of the journey at that height.

c Air resistance would increase the downward acceleration; the ball would take longer to reach the given height (if it would even get that high) and the speed at that point would be lower.

5 Taking the dominant direction as upward, $s = 5.6$, $v = 3.5$, $a = -9.8$. Find u.

$u = \sqrt{v^2 - 2as} = \sqrt{3.5^2 + 2 \times 9.8 \times 5.6} = 11.0\,\text{m s}^{-1}$

6 Taking the dominant direction as upward, $u = 28$, $a = -9.8$.

a When $v = 0$, find s.

$s = \dfrac{(v^2 - u^2)}{2a} = \dfrac{-28^2}{-2 \times 9.8} = 40\,\text{m}$

The ball has risen $40\,\text{m}$ from its initial position and so is $55\,\text{m}$ above ground.

b When $s = -15$, find t and v.

$v = -\sqrt{u^2 + 2as} = -\sqrt{28^2 + 2 \times 9.8 \times 15} = -32.8\,\text{m s}^{-1}$

Select negative root for the velocity as it returns to the ground; the positive root would represent the necessary velocity at ground level for the object to have passed a height $15\,\text{m}$ above ground level at velocity u.

$t = \dfrac{(v - u)}{a} = \dfrac{-32.8 - 28}{-9.8} = 6.21\,\text{s}$

The ball hits the ground $6.21\,\text{s}$ after being projected, travelling downwards at a speed $32.8\,\text{m s}^{-1}$.

7 **a** Taking the dominant direction as upward, $u = 6.5$, $a = -9.8$, $s = 8.3$. Determine if there is a solution for v in this situation.

$v = \sqrt{u^2 + 2as} = \sqrt{6.5^2 - 2 \times 9.8 \times 8.3} = \sqrt{-120}$

There is no solution for v, which means that the flowers do not reach the window.

b Set $v = 0$ for the same acceleration and displacement and find u, the initial velocity for which the flowers just reach the window.

$u = \sqrt{v^2 - 2as} = \sqrt{0 + 2 \times 9.8 \times 8.3} = 12.8\,\text{ms}^{-1}$

If the flowers are thrown upwards at an initial speed of at least $12.8\,\text{m s}^{-1}$, they will reach the window.

c With air resistance, the downward acceleration would be greater, so the value for question 7b would increase.

8 Taking the dominant direction as upward, $u = 16$, $a = -9.8$.

a Find the value of t for which $v = 0$: $t = \dfrac{(v - u)}{a} = \dfrac{-16}{-9.8} = 1.63\,\text{s}$

b Find the value of t for which $s = -12$: $s = ut + \dfrac{1}{2}at^2$ so $-12 = 16t - 4.9t^2$

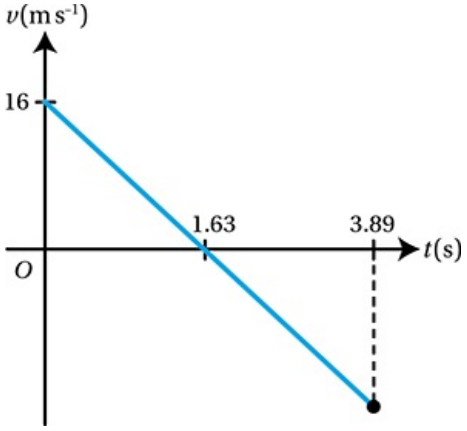
$4.9t^2 - 16t - 12 = 0$ therefore $t = \dfrac{16 + \sqrt{16^2 + 4 \times 4.9 \times 12}}{2 \times 4.9} = 3.89\,\text{s}$

Tip

Note that the other solution to the quadratic, which gives a time $-0.629\,\text{s}$, represents the time the particle would have to be projected from ground level to reach the window height at $t = 0$.

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c



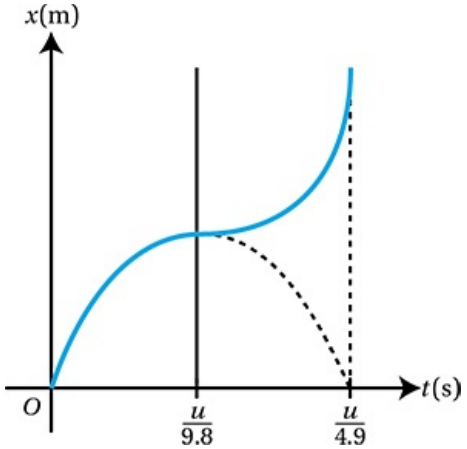
9 **a** Because you know that the downward speed for a particular height will equal the upward speed at that height (due to symmetry in the displacement–time graph), if the particle is launched from ground level with upward velocity u, it will return with downward velocity $-u$.

$$t = \frac{(v-u)}{a} = \frac{-u-u}{-9.8} = \frac{u}{4.9}\,\text{s}$$

b $s = ut + \frac{1}{2}at^2 = ut - 4.9t^2$

Displacement follows a negative quadratic with roots at $t = 0$ and $\frac{u}{4.9}$ s and a maximum at $\frac{u}{9.8}$ s.

Distance travelled will mimic the upward part of this, and then will flip the downward track of the displacement curve to continue rising.



10 Taking the dominant direction as upward, $u = 8$, $a = -9.8$, $v = -18$. Find s:

$$s = \frac{(v^2-u^2)}{2a} = \frac{(18^2-8^2)}{-2\times 9.8} = -13.3\,\text{m}$$

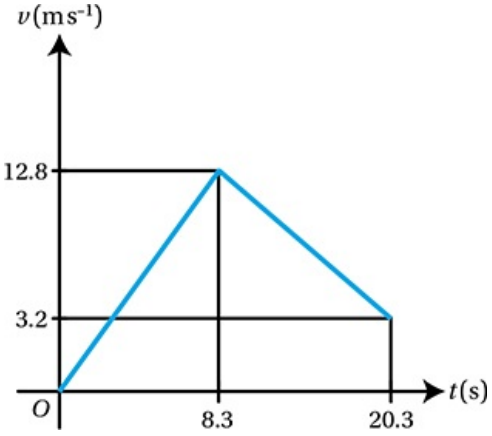
The stone hits the ground $13.3\,\text{m}$ below the height of the cliff, so the cliff rises $13.3\,\text{m}$ above ground level.

EXERCISE 20D

1 End speed $v = u + at = 12.8 - 0.8\times 12 = 3.2\,\text{m s}^{-1}$

Velocity–time graph is as shown below.

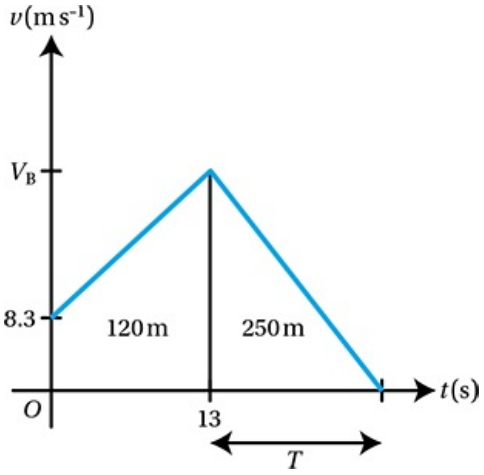
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Total distance travelled equals area under the graph

$= \frac{1}{2} \times 8.3 \times 12.8 + \frac{(3.2 + 12.8)}{2} \times 12 = 149.12\,\text{m} \approx 149\,\text{m}$

2 Velocity–time graph is as shown.



a Using the fact that distance travelled equals the area under the graph:

$120 = 13 \times \frac{(8.3 + v_B)}{2}$ therefore $v_B = \frac{2 \times 120}{13} - 8.3 = 10.2\,\text{m s}^{-1}$.

b For the second part of the journey: $250 = \frac{1}{2} \times v_B \times T$ so $T = \frac{500}{v_B}$

The gradient of the line gives the (negative) acceleration.

$a = -\frac{v_B}{T} = -\frac{{v_B}^2}{500} = -0.207\,\text{ms}^{-2}$

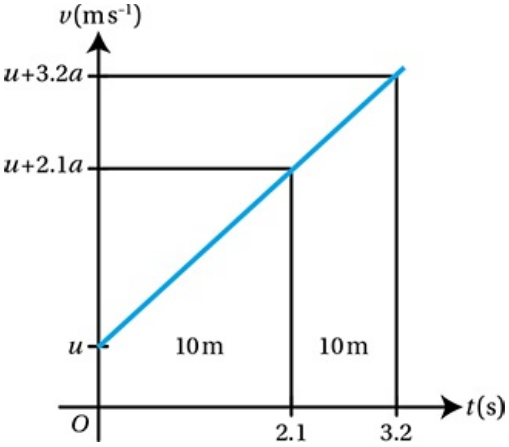
The deceleration of the fox is $0.207\,\text{m s}^{-2}$.

Tip

Notice that the duration of the second part of the journey is used in the calculation but does not actually need to be evaluated to reach the end answer. You can often save time keeping simple expressions in your working and evaluating the final answer in one go.

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3



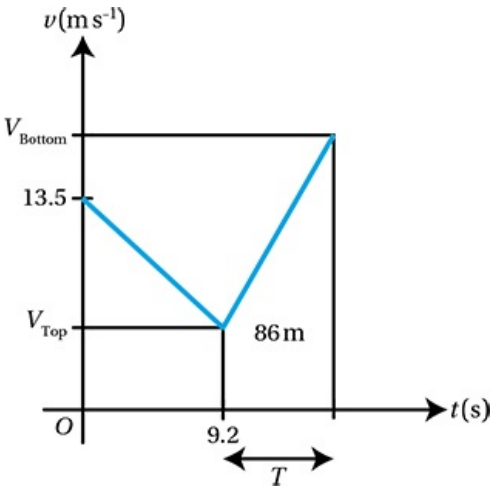
Area under the velocity–time graph is equal to distance travelled.

$$\begin{cases} 10 = \dfrac{(u + u + 2.1a)}{2} \times 2.1 \quad (1) \\ 10 = \dfrac{(u + 2.1a + u + 3.2a)}{2} \times 1.1 \ (2) \end{cases}$$

$$\begin{cases} 20 = 2.1(2u + 2.1a) = 4.2u + 4.41a \ (1) \\ 20 = 1.1(2u + 5.3a) = 2.2u + 5.83a \ (2) \end{cases}$$

$5.83(1) - 4.41(2) : 28.4 = 14.784u$ therefore $u = 1.92\,\text{ms}^{-1}$

4



$v_{\text{Top}} = 13.5 - 0.9 \times 9.2 = 5.22\,\text{m s}^{-1}$

$v_{\text{Bottom}} = v_{\text{Top}} + 1.6T$

Using the fact that area under the graph equals distance covered, for the downhill journey:

$$86 = \frac{(v_{\text{Top}} + v_{\text{Bottom}})}{2} \times T \text{ so } 172 = (2v_{\text{Top}} + 1.6T)T$$

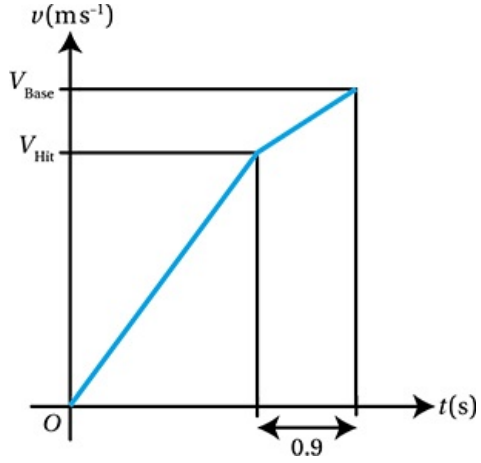
$$1.6T^2 + 10.44T - 172 = 0$$

In this context, only the positive value of T is needed.

$$T = \frac{-10.44 + \sqrt{10.44^2 + 4 \times 1.6 \times 172}}{2 \times 1.6} = 7.61\,\text{s}$$

$v_{\text{Bottom}} = v_{\text{Top}} + 1.6T = 17.4\,\text{m s}^{-1}$

5



Taking the dominant direction as downward; for the first part of the movement, $a = 9.8$, $u = 0$

Ball hits the water at speed v_{Hit} and strikes the bottom of the well at speed v_{Base}.

$v_{\text{Hit}} = \sqrt{u^2 + 2as} = \sqrt{0^2 + 2 \times 9.8 \times 2.6} = 7.14\,\text{m s}^{-1}$

For the second part of the movement, $a = 1.2$, $u = v_{\text{Hit}}$, $v = v_{\text{Base}}$, $t = 0.9$.

$s = ut + \frac{1}{2}at^2 = 7.14 \times 0.9 + \frac{1}{2} \times 1.2 \times 0.9^2 = 6.91\,\text{m}$

The water in the well is $6.91\,\text{m}$ deep.

6 There are different units in the data of this question; adjust lengths to metres, times to seconds and speeds to m s^{-1}.

Taking the dominant direction as upward, for the first part of the movement.

$a = -9.8$, $u = 260 \times \frac{1000}{3600} = 72.2$, $t = 6$. Find s and v:

$s = ut + \frac{1}{2}at^2 = 72.2 \times 6 - 4.9 \times 6^2 = 256.9\,\text{m}$

In the second part of the movement, you start a new projectile model and look for it to make up the remaining part of the upward distance.

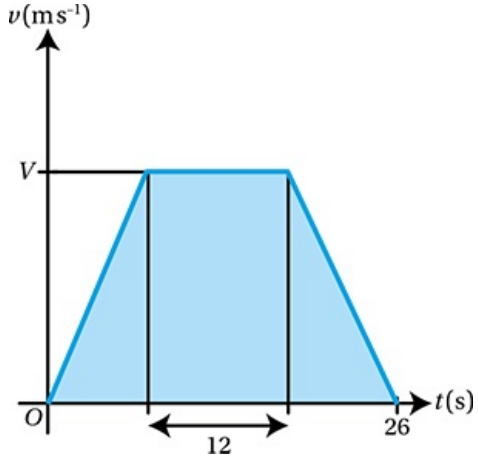
$a = 2.8$, $u = 13.4$, $s = 143.1$. Find t: $s = ut + \frac{1}{2}at^2$

$143.1 = 13.4t + 1.4t^2$ therefore $1.4t^2 + 13.4t - 143.1 = 0$

$t = \frac{-13.4 + \sqrt{13.4^2 + 4 \times 1.4 \times 143.1}}{2 \times 1.4} = 6.39\,\text{s}$

Total time taken to reach $400\,\text{m}$ is $6 + 6.39 = 12.4\,\text{s}$.

7



Use the fact that area enclosed by the velocity–time graph is distance travelled. If you take out the rectangle and push the two triangles together, there is one rectangle $12\,\text{s}$ long and $V\,\text{m s}^{-1}$ high, and one triangle with base length $14\,\text{s}$ and height $V\,\text{m s}^{-1}$.

$12V + \frac{1}{2} \times 14 \times V = 840$ so $19V = 840$ therefore $V = 44.2\,\text{m s}^{-1}$

8 Taking the dominant direction as upward; for the first part of the movement: $a = 5.2$, $u = 0$, $s = 25$. Find t and v:

$v = \sqrt{u^2 + 2as} = \sqrt{0 + 10.4 \times 25} = 16.1\,\text{m s}^{-1}$

$t = \dfrac{(v - u)}{a} = \dfrac{16.1}{5.2} = 3.10\,\text{s}$

For the second part of the movement, $a = -9.8$, $u = 16.1$. Find s when $v = 0$ and find v when $s = -25$:

$s = \dfrac{(v^2 - u^2)}{2a} = \dfrac{-16.1^2}{-2 \times 9.8} = 13.3\,\text{m}$

The maximum height of the rocket is $25 + 13.3 = 38.3\,\text{m}$.

$v = -\sqrt{u^2 + 2as} = -\sqrt{16.1^2 + 2 \times 9.8 \times 25} = -27.4\,\text{m s}^{-1}$

The rocket hits the ground at a speed of $27.4\,\text{m s}^{-1}$.

9 Measure all heights from ground level.

The dropped ball (ball 1) has initial displacement $s_1(0) = 30$, $u_1 = 0$, $a = -9.8$

$s_1 = 30 + u_1 t + \frac{1}{2}at^2 = 30 - 4.9t^2$

Balls collide at $t = T$, $s_1 = 15$

$4.9T^2 = 30 - 15$ therefore $T = \sqrt{\dfrac{15}{4.9}} = 1.75\,\text{s}$

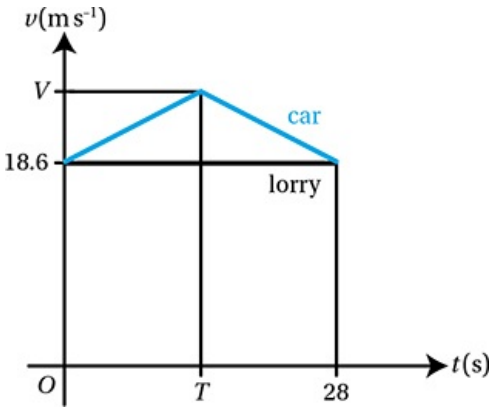
The projected ball has $a = -9.8$, $s_2 = 15$, $t = T$. Find u_2:

$u_2 = \dfrac{s_2 - \frac{1}{2}at^2}{t} = \dfrac{15 + 4.9 \times T^2}{T} = 17.1\,\text{m s}^{-1}$

10 a The car moves (relative to the back of the lorry) from a position at $-35\,\text{m}$ to $40 + 15.2 = 55.2\,\text{m}$ in front of the back of the lorry.

In total, this is an additional $90.2\,\text{m}$.

b The velocity–time graph would look like this (note that you have no information on the time T when the car reaches V; it need not be in the midpoint of the overtaking period if the acceleration and deceleration are of different magnitudes).



Area under a velocity–time graph indicates distance travelled, so the area of the upper triangle must equal the additional $90.2\,\text{m}$ travelled by the motorbike.

$\frac{1}{2} \times 28 \times (V - 18.6) = 90.2$ therefore $V = 18.6 + \dfrac{90.2}{14} = 25.0\,\text{m s}^{-1}$

MIXED PRACTICE 20

1 $u = 12$, $a = 1.6$, $v = 26$. Find s and t:

a $s = \dfrac{(v^2 - u^2)}{2a} = \dfrac{(26^2 - 12^2)}{2 \times 1.6} = 166.25\,\text{m}$

b $t = \dfrac{(v - u)}{a} = \dfrac{26 - 12}{1.6} = 8.75\,\text{s}$

2 Taking the dominant direction as upward, $u = 11$, $a = -9.8$, $t = 2$. Find v and s:

a $v = u + at = 11 - 9.8 \times 2 = -8.6\,\text{m s}^{-1}$

After $2\,\text{s}$, the speed is $8.6\,\text{m s}^{-1}$ and the stone is travelling downwards.

b $s = ut + \dfrac{1}{2}at^2 = 11 \times 2 - 4.9 \times 2^2 = 2.4\,\text{m}$

The stone is $2.4\,\text{m}$ above the ground at this time.

3 $u = 6.2$, $a = -2.1$, $s = 8$. Find v: $v = \sqrt{u^2 + 2as} = \sqrt{6.2^2 - 4.2 \times 8} = 2.2\,\text{m s}^{-1}$

4 **a** The initial velocity is upwards and the acceleration is downwards. Use constant acceleration formulae with $u = 7$, $g = -9.8$ and $s = 2.1$:

$v^2 = 7^2 - 2 \times 9.8 \times 2.1$ therefore $v = 2.8\,\text{ms}^{-1}$

b The object reaches the greatest height when $v = 0$:

$0^2 = 7^2 - 2 \times 9.8s$ therefore $s = 2.5\,\text{m}$

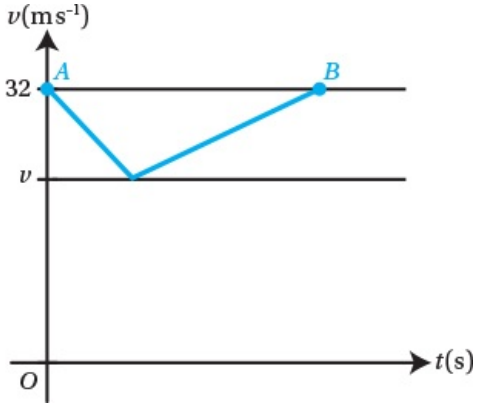
c The velocity is $v = -5.7$ (as it is downwards).

Using $v = u + at$: $-5.7 = 7 - 9.8t$ therefore $t = 1.30\,\text{s}$

5 Make units consistent: all lengths in kilometres, time in hours and speeds in kilometres per hour:

$1.2\,\text{minutes} = \dfrac{1.2}{60} = 0.02\,\text{hours}$

a



b Area under the graph equals distance travelled.

$0.02 \times \dfrac{32 + v}{2} = 0.55$ therefore $v = 55 - 32 = 23\,\text{km h}^{-1}$

c $\text{Average speed} = \dfrac{\text{total distance}}{\text{total time}} = \dfrac{0.55}{0.02} = 27.5\,\text{km h}^{-1}$

Tip

This is of course also the average of 32 and v.

6 **a** Substituting $t = \dfrac{(v - u)}{a}$ into $s = ut + \dfrac{1}{2}at^2$:

$s = \dfrac{u(v - u)}{a} + \dfrac{a(v - u)^2}{2a^2} = \dfrac{2(uv - u^2) + v^2 - 2uv + u^2}{2a}$

$2as = v^2 - u^2$ therefore $v^2 = u^2 + 2as$

b Taking the dominant direction as downward, $u = 8.5$, $v = 110$, $a = 9.8$. Find s:

$$s = \frac{(v^2 - u^2)}{2a} = \frac{110^2 - 8.5^2}{2 \times 9.8} = 614\,\text{m}$$

7 For both particles, take the dominant direction to be upward and s to be the upward displacement from ground level.

Then $u_1 = 15.4$, $a = -9.8$, $s_1 = u_1 t + \frac{1}{2}at^2$ and $u_2 = -15.4$, $a = -9.8$, $s_2 = 20 + u_2 t + \frac{1}{2}at^2$

They collide when $s_1 = s_2$: $15.4t - 4.9t^2 = 20 - 15.4t - 4.9t^2$ so $30.8t = 20$

$t = \frac{20}{30.8} = 0.649\,\text{s}$

a At $t = 0.649\,\text{s}$, $s_1 = s_2 = 7.93\,\text{m}$

b At $t = 0.649\,\text{s}$:

$v_1 = u_1 + at = 15.4 - 9.8 \times 0.649 = 9.04\,\text{ms}^{-1}$ so the first particle is moving upwards with speed $9.04\,\text{m s}^{-1}$.

$v_2 = u_2 + at = -15.4 - 9.8 \times 0.649 = -21.8\,\text{m s}^{-1}$ so the second particle is moving downwards with speed $21.8\,\text{m s}^{-1}$.

8 $v = \int a\,\text{d}t = at + c$

$v(0) = c = u$ so $v = u + at$

$s = \int v\,\text{d}t = ut + \frac{1}{2}at^2 + d$

$s(0) = d = 0$, since the displacement is measured from the particle's position at $t = 0$.

So $s = ut + \frac{1}{2}at^2$ as required.

9 The greatest height is reached when $v = 0$. You can use $v^2 = u^2 + 2as$ with u positive and a negative.

$0 = 8.42 - 2 \times 9.8s$ so $s = 3.6\,\text{m}$

a You are again looking for when $v = 0$, but now $s = 3.6 - 2 = 1.6$.

$0 = u^2 - 2 \times 9.8 \times 1.6$ so $u = 5.6\,\text{m s}^{-1}$

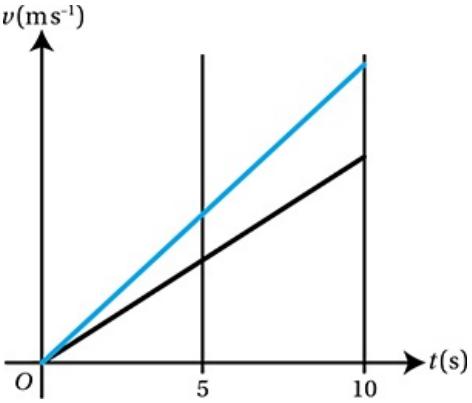
b When the particles are at the same height, the displacement of P from its starting point is $2\,\text{m}$ greater than the displacement of Q from its starting point:

$5.6t - \frac{1}{2}gt^2 + 2 = 8.4t - \frac{1}{2}gt^2 \Rightarrow t = \frac{5}{7}$

Then $v_P = 8.4 - gt = 1.4$ and $v_Q = 5.6 - gt = -1.4$

Hence they have the same speed, P is moving up and Q is moving down.

10



Area under the velocity–time graph equals distance travelled.

Using the geometry of similar triangles, if the area between the lines is $30\,\text{m}$ for the first 5 seconds it will be four times that amount for the first 10 seconds.

The first car will be $120\,\mathrm{m}$ in front.

Tip

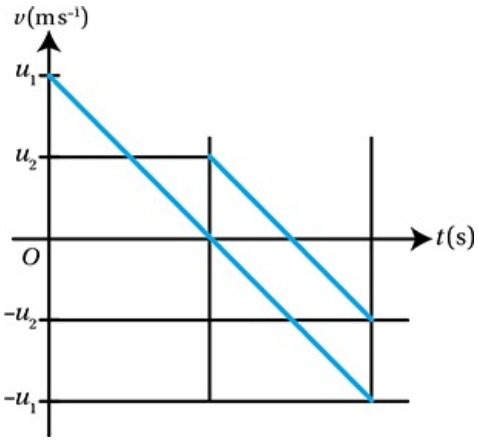
If you are uncertain about using rules of geometry in your reasoning, you can allocate unknowns; say the first car gets to speed a after 5 seconds and $2\mathrm{a}$ after 10 seconds, and the second car gets to speed b after 5 seconds and $2\mathrm{b}$ after 10 seconds.

Then using the fact that the area represents distances, $30 = \frac{1}{2} \times 5a - \frac{1}{2} \times 5b = \frac{5}{2}(a-b)$

And the distance between the cars at the end of 10 seconds is

$\frac{1}{2} \times 10 \times 2a - \frac{1}{2} \times 10 \times 2b = 10(a-b) = 4 \times \frac{5}{2}(a-b) = 120$

11 It can be very quickly seen from the velocity–time graph that the ratio $u_1 : u_2 = 2 : 1$.



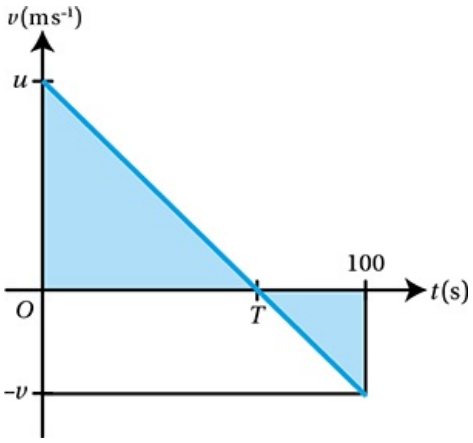
The gradient for each line is the same, since the acceleration for both particles is g.

Both lines (since movement up will be symmetrical with movement down) will end with the same speed as they begin, but with the opposite sign.

The first particle travels for twice the time experienced by the second particle, so must start and end with twice the speed.

12 $-v = u - 100 \times 2$ so $v = 200 - u$

On the velocity–time graph, total distance travelled equals the area enclosed by the curve and the horizontal axis, as shaded below.



The particle is at rest at time $T = \frac{u}{2}$.

So total distance $= \frac{1}{2}u \times T + \frac{1}{2}v \times (100 - T)$, substituting $v = 200 - u$ and $T = \frac{u}{2}$:

$\text{total distance} = \dfrac{uT + (100 - T)(200 - u)}{2} = uT - 100T - 50u + 10000 = \dfrac{u^2}{2}$
$- 100u + 10000$

$\text{Average speed} = \dfrac{\text{total distance}}{\text{total time}} = \dfrac{\dfrac{u^2}{2} - 100u + 10000}{100} = \dfrac{u^2}{200} - u + 100 = 62.5$

$u^2 - 200u + 7500 = 0$ so $(u - 150)(u - 50) = 0$

Since you require that $u > v > 0$, reject the solution $u = 50$, $v = 150$ and find that $u = 150$, $v = 50$.

Chapter 21 worked solutions

21 Force and motion

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 21A

IRANINOVIN

4 **a** Going round a bend there is a moment-to-moment change of direction, and the effective force is towards the centre of the bend. This is why as a passenger, you may feel yourself inclined to move away from the centre of the bend but pushed towards that point by the vehicle. (Remember Newton's third law—it feels as though you are pushing away from the centre, but in fact you are experiencing the vehicle pushing you towards the centre.)

b There is no known location in the Universe not affected, even if only minutely, by some forces of gravity. To a reasonable approximation, an object moving through space beyond orbit will have only small gravitational forces acting upon it and offers a good model for the first law.

c While friction and air resistance will play an important part in the long-term movement of an object, the approximation is reasonable for short-term projections if the system is reasonably smooth. A shiny ball dropping through the air will encounter air resistance, but it will have very little effect on the ball's movement. A feather on the other hand, which has a non-compact shape, will have its movement considerably affected by air resistance; ultimately, whether or not friction and air resistance can be reasonably ignored in an approximation depends on the realities of the system being modelled.

5 Convert all units to kg, m, s

Mass $900\,\text{kg}$

$\text{V}(3.5) =$ (assumed constant, since proportional to driving force) $\dfrac{15000}{3600}\,\text{m s}^{-1}$

Change in $3.5s = 1.19\,\text{ms}^{-2}$ therefore $F = ma = 900 \times 1.19 = 1070\,\text{N}$.

6 $F = ma = 0.12 \times \left(\dfrac{-3.2}{8}\right) = -0.048$

Friction force is $0.048\,\text{N}$ acting against the direction of movement.

7 $a = \dfrac{F}{m} = \dfrac{260}{28} = 9.29\,\text{m s}^{-2}$

Constant acceleration problem: $a = 9.29$, $u = 0$, $v = 2.5$. Find t.

$t = \dfrac{v-u}{a} = \dfrac{2.5}{9.29} = 0.269\,\text{s}$

8 $\mathbf{F} = m\mathbf{a} = 1.8\begin{pmatrix}0.6\\1.1\end{pmatrix} = \begin{pmatrix}1.08\\1.98\end{pmatrix}\,\text{N}$

9 **a** $\mathbf{a} = \dfrac{1}{m}\mathbf{F} = \dfrac{1}{6.5}\begin{pmatrix}-8.5\\6.5\end{pmatrix} = \begin{pmatrix}-1.31\\1\end{pmatrix}\,\text{m s}^{-2}$

b Magnitude of acceleration is $\sqrt{(-1.31)^2 + 1^2} = 1.65\,\text{m s}^{-2}$

10 Converting units to kg, m, s.

Constant acceleration problem: $u = \dfrac{50000}{3600}$, $v = \dfrac{30000}{3600}$, $s = 650$. Find a:

$a = \dfrac{v^2 - u^2}{2s} = -0.095\,\text{m s}^{-2}$

$\text{Mass} = 2300\,\text{kg}$ therefore $F = ma = 2300 \times (-0.095) = -218$

The braking force is approximately $218\,\text{N}$, applied against the direction of travel.

11 Constant acceleration problem: $u = 0$, $s = 16$, $t = 3$. Find a.

$a = \dfrac{s - ut}{\frac{1}{2}t^2} = \dfrac{16}{4.5} = 3.55\,\text{m s}^{-2}$

$m = \dfrac{F}{a} = \dfrac{23}{3.55} = 6.47\,\text{kg}$

EXERCISE 21B

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7 Total force is $180\,\text{N}$ therefore $m = \dfrac{F}{a} = \dfrac{180}{0.15} = 1200\ \text{kg}$.

8 Total effective force: $40 - 10 = 30\,\text{N}$ therefore $m = \dfrac{F}{a} = \dfrac{30}{1.5} = 20\,\text{kg}$

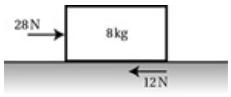
9 **a** $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \begin{pmatrix} -5 \\ -4 \end{pmatrix} + \mathbf{F}_3 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ so $\mathbf{F}_3 = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

$F_3 = \sqrt{5^2 + 6^2} = \sqrt{61}$

b Resultant motion is to the right. Angle away from the horizontal is $\tan^{-1}\left(\dfrac{6}{5}\right) = 50.2^\circ$

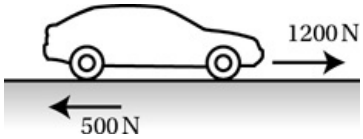
EXERCISE 21C

1



Resultant force $28 - 12 = 16 = ma$ therefore $a = \dfrac{16}{8} = 2\,\text{m s}^{-2}$

2



Resultant force $1200 - 500 = 700 = ma$ therefore $m = \dfrac{700}{2.6} = 269\,\text{kg}$.

3 $m = 6200\,\text{kg}, a = 1.2\,\text{m s}^{-2}$

$F = ma = 7440\,\text{N}$ is the resultant force producing motion.

a Resistance to the motion is $D - F = 8200 - 7440 = 760\ \text{N}$

b $a = 1.2$, $u = 0$, $v = \dfrac{40000}{3600}$. Find t: $t = \dfrac{v - u}{a} = \dfrac{11.1}{1.2} = 9.26\,\text{s}$

Without resistance, $a = \dfrac{F}{m} = \dfrac{8200}{6200} = 1.32\,\text{m s}^{-2}$

$a = 1.32$, $u = 0$, $v = \dfrac{40000}{3600}$. Find t: $t = \dfrac{v - u}{a} = \dfrac{11.1}{1.32} = 8.40\,\text{s}$.

Without resistance, the truck would take $0.86\,\text{s}$ less time to reach $40\,\text{km h}^{-1}$ from rest.

4 Acceleration vector $\mathbf{a} = \dfrac{1}{m}(\text{applied force} - \text{friction})$

So $2.5\mathbf{i} + 3.1\mathbf{j} = \dfrac{1}{8}(p\mathbf{i} + q\mathbf{j} - (6.8\mathbf{i} - 5.2\mathbf{j}))$

$p = 8 \times 2.5 + 6.8 = 26.8$

$q = 8 \times 3.1 - 5.2 = 19.6$

5 System is in equilibrium so the frictional force must equal $(9\mathbf{j} - 16\mathbf{i})\,\text{N}$ (orienting so that the $16\,\text{N}$ in the diagram as $16\mathbf{i}\,\text{N}$)

Then the magnitude of the frictional force equals $\sqrt{9^2 + 16^2} = 18.4\,\text{N}$

And the angle to the $16\,\text{N}$ force equals $180^\circ + \tan^{-1}\left(\dfrac{9}{-16}\right) = 151^\circ$

6 Acceleration vector $\mathbf{a} = \dfrac{1}{m}(\text{applied force} - \text{friction})$

Orienting the applied $62\,\text{N}$ force as acting in direction $\mathbf{i}$:

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So $\mathbf{a} = \dfrac{1}{45}(62\mathbf{i} - 18(\mathbf{i}\cos 170^\circ + \mathbf{j}\sin 170^\circ))$

$= 0.984\mathbf{i} - 0.0695\mathbf{j}$

Then $|\mathbf{a}| = \sqrt{0.984^2 + 0.0695^2} = 0.986\,\mathrm{m\,s^{-2}}$

7 $F = ma = 56 \times 0.8 = 44.8\,\mathrm{N}$

Resultant force of $44.8\,\mathrm{N} = 120 - 80 + x$

The second girl pulls with a force $x = 4.8\,\mathrm{N}$.

8 Constant speed: $a = 0$ so resultant force is zero.

$2F = 420$ therefore $F = 210\,\mathrm{N}$

9 Constant acceleration problem: $u = 15.3$, $v = 0$, $s = 120$. Find a:

$a = \dfrac{v^2 - u^2}{2s} = -\dfrac{15.3^2}{2 \times 120} = -0.975\,\mathrm{m\,s^{-2}}$

$F = ma = 950 \times (-0.975) = 926.6\,\mathrm{N}$

The braking force equals $926.6 - 320 = 607\,\mathrm{N}$

10 a $a = \dfrac{F}{m} = \dfrac{180}{45} = 4\,\mathrm{m\,s^{-2}}$

Constant acceleration problem: $a = 4, u = 0, s = 25$. Find t:

$s = ut + \dfrac{1}{2}at^2$ so $25 = 2t^2$ therefore $t = \sqrt{12.5} = 3.54\,\mathrm{s}$

It takes 3.54 seconds for the box to travel $25\,\mathrm{m}$.

b Constant acceleration problem: $u = 0$, $s = 25$, $t = 4.2$. Find a:

$a = \dfrac{s - ut}{\dfrac{1}{2}t^2} = \dfrac{25}{8.82} = 2.83$, therefore $F = ma = 45 \times 2.83 = 127.6\,\mathrm{N}$

So acting against the $180\,\mathrm{N}$ must be a resistance of $180 - 127.6 = 52.4\,\mathrm{N}$.

11 a Constant acceleration problem: $u = 12$, $v = 26$, $t = 7$. Find a:

$a = \dfrac{v - u}{t} = \dfrac{(26 - 12)}{7} = 2\,\mathrm{m\,s^{-2}}$

Resultant force producing motion: $1200 - 320 = 880\,\mathrm{N}$

$m = \dfrac{F}{a} = \dfrac{880}{2} = 440\,\mathrm{kg}$

b Now $a = 0$ so the driving force must be exactly the amount needed to counter the resistance to motion.

Driving force is $320\,\mathrm{N}$.

c New effective force is $-320\,\mathrm{N}$.

$a = \dfrac{F}{m} = -\dfrac{320}{440} = -0.727\,\mathrm{m\,s^{-2}}$

Constant acceleration problem: $u = 26$, $v = 0$, $a = -0.727$. Find s:

$s = \dfrac{v^2 - u^2}{2a} = \dfrac{-26^2}{2 \times (-0.727)} = 464.75\,\mathrm{m}$

For the first part of the journey past A, considering the area under the velocity—time graph, distance travelled equals $7 \times \left(\dfrac{12 + 26}{2}\right) + 26 \times 12 = 445\,\mathrm{m}$

Total distance $AB = 445 + 464.75 = 909.75 \approx 910\,\mathrm{m}$.

12 Effective force is $1.5\,\mathrm{N}$.

$a = \dfrac{F}{m} = \dfrac{1.5}{4.6} = 0.326\,\text{m s}^{-2}$

Constant acceleration problem: $a = 0.326$, $u = 0$, $t = 4$. Find v:

$v = u + at = 4 \times 0.326 = 1.30\,\text{m s}^{-1}$

A second constant acceleration problem: $u = 1.30$, $v = 1.24$, $s = 2$. Find a:

$a = \dfrac{v^2 - u^2}{2s} = \dfrac{1.24^2 - 1.30^2}{2 \times 2} = -0.0409\,\text{m s}^{-2}$

$F = ma = 4.6 \times (-0.0409) = 0.1883\,\text{N}$

The added force equals $1.5 + 0.1883 \approx 1.69\,\text{N}$ in the same direction as the $6.5\,\text{N}$ force.

13 a The first person's pull is east and north, and the second person's pull is east and south; the north and south elements exactly cancel out, so that their resultant pull is entirely eastward, partially countered by a resistance directed entirely westward.

b Net force is $2p - 180$. Mass is $\dfrac{1400}{9.8} = 142.9\,\text{kg}$

$a = \dfrac{F}{m} = \dfrac{2p - 180}{142.9} = 0.014p - 1.26$

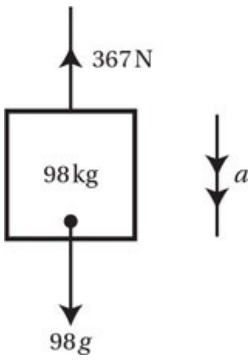
But $a = 0.003p$ and so $0.003p = 0.014p - 1.26$

$0.011p = 1.26$ therefore $p = 115$

c $p^2 + q^2 = 200^2$ so $q = \sqrt{200^2 - p^2} = 164$

EXERCISE 21D

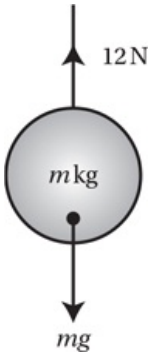
4



Resultant force downwards: $98g - 367 = 593.4\,\text{N}$

$a = \dfrac{F}{m} = \dfrac{593.4}{98} = 6.06\,\text{ms}^{-2}$

5



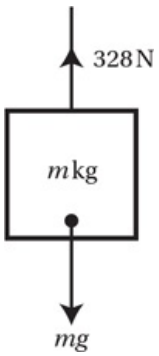
a Resultant force zero: $mg = 12$

$m = \dfrac{12}{g} = 1.22\,\text{kg}$

b Resultant force: $15 - mg = 3\,\text{N}$ upwards

$$a = \frac{F}{m} = \frac{3}{1.22} = 2.45\,\mathrm{ms}^{-2}$$

6



Resultant force zero: $mg = 328$

$$m = \frac{328}{3.7} = 88.6\,\mathrm{kg}$$

7 Let $R =$ air resistance

$1.2g - R = 1.2 \times 8.9$

$R = 1.2 \times 9.8 - 1.2 \times 8.9 = 1.08\,\mathrm{N}$

8 Constant acceleration problem: $u = 0,\ s = 24,\ t = 2.3$

From $s = ut + \frac{1}{2}at^2$

$$a = \frac{2s}{t^2} = \frac{2 \times 24}{2.3^2} \approx 9.074\,\mathrm{m\,s}^{-2}$$

Let $R =$ air resistance

$$\begin{aligned} mg - R &= ma \\ m(g-a) &= R \\ m = \frac{R}{g-a} &= \frac{0.25}{9.8 - 9.074} = 0.344\,\mathrm{kg}\,(3\,\mathrm{s.f.}) \end{aligned}$$

9 **a** $s = ut + \frac{1}{2}at^2$, $s = 25,\ u = 0,\ \mathrm{a} = g = 9.8$

$$t = +\sqrt{\frac{2s}{g}} = +\sqrt{\frac{2 \times 25}{9.8}} = 2.26\,\mathrm{s}\,(3\,\mathrm{s.f.})$$

b Calculating for a with $t = 2.42$:

$$a = \frac{2s}{t^2} = \frac{2 \times 25}{2.42^2} \approx 8.538\,\mathrm{m\,s}^{-2}$$

Let $R =$ air resistance

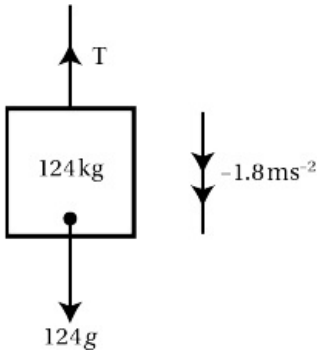
$$\begin{aligned} mg - R &= ma \\ R &= m(g-a) \\ &= 1.2(9.8 - 8.538) \\ &= 1.51\mathrm{N}(3\mathrm{s.f.}) \end{aligned}$$

10 Constant acceleration problem: $u = 0,\ s = 84,\ t = 6$. Find a: $a = \dfrac{s - ut}{\frac{1}{2}t^2} = \dfrac{84}{18} = 4.67\,\mathrm{m\,s}^{-2}$

Resultant upward force $T - mg = ma$

$T = m(a+g) = 350(4.67 + 9.8) = 5064.5 \approx 5060\,\mathrm{N}$

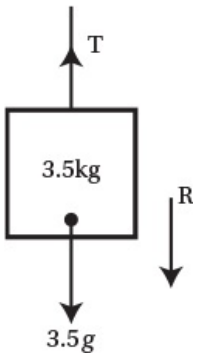
11



$124g - T = 124 \times (-1.8)$ therefore $T = 124(g + 1.8) = 1438.4\,\mathrm{N}$

Tension is approximately $1440\,\mathrm{N}$.

12



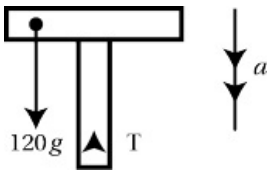
a Forces in equilibrium with $R = 5\,\mathrm{N}$

$T = R + 3.5g = 39.3\,\mathrm{N}$

b R falls to zero, so the resultant force is now $5\,\mathrm{N}$ upwards.

$$a = \frac{F}{m} = \frac{5}{3.5} = 1.43\,\mathrm{m\,s^{-2}}$$

13



$$a = \frac{-3.2}{5} = -0.64\,\mathrm{m\,s^{-2}}$$

$120g - T = 120 \times (-0.64)$ so $T = 120(g + 0.64) = 1252.8 \approx 1250\,\mathrm{N}$.

EXERCISE 21E

4 Horizontal forces in equilibrium: $-8 + 4 + \mathbf{F}_1 = 0$, $\mathbf{F}_1 = 4\,\text{N}$

Vertical forces in equilibrium: $3 + 2 - \mathbf{F}_2 = 0$, $\mathbf{F}_2 = 5\,\text{N}$

5 Horizontal forces in equilibrium: $3 - 5 + x = 0$, $x = 2$

Vertical forces in equilibrium: $-6 + 2 + y = 0$, $y = 4$

6 Horizontal forces in equilibrium: $16 + 2 - a - b = 0$ so $a + b = 18$ $\quad(1)$

Vertical forces in equilibrium: $2 + a - b = 0$, so $b = a + 2$ $\quad(2)$

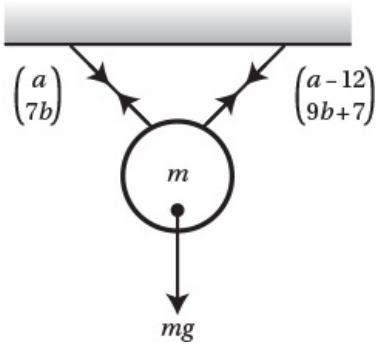
Substituting (2) into (1): $2a = 2 = 18$ so $a = 8$, $b = 10$.

7 Vertical forces in equilibrium: $T + 1.4g = 21$ so $T = 21 - 1.4g = 7.28\,\text{N}$

8 Friction is exactly countering the resultant from the two people's efforts, so the friction force is $28 - 16 = 12\,\text{N}$ (in the direction of the $16\,\text{N}$ pull).

9 There is a thrust of $19\,\text{N}$ in the rod so that the resultant pull on each side is $63\,\text{N}$ and the system is in equilibrium.

10



Horizontal forces must be in equilibrium: $a + a - 12 = 0$ therefore $a = 6$

Vertical forces must be equal in the two strings, since they are identical:

$7b = 9b + 7$ so $2b = -7$ therefore $b = -3.5$

So the total force applied to the ceiling is $\begin{pmatrix} 6 \\ -24.5 \end{pmatrix} + \begin{pmatrix} -6 \\ -24.5 \end{pmatrix} = \begin{pmatrix} 0 \\ -49 \end{pmatrix}$

This results from the weight of the ball (opposed by the tension in the strings which communicate and distribute the force up to the ceiling).

$mg = 49$ therefore $m = \dfrac{49}{g} = 5$

11 Resultant of the pulling is $(57\mathbf{i} - 6\mathbf{j})\,\text{N}$ so for the box to remain at rest, the friction must be exactly opposing this force. Friction $\mathbf{F} = (-57\mathbf{i} + 6\mathbf{j})\,\text{N}$

12 a Equilibrium: $\mathbf{F} + \begin{pmatrix} -17 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 25 \end{pmatrix} = 0$ therefore $\mathbf{F} = \begin{pmatrix} 17 \\ -25 \end{pmatrix}$

b Magnitude is $\sqrt{17^2 + 25^2} = 30.2\,\text{N}$

Angle is $\tan^{-1}\left(\dfrac{-25}{17}\right) = 55.8^\circ$ below the horizontal.

MIXED PRACTICE 21

1 a Constant acceleration problem: $u = 0$, $v = 12.6$, $t = 8$. Find a:

$a = \dfrac{v - u}{t} = \dfrac{12.6}{8} = 1.575\,\text{m s}^{-2}$

$F = ma$ therefore $m = \dfrac{F}{a} = \dfrac{1360}{1.575} = 863\,\text{kg}$

b With resistance, the effective force would be lower so to achieve the same acceleration would require the car's mass to be lower.

2 Constant acceleration problem: $u = 2.6$, $v = 0$, $s = 3.7$. Find a:

$a = \dfrac{v^2 - u^2}{2s} = \dfrac{-2.6^2}{2 \times 3.7} = -0.914\,\mathrm{m\,s^{-2}}$

Frictional force $F = ma = 13 \times -0.914 = -1.9$

The frictional force is $1.9\,\mathrm{N}$ opposing the movement.

3 **a** Resultant force causing acceleration: $2170 - 865 = 1305\,\mathrm{N}$

$a = \dfrac{F}{m} = \dfrac{1305}{1600} = 0.816\,\mathrm{m\,s^{-2}}$

b Constant acceleration problem: $a = 0.816$, $u = 0$, $s = 260$. Find t:

$s = ut + \dfrac{1}{2}at^2$ so $260 = \dfrac{1}{2} \times 0.816 \times t^2$ therefore $t = \sqrt{\dfrac{260}{0.408}} = 25.2\,\mathrm{s}$

4 **a** $m = \dfrac{W}{g} = \dfrac{268}{9.8} = 27.3\,\mathrm{kg}$

b Resultant vertical force $T - mg = ma$ so $T = m(a + g) = 27.3 \times 10.7 = 293\,\mathrm{N}$

c $a = 0$ so the system is in equilibrium and the tension equals the weight.

$T = 268\,\mathrm{N}$

d **i** The mass of the rope has been ignored when considering the system's total mass.

ii The rope is considered to have a constant tension throughout it and perfectly transmits all tension forces applied to it.

5 **a** Equilibrium: the sum of the forces is zero.

$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \begin{pmatrix} 2.6 + p \\ 3.8 + q \end{pmatrix}$ therefore $p = -2.6$, $q = -3.8$

b $\mathbf{F} = m\mathbf{a}$ therefore $\mathbf{a} = \dfrac{1}{m}\mathbf{F} = \dfrac{1}{2.5}\begin{pmatrix} 2.6 \\ 3.8 \end{pmatrix} = \begin{pmatrix} 1.04 \\ 1.52 \end{pmatrix}\mathrm{m\,s^{-2}}$

6 **a** Use constant acceleration formulae: $v = 0$, $t = 14$, $s = 32$; find a:

You can find u: $s = \dfrac{u + v}{2}t \Rightarrow \dfrac{u + 0}{2} \times 14 = 32 \Rightarrow u = 4.57$

Then use $v = u + at$: $0 = 4.57 + 14a \Rightarrow a = -0.327$

So the deceleration is $0.327\,\mathrm{ms^{-2}}$.

b Friction is the only force acting on the stone, so the force is $F = ma = 21 \times 0.327 = 6.86\,\mathrm{N}$

7 **a** Constant acceleration problem: $u = \dfrac{30\,000}{3600}$, $v = \dfrac{40\,000}{3600}$, $s = 200$. Find a:

$a = \dfrac{v^2 - u^2}{2s} = 0.135\,\mathrm{m\,s^{-2}}$

Resultant force producing acceleration $F = ma = 750 \times 0.135 = 101\,\mathrm{N}$

Driving force of car $D = F + R = 101 + 380 = 481\,\mathrm{N}$

b Instead of D in the direction of travel, braking force $B = 620\,\mathrm{N}$ against the direction of travel.

$F = -B - R = -620 - 380 = -1000\,\mathrm{N}$

$a = \dfrac{F}{m} = -\dfrac{1000}{750} = -1.33\,\mathrm{m\,s^{-2}}$

Constant acceleration problem: $u = \dfrac{40\,000}{3600}$, $a = -1.33$, $v = 0$. Find t:

$t = \dfrac{v - u}{a} = \dfrac{-11.1}{-1.33} = 8.33\,\mathrm{s}$

Car stops 8.33 seconds after it begins braking.

8 For orientation, consider the $102\,\mathrm{N}$ force is pulling to the north. Let the angle between this force and $\mathbf{T}$ be x.

North–south forces in equilibrium: $102 - 150 + T\cos x = 0$ so $T\cos x = 48$ $\quad(1)$

East–west forces in equilibrium: $86 - T\sin x = 0$ so $T\sin x = 86$ $\quad(2)$

$(1)^2 + (2)^2$: $T^2 = 48^2 + 86^2 = 9700$ therefore $T = \sqrt{9700} = 98.5\,\mathrm{N}$

$(2) \div (1)$: $\tan x = \dfrac{86}{48}$ therefore $x = 60.8^\circ$

Force $\mathbf{T}$ has magnitude $98.5\,\mathrm{N}$ and is at an angle 60.8° from the $102\,\mathrm{N}$ force.

9 Equilibrium requires $\mathbf{F} = (-15\mathbf{i} + 12\mathbf{j})\,\mathrm{N}$

Magnitude of $\mathbf{F}$: $\sqrt{15^2 + 12^2} = 19.2\,\mathrm{N}$

Direction of $\mathbf{F}$: $\tan^{-1}\left(\dfrac{12}{-15}\right) = -38.7^\circ$

The force makes an angle of 141° with the direction of the $15\,\mathrm{N}$ force.

10 Write $\mathbf{X}$ and $\mathbf{Y}$ in component form, with $\mathbf{i}$ on a bearing of 090° and $\mathbf{j}$ on a bearing of 000°.

Then $\mathbf{X} = 12\mathbf{i}$ and $\mathbf{Y} = 15\mathbf{j}$.

i $\mathbf{X} + \mathbf{Y} = 15\mathbf{i} + 12\mathbf{j}$

The magnitude is $\sqrt{12^2 + 15^2} = 19.2\,\mathrm{N}$

Bearing: $\tan\theta = \dfrac{12}{15} \Rightarrow \theta = 038.7^\circ$

ii The force $\mathbf{E}$ has the same magnitude and opposite direction from $\mathbf{X} + \mathbf{Y}$.

The magnitude of $\mathbf{E}$ is $19.2\,\mathrm{N}$ and the bearing is $39 + 180 = 219^\circ$.

11 Horizontal forces in equilibrium: $-102 + a + x = 0$ so $x = 102 - a$

Vertical forces in equilibrium: $150 + a - 24 \times 9.8 = 0$ so $a = 235.2 - 150 = 85.2$

Therefore $x = 102 - 85.2 = 16.8\,\mathrm{N}$

The rod supplies $16.8\,\mathbf{i}\,\mathrm{N}$, so the force is a thrust in the rod.

12 a Total forces acting on the particle: $12\mathbf{i} + 10\mathbf{j} + 18\mathbf{i} - 13\mathbf{j} = 30\mathbf{i} - 3\mathbf{j}$

$\mathbf{F} = m\mathbf{a}$ so: $\mathbf{a} = \dfrac{1}{m}\mathbf{F} = \dfrac{1}{4}(30\mathbf{i} - 3\mathbf{j}) = (7.5\mathbf{i} - 0.75\mathbf{j})\,\mathrm{m\,s^{-2}}$

b Forces in equilibrium: require additional force $(-30\mathbf{i} + 3\mathbf{j})\,\mathrm{N}$

Magnitude: $\sqrt{30^2 + 3^2} = 30.1\,\mathrm{N}$

Angle above the horizontal: $\tan^{-1}\left(\dfrac{3}{30}\right) = 5.71^\circ$

13 Total forces acting on the particle: $-160\mathbf{i} + 480\mathbf{j} - 53 \times 9.8\mathbf{j} + T = 0$ for equilibrium

$\mathbf{T} = 160\mathbf{i} + 39.4\mathbf{j}$

Magnitude: $\sqrt{160^2 + 39.4^2} = 165\,\mathrm{N}$

Angle above horizontal: $\tan^{-1}\left(\dfrac{39.4}{160}\right) = 13.8^\circ$

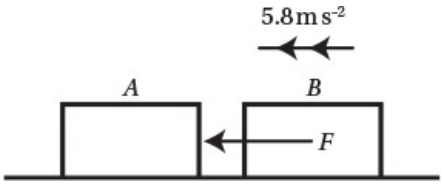
Chapter 22 worked solutions

22 Objects in contact

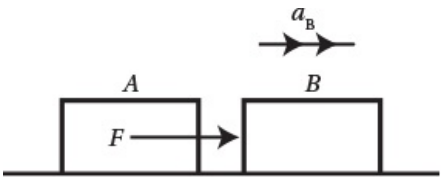
Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

EXERCISE 22A

1



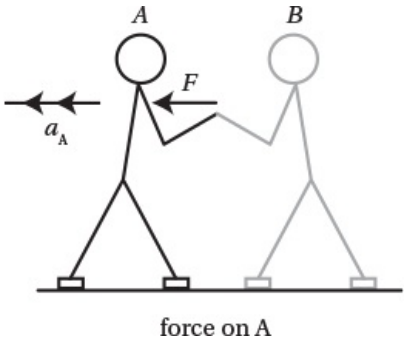
Force on A: $F = m_A a_A = 265 \times 5.8 = 1537\,\text{N}$



Equals force on B: $F = m_B a_B = 280 \times a_B = 1537\,\text{N}$

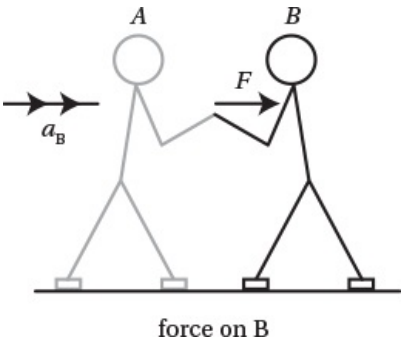
$a_B = \dfrac{1537}{280} = 5.49\,\text{m s}^{-2}$ away from A

2



force on A

Force on A: $F = m_A a_A = 78 \times 3.5 = 273\,\text{N}$



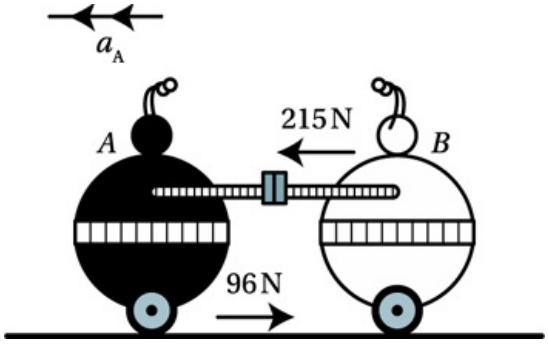
force on B

Force on B:

$F = m_B a_B = m_B \times 4.2 = 273\,\text{N}$

$\text{m}_B = \dfrac{273}{4.2} = 65\,\text{kg}$

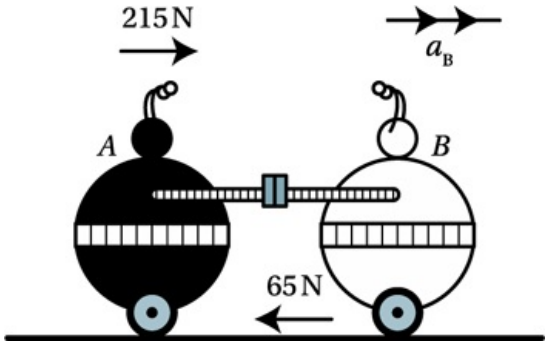
3 Forces acting on robot A:



$F = 215 - 96 = m_A a_A = 120 a_A$

$a_A = \dfrac{119}{120} = 0.992\,\text{ms}^{-2}$

Forces acting on robot B:



$F = 215 - 65 = m_B a_B = 90 a_B$

$a_B = \dfrac{150}{90} = 1.67\,\text{m s}^{-2}$

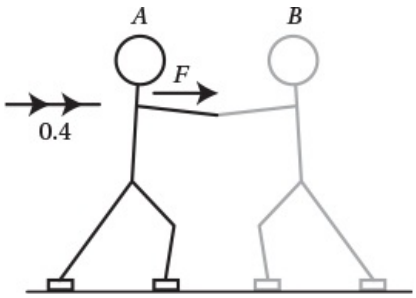
Separation acceleration $a = a_A + a_B = 2.66\,\text{m s}^{-2}$

$u = 0$, $a = 2.66$, $t = 2$. Find s:

$s = ut + \dfrac{1}{2}at^2 = 1.33 \times 2^2 = 5.32\,\text{m}$

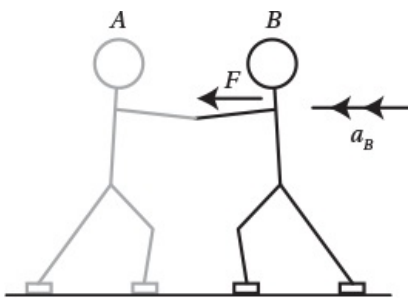
After 2 seconds, the robots are $5.32\,\text{m}$ apart.

4 Forces acting on A:



a $F = m_A a_A = 52 \times 0.4\,\text{m s}^{-2} = 20.8\,\text{N}$

Forces acting on B:



$F = m_B a_B = 68 a_B = 20.8\,\text{N}$

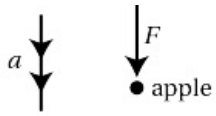
$a_B = \dfrac{20.8}{68} = 0.306\,\text{m s}^{-2}$

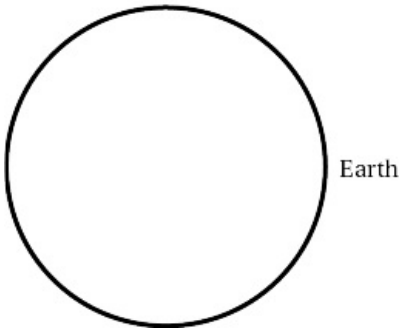
b Approach acceleration is $0.4 + 0.306 = 0.706\,\text{m s}^{-2}$

$u = 0$, $a = 0.706$, $s = 1.2$. Find t:

$s = ut + \dfrac{1}{2}at^2$ so $1.2 = \dfrac{1}{2} \times 0.706 \times t^2$ therefore $t = \sqrt{\dfrac{2.4}{0.706}} = 1.84\,\text{s}$

5 a



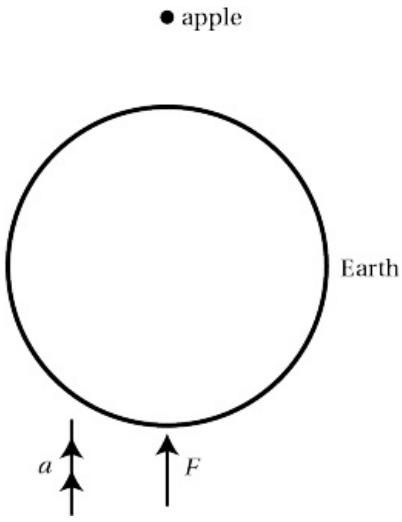


Force on apple: $F = m_{\text{apple}}\, a_{apple} = 0.12 \times 9.8 = 1.18\,\text{N}$

b Taking the dominant direction of movement as downward: $u = 0$, $a = 9.8$, $s = 10$. Find t:

$s = ut + \dfrac{1}{2}at^2$ so $10 = 4.9t^2$ therefore $t = \sqrt{\dfrac{10}{4.9}} = 1.43\,\text{s}$

c



Equal force acting on the Earth: $1.18\,\text{N}$

d $F = m_{\text{Earth}} a_{\text{Earth}} = 5.97 \times 10^{24} \times a_{\text{Earth}} = 1.18$

$a_{\text{Earth}} = \dfrac{1.18}{5.97 \times 10^{24}} = 1.97 \times 10^{-25}\,\text{m s}^{-2}$

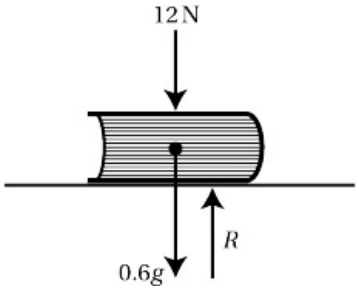
If the Earth were stationary at the start of this scenario then: $u = 0$, $a = 1.97 \times 10^{-25}$, $t = 1.43$

$s = ut + \dfrac{1}{2}at^2 = \dfrac{1}{2} \times 1.97 \times 10^{-25} \times 1.43^2 = 2.02 \times 10^{-25}\,\text{m}$

The Earth would move about $2.02 \times 10^{-25}\,\text{m}$; in context, the size of an atom is in the order of $10^{-10}\,\text{m}$, so this is many orders of magnitude smaller than an atom.

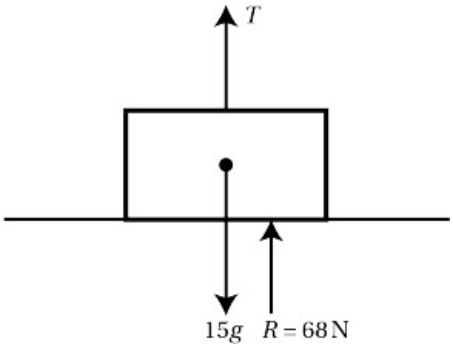
EXERCISE 22B

3



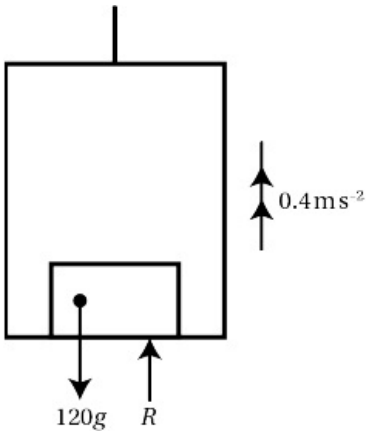
Forces on book in equilibrium: $R = 0.6g + 12\,\mathrm{N}$ therefore $R = 17.9\,\mathrm{N}$.

4



Forces on box in equilibrium: $T = 15g - R = 15g - 68\,\mathrm{N}$ therefore $T = 79\,\mathrm{N}$.

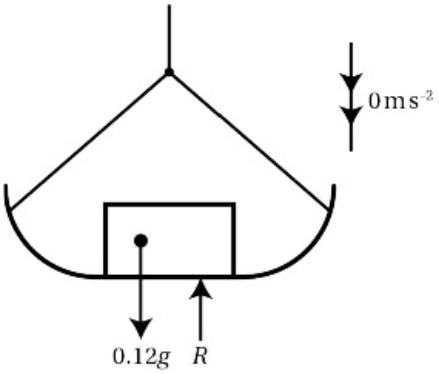
5



$\text{Forces on box} = \text{mass} \times \text{acceleration}$

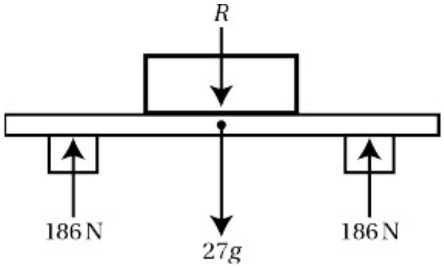
$R - 120g = 120 \times 0.4$ therefore $R = 1224\,\mathrm{N}$

6 Considering only the forces on the box:

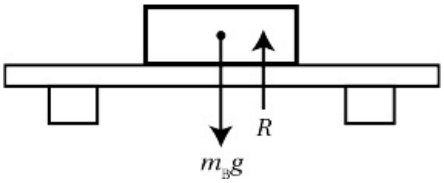


$0.12g - R = 0$ therefore $R = 1.18\,\mathrm{N}$

7



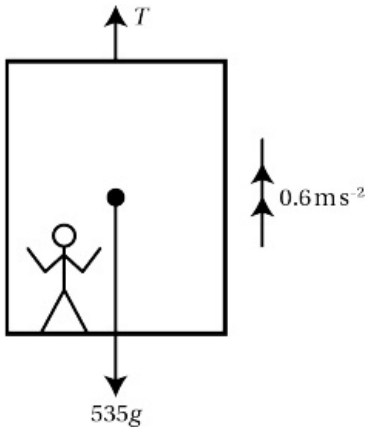
Forces acting on shelf are in equilibrium: $R + 27g = 2 \times 186\,\text{N}$ so $R = 107.4\,\text{N}$



Forces acting on box are in equilibrium:

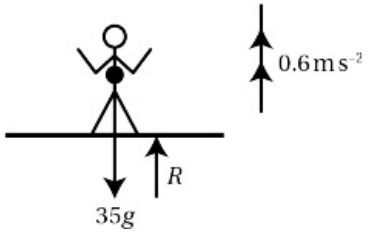
$R = m_B g$ therefore $m_B = \dfrac{107.4}{9.8} = 11.0\,\text{kg}$

8 **a** Considering the lift and child as a single object to assess external forces:



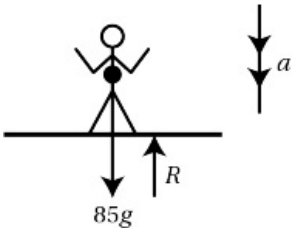
$T - 535g = 535 \times 0.6$ therefore $T = 5564\,\text{N}$

b Considering only the child:



$R - 35g = 35 \times 0.6$ therefore $R = 364\,\text{N}$

9 Considering only forces acting on the person:

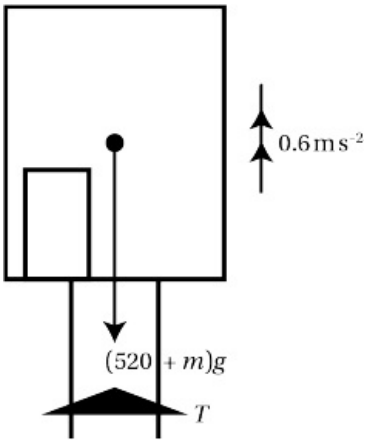


$85g - R = 85a$

a $a = -4.2$ therefore $R = 1190\,\text{N}$

b $a = 0$ therefore $R = 833\,\text{N}$

10 Considering the lift and load as a single body of mass $(520 + m)\,\text{kg}$:



The downward acceleration shown in the diagram can vary from $-7.8\,\text{m s}^{-2}$ (downward deceleration) to $7.8\,\text{m s}^{-2}$ (upward deceleration).

Using Newton's second law:

$(520 + m)g - T = (520 + m)a$ so $T = (520 + m)(g - a)$

$520 + m = \dfrac{T}{g - a}$ therefore $m = \dfrac{T}{g - a} - 520$

Maximum permissible load m must be viable for all values of a allowed in the model.

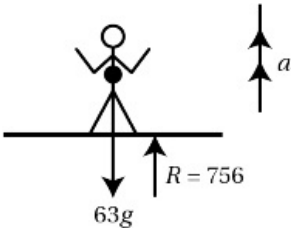
$$m \leqslant \frac{15400}{9.8 - a} - 520$$

The lowest value for this is when $a = -7.8$. $m \leqslant 355\,\text{kg}$

Tip

This may seem perverse, as you can get a higher value for m by selecting $a = +7.8$, but this would find the maximum load the lift can withstand under the best possible acceleration situation. As soon as the lift began moving differently, the rod would be overstressed.

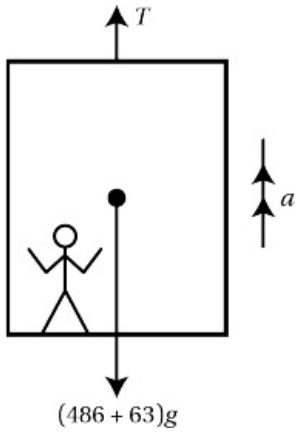
11 a Considering forces acting on woman:



$756 - 63g = 63a$ therefore $a = \dfrac{756 - 63g}{63} = 2.2$

The lift is accelerating upwards.

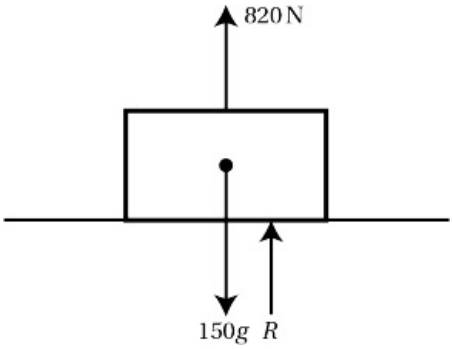
b Considering forces acting on the whole system of lift and woman together.



$T - (486 + 63)g = (486 + 63)a$ so $T = 549(g + 2.2)$ therefore $T = 6588\,\text{N}$

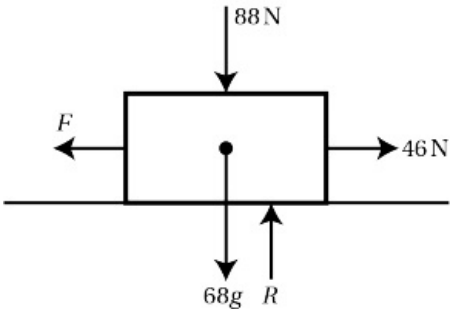
EXERCISE 22C

1



Forces in equilibrium: $820\,\text{N} + R = 150g$ therefore $R = 650\,\text{N}$

2 Complete the diagram for all forces on the box:



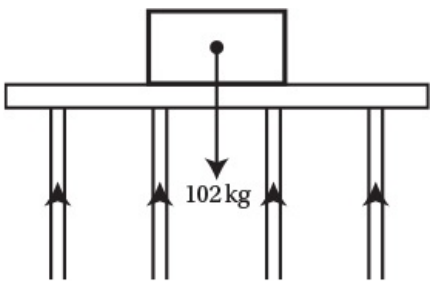
Forces in equilibrium

Horizontal forces: $F = 46\,\text{N}$

Vertical forces: $88\,\text{N} + 68g = R$

$R = 754.4\,\text{N}$

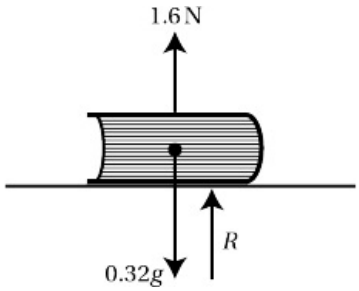
3 Taking the box and table as a complete system of $16\,\text{kg} + 86\,\text{kg} = 102\,\text{kg}$



Forces in equilibrium:

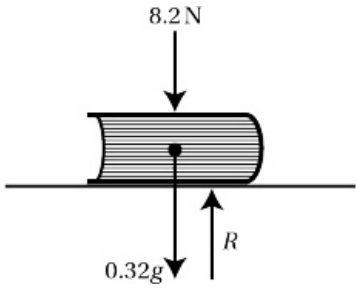
$4T = 102g$, therefore $T = 250\,\text{N}$

4 **a** Forces in equilibrium:



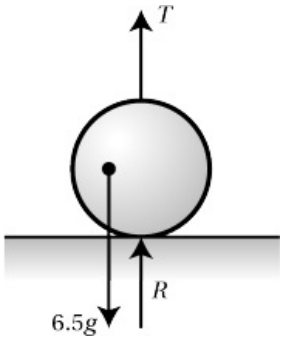
$R + 1.6 = 0.32\text{g}$ therefore $R = 1.54\,\text{N}$

b Forces in equilibrium:



$R = 0.32g + 8.2\,\text{N}$ therefore $R = 11.3\,\text{N}$

5



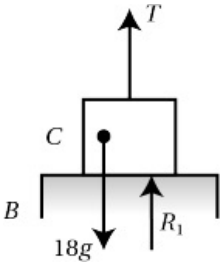
a $T = 38$, forces in equilibrium:

$R + T = 6.5g$ so $R = 6.5g - 38$ therefore $R = 25.7\,\text{N}$

b Ball on the point of lifting off table: $R = 0$ but forces in equilibrium (no acceleration)

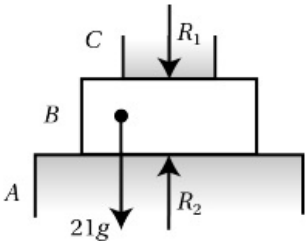
$T = 6.5g$ therefore $T = 63.7\,\text{N}$

6 Consider forces on C:



Forces in equilibrium: $R_1 + T = 18g$

Now consider forces on B:



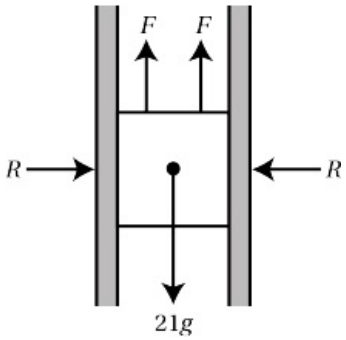
The normal force from C must be R_1 due to Newton's third law

Forces in equilibrium: $21g + R_1 = R_2$

Given $R_2 = 260\,\mathrm{N}$, $R_1 = R_2 - 21g = 54.2\,\mathrm{N}$

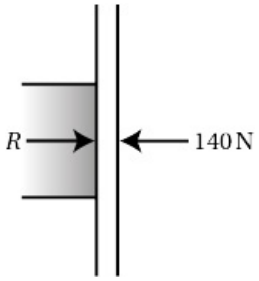
Then from the previous equation: $T = 18g - R_1 = 122.2\,\mathrm{N}$

7 **a** Considering vertical forces on the box:



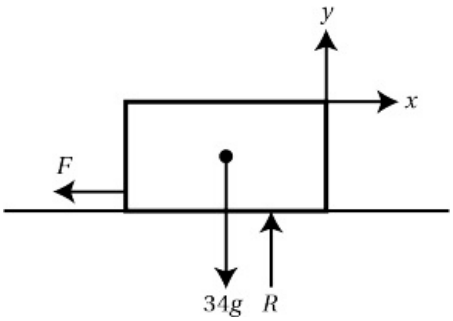
$2F = 1.2g$ therefore $F = 5.88\,\mathrm{N}$

b Considering horizontal forces on a plank:



$R = 140\,\mathrm{N}$

8 Splitting the tension into horizontal and vertical components:



Considering vertical forces: $y + R = 34g$

R is given as $180\,\mathrm{N}$

$y = 34g - 180 = 153.2\,\mathrm{N}$

Considering horizontal forces: $x = F$

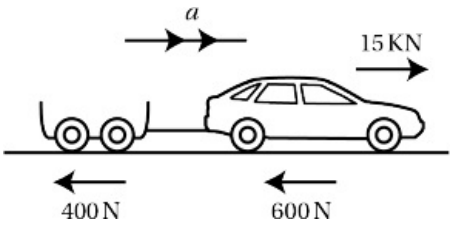
F is given as $220\,\mathrm{N}$

$x = 220\,\mathrm{N}$

Tension force has magnitude $\sqrt{x^2 + y^2} = 268\,\text{N}$.

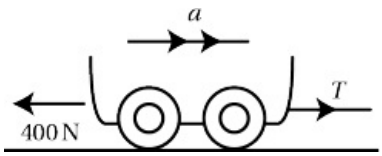
EXERCISE 22D

1 System as a whole:



a $15\,000 - 400 - 600 = (750 + 350)a$ therefore $a = \dfrac{14\,000}{1100} = 12.7\,\text{m s}^{-2}$

b



$T - 400 = 350a$ therefore $T = 400 + 350 \times 12.7 = 4855\,\text{N}$

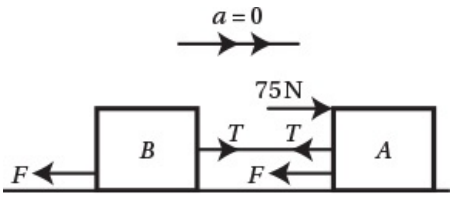
> **Tip**
>
> There is no need to consider the car separately as well, it will produce the same result, so can be used as a check: $15\,000 - 600 - T = 750a$ should be consistent with the answers in questions 1a and b.

2 Considering forces only on P: $D - T = 4.5a = 9.45$

Considering forces only on Q: $T = 6a = 12.6\,\text{N}$

So $D = 9.45 + T = 22.05\,\text{N}$

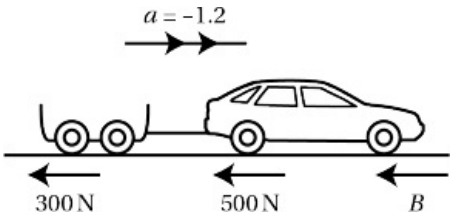
3



Considering external forces acting on the two boxes as a whole, forces are in equilibrium (no acceleration):

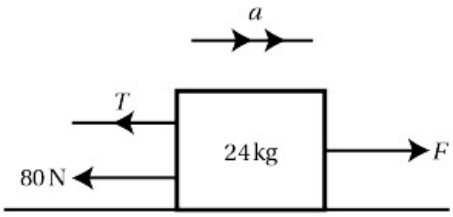
$75 - 2F = 0$ therefore $F = 37.5\,\text{N}$

4 Taking the car and trailer as a single system with mass $1200\,\text{kg} + 400\,\text{kg} = 1600\,\text{kg}$



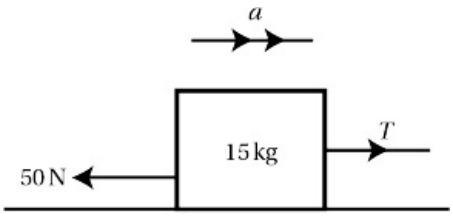
$-B - 300 - 500 = -1.2 \times 1600$ therefore $B = 1920 - 800 = 1120\,\text{N}$

5 Considering only the first box:



$F - T - 80 = 24a$

Considering only the second box:



$T - 50 = 15a$

From the system overall (the sum of these two equations):

$F - 130 = 39a$ therefore $F = 13(3a + 10)$

From the second equation: $T = 15a + 50 = 5(3a + 10)$

a Since you require $T \leqslant 120$: $5(3a + 10) \leqslant 120$ so $3a + 10 \leqslant 24$

$F \leqslant 13 \times 24 = 312\,\text{N}$

b Considering forces on the second box and assuming that the frictional force on the second box is constant as long as it is in motion:

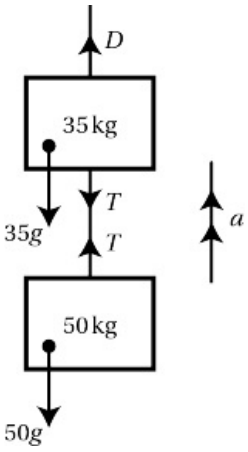
$-50 = 15a$ therefore $a = -\dfrac{10}{3}\,\text{m s}^{-2}$

Also, $u = 2.4\,\text{m s}^{-1}$, $v = 0\,\text{m s}^{-1}$. Find t:

Constant acceleration problem $t = \dfrac{v - u}{a} = -\dfrac{2.4}{-3.33} = 0.72\,\text{s}$

The second box will stop after 0.72 seconds.

6



Considering only external forces acting on the two-crate $85\,\text{kg}$ system :

$D - 85g = 85a$

$D = 85(g + a)$

Considering only forces acting on the lower crate:

$T - 50g = 50a$ therefore $T = 50(g + a)$

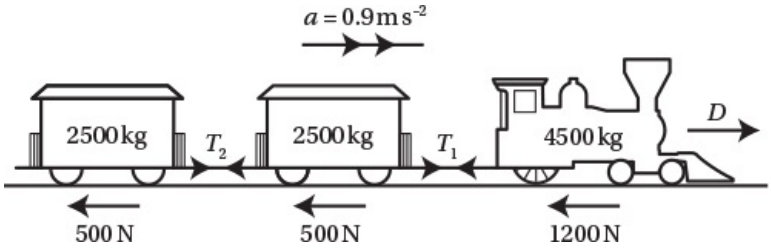
a $a = 0.8$

$D = 901\,\text{N}$, $T = 530\,\text{N}$

b $a = 0$

$D = 833\,\text{N}$, $T = 490\,\text{N}$

7



a Considering the system as a whole:

$D - 2200 = 9500 \times 0.9$ therefore $D = 10.75\,\text{kN}$

b Considering only the locomotive:

$D - 1200 - T_1 = 4500 \times 0.9$ therefore $T_1 = D - 1200 - 4500 \times 0.9 = 5.5\,\text{kN}$

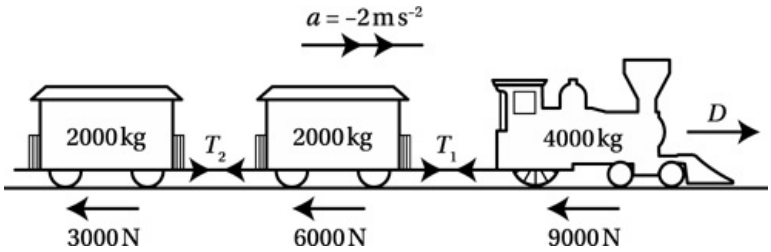
Considering only the rear carriage:

$T_2 - 500 = 2500 \times 0.9$ therefore $T_2 = 2.75\,\text{kN}$

Tip

Considering only the front carriage (to check): $T_1 - T_2 - 500 = 2500 \times 0.9$ Consistent with $T_1 = 5.5\,\text{kN}$, $T_2 = 2.75\,\text{kN}$.

8 Assume driving; if the train is braking, then driving force D will be found to be negative.



a Considering external forces acting on the $8000\,\text{kg}$ system :

$D - 18000 = 8000 \times (-2)$ therefore $D = 2000\,\text{N}$

The locomotive is providing a driving force (of $2\,\text{kN}$).

b Considering forces acting on the locomotive only:

$D - T_1 - 9000 = 4000 \times (-2)$ therefore $T_1 = 1000\,\text{N}$

There is a tension of $1\,\text{kN}$ between the locomotive and the first carriage.

Considering forces acting on the second carriage only:

$T_2 - 3000 = 2000 \times (-2)$ therefore $T_2 = -1000\,\text{N}$

There is a thrust of $1\,\text{kN}$ between the two carriages.

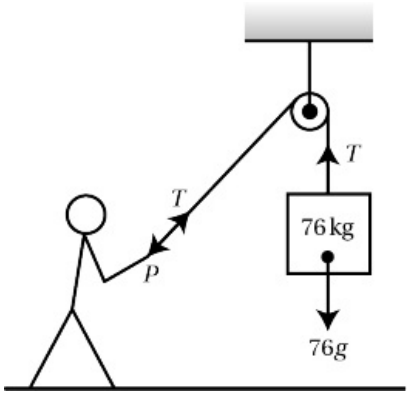
Tip

Considering forces acting on the first carriage only (to check):
$T_1 - T_2 - 6000 = 2000 \times (-2)$ consistent with $T_1 = 1000$, $T_2 = -1000$

EXERCISE 22E

IRANINOVIN

1



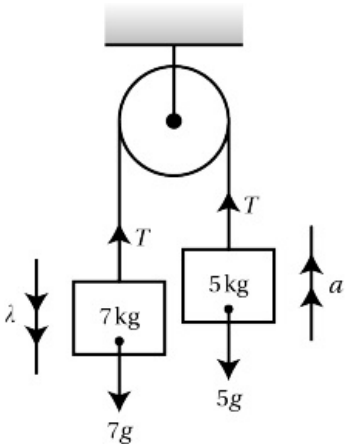
Box in equilibrium: $T = 76g = 744.8\,\mathrm{N}$

Rope stationary at the end man is pulling: $P = T$

The man is pulling with a force of $744.8\,\mathrm{N}$

(That is, he is applying the same force as he would do to lift the block directly.)

2



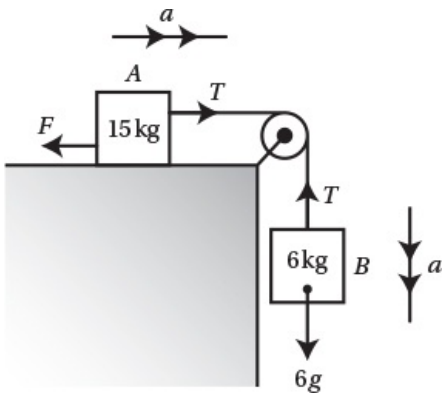
Forces applying to the $5\,\mathrm{kg}$ particle: $T - 5g = 5a$ $\quad (1)$

Forces applying to the $7\,\mathrm{kg}$ particle: $7g - T = 7a$ $\quad (2)$

$(1) + (2) : 2g = 12a$ so $a = \dfrac{g}{6} = 1.63\,\mathrm{m\,s^{-2}}$

Substituting into (1): $T = 5(g + a) = 57.2\,\mathrm{N}$

3



Vertical forces on B: $6g - T = 6a$ $\quad (1)$

Horizontal forces on A: $T - F = 15a$ $\quad (2)$

Tip

You are routinely ignoring forces not in the direction of travel, such as the weight of A and the normal reaction from the table, since these cancel out. If you study this topic in Student Book 2 you will need to assess all forces, since the frictional element is closely

related to the normal reaction force. For Student Book 1, simply specify 'horizontal' or 'vertical' forces to be clear that you know you are ignoring some forces in the model.

$(1) + (2) : 6g - F = 21a$

a For system in equilibrium, $a = 0$ so $F = 6g = 58.8\,\text{N}$

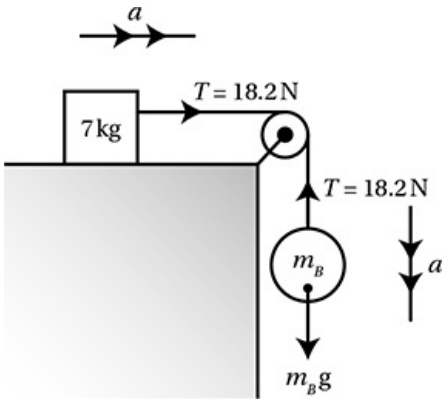
b Smooth contact means $F = 0$

$(2) : T = 15a$

Substitute into $(1) : 6g - 15a = 6a$

$a = \frac{6}{21}g = 2.8\,\text{m s}^{-2}$ so $T = 42\,\text{N}$

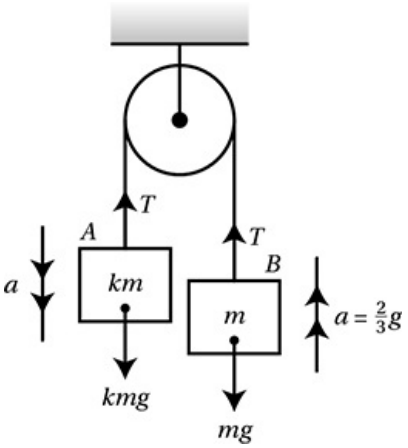
4



Forces on box: $T = 7a = 18.2$ therefore $a = 2.6\,\text{m s}^{-2}$

Forces on ball: $m_B g - T = m_B a$ so $m_B = \frac{T}{g - a} = \frac{18.2}{9.8 - 2.6} = 2.53\,\text{kg}$

5 Call the greater mass (km) particle A and the other particle B:



Forces on A: $kmg - T = kma$ $\quad (1)$

Forces on B: $T - mg = ma$ $\quad (2)$

$(1) + (2) : (k - 1)mg = (k + 1)ma$

Since $a = \frac{2}{3}g, k - 1 = \frac{2}{3}(k + 1)$ so $\frac{1}{3}k = \frac{5}{3}$ therefore $k = 5$

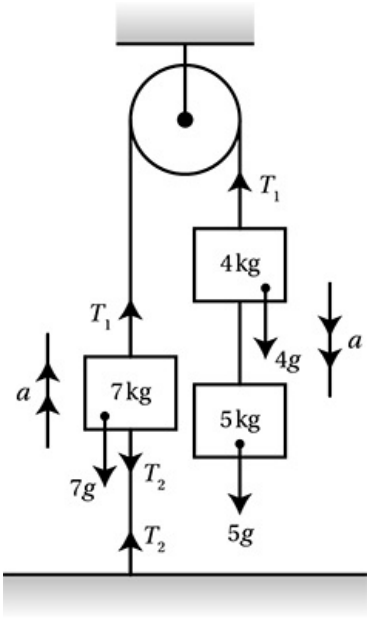
Tip

If the question had not specified that $k > 1$, you would not have known at the start which particle was heavier, so would have had to allow that you might have the acceleration misdirected.

Within the solution, this could be accounted for by allowing $a = \pm\frac{2}{3}g$, which gives rise to

the additional result $k = \frac{1}{5}$.

6



Taking the $4\,\text{kg}$ and $5\,\text{kg}$ particles as a single system of mass $9\,\text{kg}$, since you have no interest in the tension of the string linking them:

Forces on the $9\,\text{kg}$ system : $9g - T_1 = 9a$ $\quad (1)$

Forces on the $7\,\text{kg}$ particle : $T_1 - 7g - T_2 = 7a$ $\quad (2)$

$(1) + (2) : 2g - T_2 = 16a$

a With the string intact, $a = 0$ so $T_2 = 2g = 19.6\,\text{N}$

Tip

That is, the string provides the offset for the system, which is $2\,\text{kg}$ out of balance.

b With the string removed, $T_2 = 0$ so $a = \frac{2g}{16} = 1.225\,\text{m s}^{-2}$

7 The system is in equilibrium so the magnitude of the tension must equal the weight of the crate.

$mg = \sqrt{15^2 + 86^2} = 87.3\,\text{N}$ so $m = \frac{87.3}{9.8} = 8.91\,\text{kg}$

8 **a** $\left|\text{T}\right| = \sqrt{15^2 + 12^2} = 19.2\,\text{N}$

The system is in equilibrium so the weight of Q must equal $19.2\,\text{N}$

$mg = 19.2$ so $m = 1.96\,\text{kg}$

b **i** Resolving vertical forces at P: $(\uparrow) : 12 + R - 8.5g = 0$ so $R = 8.5g - 12 = 71.3\,\text{N}$

ii Resolving horizontal forces at P: $(\rightarrow) : 15 - F = 0$ so frictional force $F = 15\,\text{N}$

9 Tension in the string must equal the weight of the hanging particle: $T = 3.5\,g = 34.3\,\text{N}$

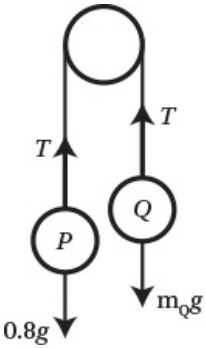
Resolving horizontal forces at the box: $(\rightarrow) : p - 34.3 = 0$ so $p = 34.3$

Resolving vertical forces at the box: $(\uparrow) : 28 + q - 6g = 0$ so $q = 6g - 28 = 30.8$

The magnitude of $\mathbf{F}$ is therefore $|\mathbf{F}| = \sqrt{34.3^2 + 30.8^2} = 46.1\,\text{N}$

IRANINOVIN

10



Define positive movement direction with P rising and Q falling.

Considering forces acting on P: $T - 0.8g = 0.8a \qquad (1)$

Considering forces acting on Q: $m_Q g - T = m_Q a \qquad (2)$

$(1) + (2)$: $(m_Q - 0.8)g = (m_Q + 0.8)a$ so $a = \dfrac{(m_Q - 0.8)}{(m_Q + 0.8)}g$

Constant acceleration problem: $u = 0$, $s = \pm 2$, $t = 1.6$. Find a:

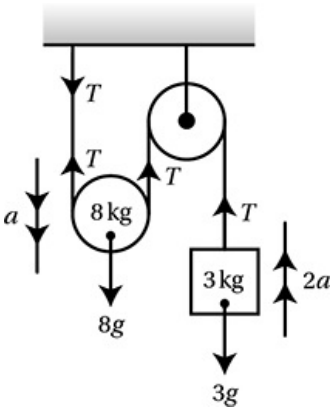
$a = 2\dfrac{s - ut}{t^2} = \dfrac{\pm 4}{1.6^2} = \pm 1.5625$

Equating the two expressions for acceleration:

$\dfrac{(m_Q - 0.8)}{(m_Q + 0.8)}g = \pm 1.5625$ so $m_Q(9.8 \mp 1.5625) = 0.8(9.8 \pm 1.5625)$

$m_Q = 0.8\dfrac{(9.8 \pm 1.5625)}{(9.8 \mp 1.5625)}$ therefore $m_Q = 1.10$ or 0.580

11



a Cylinder moves down distance x; both the pulley and the tape end attached to the ceiling are fixed, so both sides of the cylinder the tape must increase by x. Since this can only come from the single free tape end, the box will rise by $2x$.

b If every movement of the box is double the movement of the cylinder (in the opposite direction), then its speed and acceleration must also be double those of the cylinder.

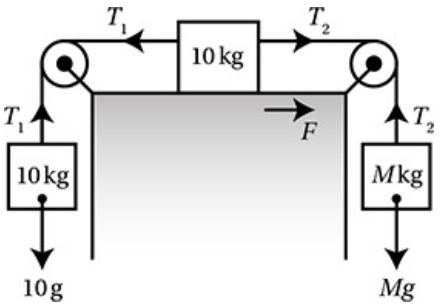
$\text{Acceleration of the box} = 2a$

c Forces on the box: $T - 3g = 3 \times 2a = 6a \qquad (1)$

Forces on the cylinder: $8g - 2T = 8a \qquad (2)$

$4(1) - 3(2) : 10T - 36g = 0$ therefore $T = 3.6g = 35.28\,\text{N}$

12



Since you do not know whether M is greater or less than $10\,\text{kg}$, you do not know the direction of F so must allow that its extreme values are $\pm 49\,\text{N}$ when considering the limits of the equilibrium.

The diagram depicts $M < 10$ for a positive value of F.

Forces on the suspended $10\,\text{kg}$ box : $10g - T_1 = 0 \qquad (1)$

Forces on the $10\,\text{kg}$ box on the table: $T_1 - T_2 - F = 0 \qquad (2)$

Forces on the $M\,\text{kg}$ box : $T_2 - Mg = 0 \qquad (3)$

$(1) + (2) + (3) : 10g - Mg - F = 0$ so $M = 10 - \dfrac{F}{g}$

At the limits of the values available for F you see $M = 10 - \dfrac{49}{g} = 5$ and $M = 10 + \dfrac{49}{g} = 15$

The system will remain in equilibrium for $5 \leqslant M \leqslant 15$.

13 a) Connection between cars not rigid.

b) Mass of steel chain not negligible compared to mass of chair.

c) Mass of steel chain compared to mass of car.

d) Mass of rod of pendulum compared to mass of bob on the end.

e) Cable needs to be able to stretch.

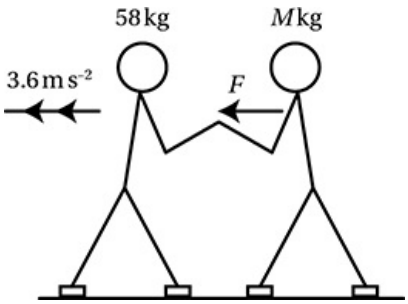
f) Bottle will be drawn out between the pliers.

g) Mass of tow bar compared to car and caravan.

h) Steel cable needs to stretch when under tension.

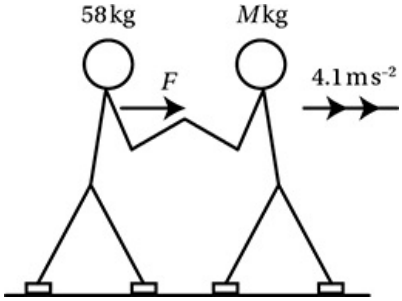
MIXED PRACTICE 22

1 Forces on $58\,\text{kg}$ skater :



$F = 58 \times 3.6$

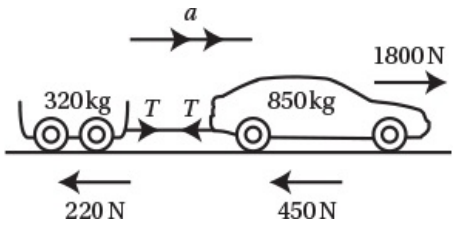
Forces on $M\,\text{kg}$ skater :



$F = M \times 4.1$

Putting these two equations equal to each other: $4.1M = 208.8$ so $M = 50.9\,\text{kg}$

2



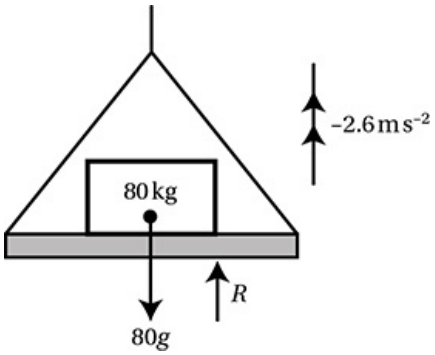
a External forces acting on the total $1170\,\text{kg}$ system :

$1800 - 220 - 450 = 1170a$ therefore $a = 0.966\,\text{m s}^{-2}$

b Forces acting solely on the trailer:

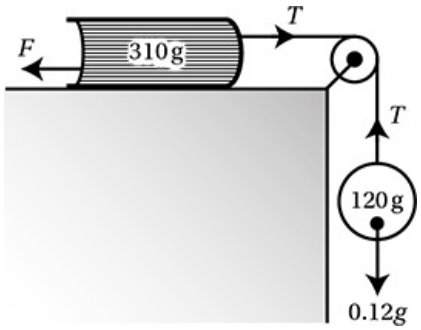
$T - 220 = 320a$ therefore $T = 220 + 320a = 529\,\text{N}$

3



Forces acting on crate: $R - 80g = 80 \times (-2.6)$ so $R = 80(g - 2.6) = 576\,\text{N}$

4



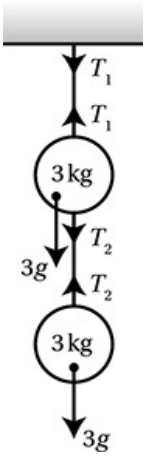
System in equilibrium

Vertical forces acting on the ball: $T = 0.12g$

Horizontal forces acting on the book: $T = F$

Therefore $F = 0.12g = 1.18\,\text{N}$

5



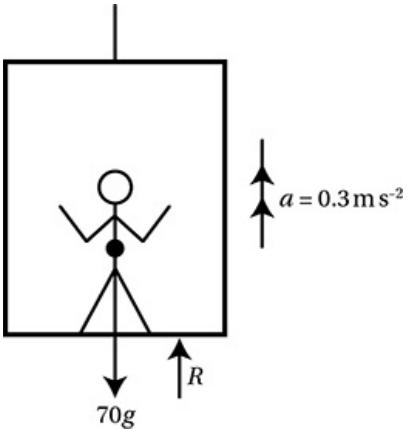
System in equilibrium.

Forces on lower ball: $3g = T_2 = 29.4$

Forces on upper ball: $3g + T_2 = T_1 = 58.8$

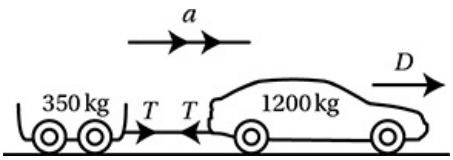
So the tensions are $29.4\,\mathrm{N}$ in the lower string and $58.8\,\mathrm{N}$ in the upper string.

6



Vertical forces acting on man: $R - 70g = 70 \times 0.3$ so $R = 70(g + 0.3) = 707\,\mathrm{N}$

7



External forces acting on the $(1550\,\mathrm{kg})$ system as a whole:

$D = 1550a$

a When $a = 1.8\,\mathrm{m\,s^{-2}}$, $D = 2790\,\mathrm{N}$

b

Tip

To work out forces, you need to know acceleration (and mass, but that is unchanged). You have to use the speed, distance and time elements with constant acceleration equations to determine a before you can address the force question.

While accelerating: $u = 9.2$, $a = 1.8$, $t = 4$. Find v:

$v = u + at = 9.2 + 1.8 \times 4 = 16.4\,\mathrm{m\,s^{-1}}$

While braking, $u = 16.4$, $v = 0$, $s = 26$. Find a:

$$a = \frac{v^2 - u^2}{2s} = -\frac{16.4^2}{2 \times 26} = -5.17\,\mathrm{m\,s^{-2}}$$

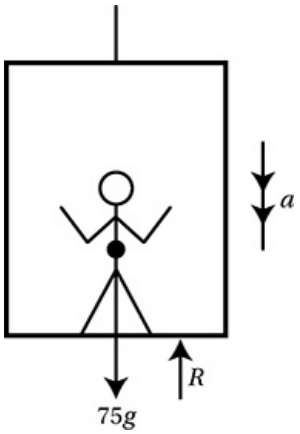
Considering only forces acting on the trailer (keeping T as tension as in the diagram, but

knowing that you should get a negative value – i.e. thrust):

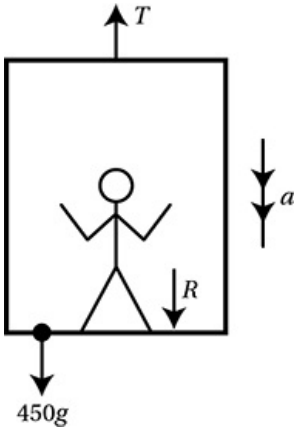
$T = 350 \times (-5.17) = -1810\,\text{N}$

There is a thrust of $1810\,\text{N}$ in the tow bars during braking.

8 Forces acting on person: $75g - R = 75a$



Forces acting on lift: $R + 450g - T = 450a$



a $a = -5.2\,\text{m s}^{-2}$

From the first equation: $R = 75(g - a) = 1125\,\text{N}$

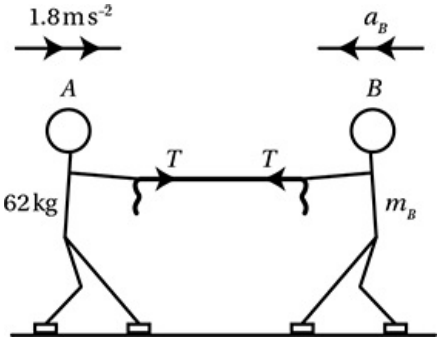
b **i** $R = 577.5\,\text{N}$

From the first equation: $a = g - \dfrac{R}{75} = 2.1$

The lift is accelerating downwards at $2.1\,\text{m s}^{-2}$

ii From the second equation: $T = R + 450(g - a) = 4042.5\,\text{N}$

9



Forces acting on Skater A: $T = 62 \times 1.8 = 111.6\,\text{N}$

Forces acting on Skater B: $T = m_B \times a_B$

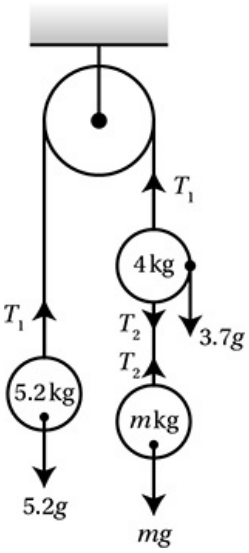
Constant acceleration problem: $u = 0$, $t = 1.5$, $s = 5$. Find a:

$$a = \frac{s - ut}{\frac{1}{2}t^2} = \frac{5}{1.125} = 4.44\,\text{m s}^{-2}$$

So the approach acceleration $1.8 + a_B = 4.44$ therefore $a_B = 2.64\,\text{m s}^{-2}$

Then $m_B = \dfrac{T}{a_B} = \dfrac{111.6}{2.64} = 42.2\,\text{kg}$

10



a Forces in equilibrium.

Forces acting on $5.2\,\text{kg}$ ball : $T_1 = 5.2g = 50.96\,\text{N}$

Forces acting on $3.7\,\text{kg}$ ball : $T_1 = T_2 + 3.7g$

$T_2 = T_1 - 3.7g = 14.7\,\text{N}$

Forces acting on $m\,\text{kg}$ ball :

$T_2 = mg$ so $m = \dfrac{T_2}{g} = 1.5$ therefore $m = 1.5$

b Tensions are:

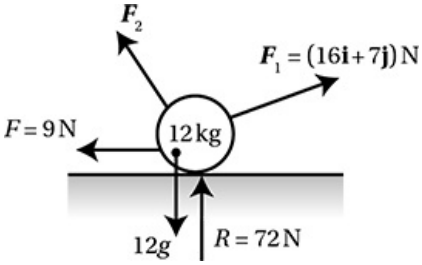
$50.96\,\text{N}$ between the $5.2\,\text{kg}$ and $3.7\,\text{kg}$ balls

$14.7\,\text{N}$ between the $3.7\,\text{kg}$ and $1.5\,\text{kg}$ balls.

Tip

You should realise that given this frictionless system is in equilibrium, the total mass on the right has to equal the mass on the left, so $m = 1.5$ from a brief inspection of the diagram. You can argue this in the exam, but state your reasoning clearly. The question says ‘find’ not ‘write down’ so the examiner is clearly expecting more than just the answer alone.

11



Note that you do not know the direction of the frictional force so must consider two scenarios, $F = 9\,\text{N}$ or $F = -9\,\text{N}$, which would represent friction acting to the right against a tendency to leftward movement.

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Let $\mathbf{F}_2 = (x\mathbf{i} + y\mathbf{j})\,\mathrm{N}$

Considering vertical forces (in equilibrium): $y + 7 + 72 = 12g = 117.6$ so $y = 38.6$

Considering horizontal forces: $F - x - 16 = 0$ so $x = F - 16$

So $x = -7$ or $x = -25$, for the two possible frictions

Magnitude of $\mathbf{F}_2$ is $\sqrt{x^2 + y^2} = 39.2\,\mathrm{N}$ or $46.0\,\mathrm{N}$

12 a Resolving vertical forces at the $4.2\,\mathrm{kg}$ particle :

$(\uparrow) : 8 + R - 4.2g = 0$ so $R = 4.2g - 8 = 33.2\,\mathrm{N}$

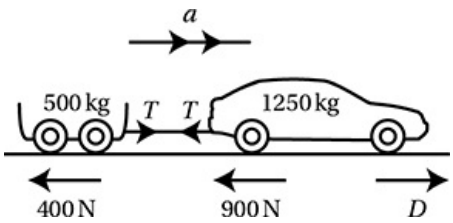
b It is not stated in which direction the frictional force is acting. Given that uncertainty, resolve horizontal forces at the $4.2\,\mathrm{kg}$ particle :

$(\rightarrow) : -22 + T \pm 4 = 0$ so $T = 22 \pm 4 = 18$ or 26

c With the system in equilibrium, the weight of the hanging particle equals the tension:

$mg = 18$ or 26 so $m = 1.84\,\mathrm{kg}$ or $2.65\,\mathrm{kg}$

13



External horizontal forces acting on the total $1750\,\mathrm{kg}$ system : $D - 1300 = 1750a$ (1)

Horizontal forces acting just on the trailer: $T - 400 = 500a$ (2)

a If $a = 0\,\mathrm{m\,s^{-2}}$ then :

$(1) :\ D = 1300\,\mathrm{N}.\ (2) : T = 400\,\mathrm{N}$

Driving force is $1300\,\mathrm{N}$ and towbar tension is $400\,\mathrm{N}$.

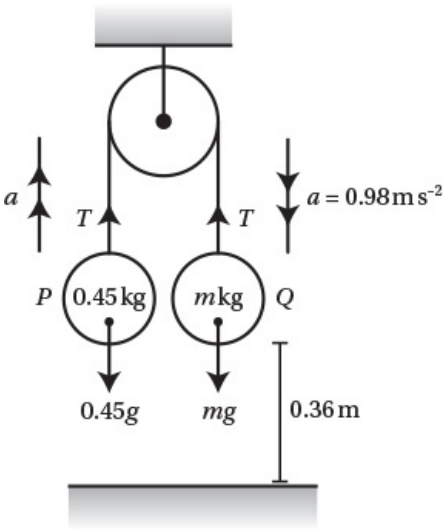
b If $a = 0.6\,\mathrm{m\,s^{-2}}$ then :

$(1) : D = 1300 + 1750 \times 0.6 = 2350\,\mathrm{N}$

$(2) : T = 400 + 500 \times 0.6 = 700\,\mathrm{N}$

Driving force is $2350\,\mathrm{N}$ and towbar tension is $700\,\mathrm{N}$.

14



Considering forces acting on P: $T - 0.45g = 0.45a$ (1)

Considering forces acting on Q: $mg - T = ma$ (2)

a $a = 0.98$ so $(1) : T = 0.45(g + a) = 4.851\,\mathrm{N}$

b $(2):\ m(g-a)=T$ so $m=\dfrac{T}{g-a}=\dfrac{4.851}{9.8-0.98}=0.55$

c Constant acceleration problem: $a=0.98$, $u=0$, $s=0.36$. Find v:

$v=\sqrt{u^2+2as}=\sqrt{2\times 0.98\times 0.36}=0.84$

The speed at which Q strikes the surface is $0.84\,\mathrm{m\,s^{-1}}$

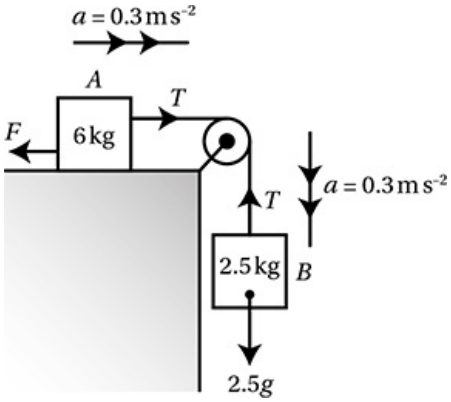
d With Q on the ground, acceleration of P becomes equal to g

New constant acceleration problem:

$u=0.84$, $a=-9.8$, $v=0$. Find s: $s=\dfrac{v^2-u^2}{2a}=\dfrac{-0.84^2}{2\times(-9.8)}=0.036$

In total, P starts $0.36\,\mathrm{m}$ above the surface, rises a further $0.36\,\mathrm{m}$ as Q falls and then $0.036\,\mathrm{m}$ while Q is on the surface, for a total maximum height of $0.756\,\mathrm{m}$ above the surface.

15



a Considering horizontal forces on A: $T-F=6\times 0.3$ $\qquad(1)$

Considering vertical forces on B: $2.5g-T=2.5\times 0.3$ $\qquad(2)$

$(1)+(2):\ 2.5g-F=8.5\times 0.3=2.55$ so $F=21.95\,\mathrm{N}$

b After $1.2\,\mathrm{m}$ of travel at constant acceleration, tension is removed.

Constant acceleration problem: $a=0.3$, $u=0$, $s=1.2$. Find v:

$v=\sqrt{u^2+2as}=\sqrt{2\times 0.3\times 1.2}=0.849\,\mathrm{m\,s^{-1}}$

At the moment B hits the floor, A is travelling horizontally with velocity $0.849\,\mathrm{m\,s^{-1}}$.

Thereafter, A is affected by only one horizontal force, $F=21.95\,\mathrm{N}$ against the direction of travel.

$-21.95=6\times a$ therefore $a=-\dfrac{21.95}{6}=-3.66\ \mathrm{m\,s^{-2}}$

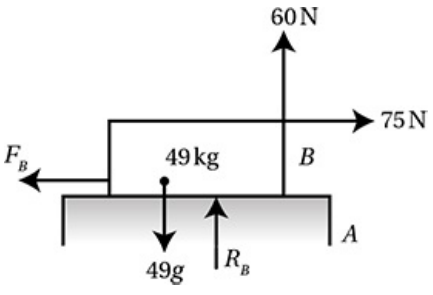
In the absence of interruption, how far will A travel before coming to rest?

Constant acceleration problem: $a=-3.66$, $u=0.849$, $v=0$. Find s:

$s=\dfrac{v^2-u^2}{2a}=\dfrac{-0.849^2}{2\times(-3.66)}=0.0984\,\mathrm{m}$

Box A will only travel $0.0984\,\mathrm{m}$ of the available $1.8\,\mathrm{m}$, so will not reach the pulley.

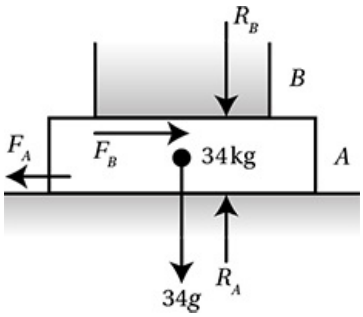
16 Considering just forces acting on Box B:



Vertical forces in equilibrium: $60 + R_B = 49g$ so $R_B = 49g - 60 = 420.2\,\text{N}$

Horizontal forces in equilibrium: $75 - F_B = 0$ so $F_B = 75\,\text{N}$

Considering just forces acting on Box A, and using Newton's third law where appropriate:



Vertical forces in equilibrium: $R_A = 34g + R_B$ so $R_A = 753.4\,\text{N}$

Horizontal forces in equilibrium: $F_A = F_B = 75\,\text{N}$

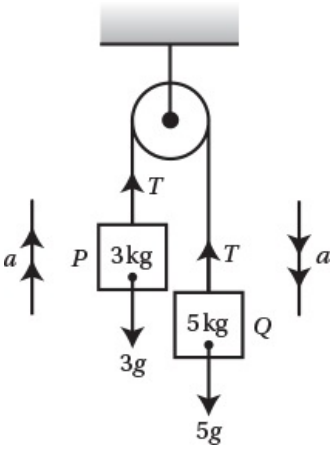
a Normal reaction force between A and the ground $R_A = 753.4\,\text{N}$

b Friction force between A and the ground $F_A = 75\,\text{N}$

c To resist movement, the frictional forces have to oppose the horizontal component of the string tension, which is now $75k\,\text{N}$.

The maximum friction is lower between the two boxes than between A and the ground, so this is the one which will be exceeded first as k increases; B will move on A but A will stay still.

17



Considering forces acting only on P: $T - 3g = 3a \qquad (1)$

Considering forces acting only on Q: $5g - T = 5a \qquad (2)$

$(1) + (2) : 2g = 8a$ therefore $a = \dfrac{2g}{8} = 2.45\,\text{m s}^{-2}$

a Constant acceleration problem: $a = 2.45$, $s = 2.5$, $u = 0$. Find t:

$s = ut + \dfrac{1}{2}at^2$ so $2.5 = 0 + \dfrac{1}{2} \times 2.45 \times t^2$ therefore $t = \sqrt{\dfrac{2.5}{1.225}} = 1.43\,\text{s}$

It takes 1.43 seconds for Q to reach the ground.

b Also need to know v at the end of this initial movement.

Constant acceleration problem: $a = 2.45$, $s = 2.5$, $u = 0$. Find v:

$v = \sqrt{u^2 + 2as} = \sqrt{2 \times 2.45 \times 2.5} = 3.5\,\text{m s}^{-1}$

So with Q on the ground, the only force acting on P is its weight. Continuing to use upward as the dominant direction for P, $a = -g$

Constant acceleration problem: $a = -9.8$, $u = 3.5$, $v = 0$. Find s:

$$s = \frac{v^2 - u^2}{2a} = \frac{-3.5^2}{2 \times (-9.8)} = 0.625\,\text{m}$$

So P rises a further $0.625\,\text{m}$, for a total of $3.125\,\text{m}$ above its starting point.

c Using the same constant acceleration data, find t when $s = 0$ (or $v = -u$):

$$t = \frac{v - u}{a} = \frac{-3.5 - 3.5}{-9.8} = 0.714\,\text{s}$$

At this time, a total of $1.43 + 0.714 = 2.14\,\text{s}$ after the system was released, P has dropped back to the position it had reached when Q touched the ground, and the string is again taut.

Cross topic review exercise 1 worked solutions

Cross-topic review exercise 1

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

1 a $\log(5^3 \div 20 \times 16) = \log\left(\dfrac{5 \times 5 \times 5 \times 4 \times 4}{5 \times 4}\right)$

$= \log(5 \times 5 \times 4)$

$= \log(100)$

$= 2$

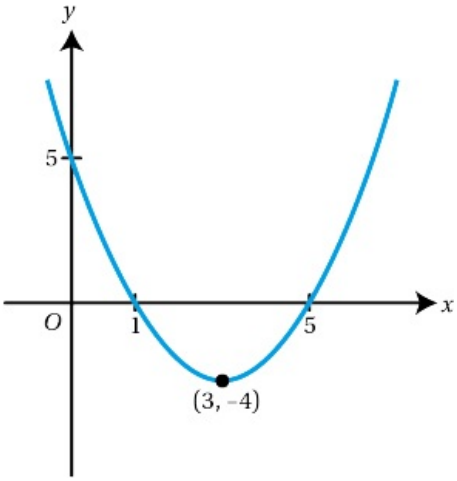
b $\ln\left(\dfrac{45}{4}\right) = \ln\left(\dfrac{5 \times 3^2}{2^2}\right) = \ln 5 + 2\ln 3 - 2\ln 2 = z + 2y - 2x$

c $\ln K = 2 - \ln c \Leftrightarrow K = \mathrm{e}^{2-\ln c} = \dfrac{\mathrm{e}^2}{\mathrm{e}^{\ln c}} = \dfrac{\mathrm{e}^2}{c}$

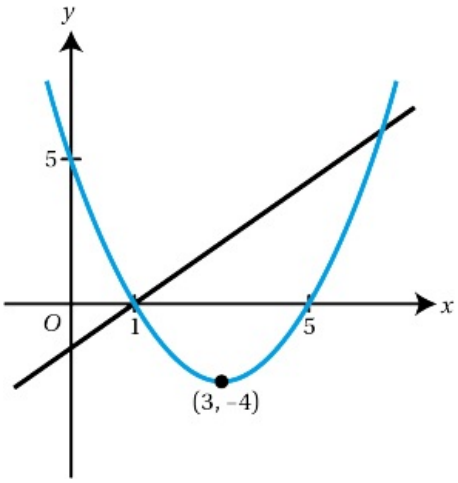
2 a i $(x-3)^2 - 4$

ii Translation by $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

iii $x^2 - 6x + 5 = (x-1)(x-5)$, so x-intercepts are $(1, 0)$ and $(5, 0)$. The y-intercept is $(0, 5)$. From question 2ai, the minimum point is $(3, -4)$.

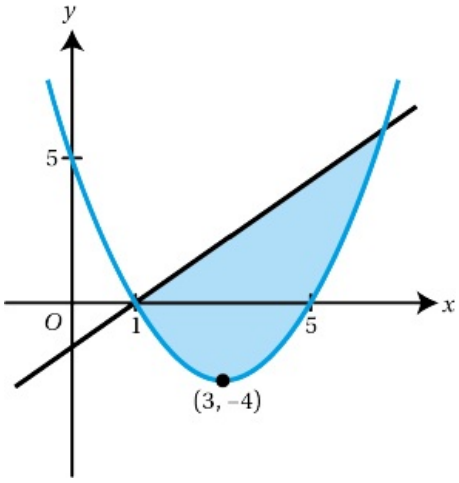


b i



ii $x^2 - 6x + 5 = x - 1 \Leftrightarrow x^2 - 7x + 6 = 0 \Leftrightarrow (x-1)(x-6) = 0 \therefore x = 1$ or 6

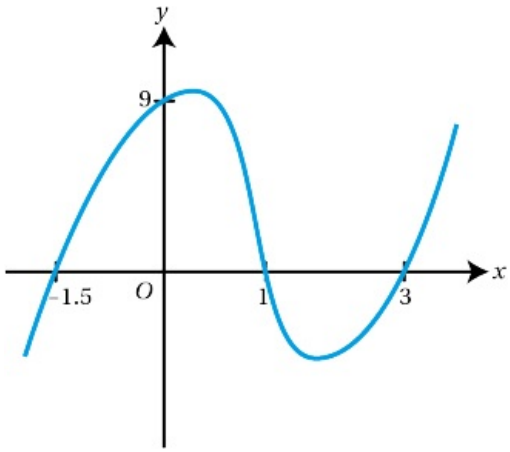
iii The required region is above the curve and below the line.



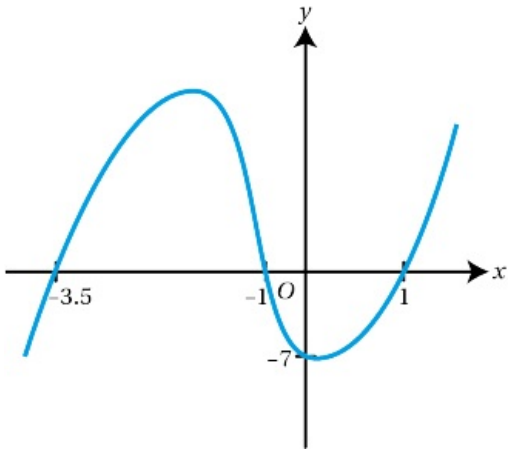
3 **a** $p(3) = 2 \times 27 - 5 \times 9 - 6 \times 3 = 0$

b $p(x) = (x-3)(2x^2 + x - 3) = (x-3)(2x+3)(x-1)$

c Cubic graph with x-intercepts -1.5, 1, 3 and y-intercept 9.



d The graph has been translated 2 units to the left.



4 The equation of the circle is $(x-12)^2 + (y-9)^2 = 145$.

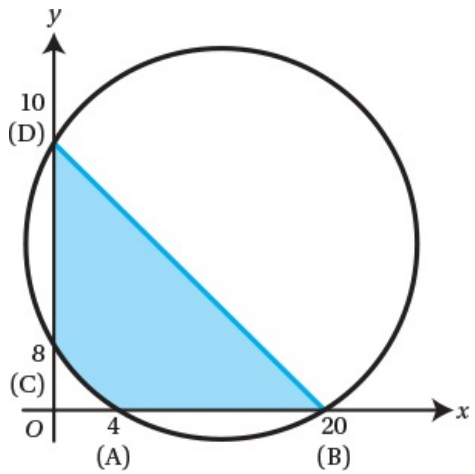
Intersections with x-axis: when $y = 0$:

$(x-12)^2 + 9^2 = 145 \Rightarrow x - 12 = \pm\sqrt{64} = \pm 8 \Rightarrow x = 4$ or 20

Intersections with y-axis: when $x = 0$:

$12^2 + (y-9)^2 = 145 \Rightarrow y - 9 = \pm\sqrt{1} = \pm 1 \Rightarrow y = 8$ or 10

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The required area equals (area of triangle OBD) − (area of triangle OAC)

$\text{Area} = \dfrac{20 \times 10}{2} - \dfrac{4 \times 8}{2} = 84$

5 Substitute $y = \mathrm{e}^x$: $3y^2 - 7y + 2 = 0$:

$\Leftrightarrow (3y - 1)(y - 2) = 0$

$\Leftrightarrow y = \dfrac{1}{3}$ or $y = 2 \Leftrightarrow x = \ln\left(\dfrac{1}{3}\right) = -\ln 3$ or $x = \ln 2$.

6 **a** $C = 2,\ 2 + A = 7 \Rightarrow A = 5$

b $3 = 2 + 5\mathrm{e}^{-k\times 2} \Leftrightarrow \mathrm{e}^{-2k} = \dfrac{1}{5} \Leftrightarrow \mathrm{e}^{2k} = 5 \Leftrightarrow k = \dfrac{\ln 5}{2}$

7 **a** $(1 + 4x + 6x^2 + 4x^3 + x^4) + (1 - 4x + 6x^2 - 4x^3 + x^4) = 2 + 12x^2 + 2x^4$.

b Let $x = \sqrt{2}$: Notice that $\sqrt{2} > 1$, but because the power is even, $\left(\sqrt{2} - 1\right)^4$ is the same as $\left(\sqrt{2} - 1\right)^4$.

$\left(\sqrt{2} + 1\right)^4 + \left(\sqrt{2} - 1\right)^4 = 2 + 12\left(\sqrt{2}\right)^2 + 2\left(\sqrt{2}\right)^4 = 2 + 12(2) + 2(4) = 34$

8 **a** $x = 3 - \dfrac{1}{x} \quad \therefore x + \dfrac{1}{x} = 3.$

b **i** $\left(x + \dfrac{1}{x}\right)^2 = x^2 + 2x\left(\dfrac{1}{x}\right) + \left(\dfrac{1}{x}\right)^2 = x^2 + 2 + \dfrac{1}{x^2}$

$\left(x + \dfrac{1}{x}\right)^3 = x^3 + 3x^2\left(\dfrac{1}{x}\right) + 3x\left(\dfrac{1}{x}\right)^2 + \left(\dfrac{1}{x}\right)^3$

$= x^3 + 3x + \dfrac{3}{x} + \dfrac{1}{x^3}$

ii $\left(x + \dfrac{1}{x}\right)^2 = 9 \Rightarrow 9 = x^2 + 2 + \dfrac{1}{x^2} \Rightarrow x^2 + \dfrac{1}{x^2} = 7$

$\left(x + \dfrac{1}{x}\right)^3 = 27 \Rightarrow 27 = x^3 + 3x + \dfrac{3}{x} + \dfrac{1}{x^3}$

$\Rightarrow x^3 + \dfrac{1}{x^3} = 27 - 3\left(x + \dfrac{1}{x}\right) = 27 - 3(3) = 18$

9 **a** The horizontal stretch scale factor is $\dfrac{1}{k}$.

b $\ln kx = \ln k + \ln x$, so the vertical translation is by $\ln k$ units up.

10 $\dfrac{n(n-1)}{2} = k \Leftrightarrow n^2 - n - 2k = 0$ so $n = \dfrac{1 + \sqrt{1 + 8k}}{2}$.

(You take the positive sign because $n > 0$.)

11 **a** Now: $t = 0,\ N = 42$; 9 years later: $t = 9,\ N = 84$.

A is the initial value, so $A = 42$.

$84 = 42 \times 2^{9k} \Rightarrow 2^{9k} = 2^1 \Rightarrow k = \dfrac{1}{9}$

$84 = 42\mathrm{e}^{9m} \Rightarrow \mathrm{e}^{9m} = 2 \Rightarrow m = \dfrac{1}{9}\ln 2 \approx 0.077$

b When $N = 100 : 100 = 42 \times \mathrm{e}^{mt}$

$\Rightarrow mt = \ln\left(\dfrac{100}{42}\right) \Rightarrow t = \dfrac{0.8675}{0.077} = 11.3$

c Using the second equation, the rate of increase equals mN.

When $t = 35 : N = 42\mathrm{e}^{35m} = 622.2$, so the rate of increase is $622.2 \times 0.077 = 47.9$.

12 The discriminant is: $(m+3)^2 - 4(m+1) = m^2 + 2m + 5 = (m+1)^2 + 4 > 0$ for all m.

Hence the graph always has two x-intercepts.

13 a $y = \mathrm{e}^x + \dfrac{1}{\mathrm{e}^x} \Rightarrow \mathrm{e}^{2x} - y\mathrm{e}^x + 1 = 0$

This is a quadratic equation in e^x

$\Rightarrow \mathrm{e}^x = \dfrac{y \pm \sqrt{y^2 - 4}}{2} \Rightarrow x = \ln\left(\dfrac{y + \sqrt{y^2 - 4}}{2}\right)$ or $\ln\left(\dfrac{y - \sqrt{y^2 - 4}}{2}\right)$

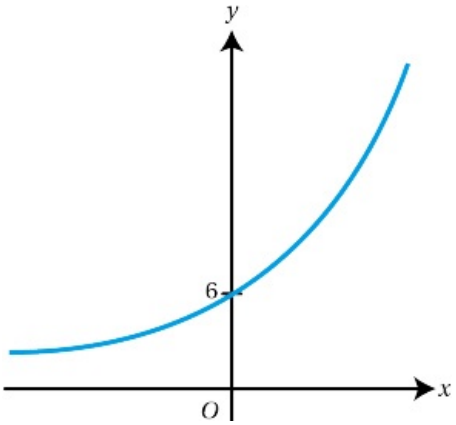
b i $y > 0$ because it is a sum of exponentials.

In the quadratic formula you need $y^2 - 4 > 0$, so $y > 2$.

ii There are two possible values of x.

$$\begin{aligned} \Rightarrow x_1 + x_2 &= \ln\left(\frac{y + \sqrt{y^2 - 4}}{2}\right) + \ln\left(\frac{y - \sqrt{y^2 - 4}}{2}\right) \\ &= \ln\left(\frac{\left(y + \sqrt{y^2 - 4}\right)\left(y - \sqrt{y^2 - 4}\right)}{4}\right) \\ &= \ln\left(\frac{4}{4}\right) = 0 \end{aligned}$$

14 a Since $b > 1$, this is exponential growth. The y-intercept is 6, and the x-axis is an asymptote.



b $54b^2 = 6b^x \Leftrightarrow b^x = 9b^2 \Leftrightarrow x = \log_b(9b^2)$

$= \log_b 9 + \log_b(b^2) = 2\log_b 3 + 2 = 2(\log_b 3 + 1)$

Cross topic review exercise 2 worked solutions

Cross-topic review exercise 2

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

1 $\mathrm{f}'(x) = 3ax^2 + 2bx + 4$

$\mathrm{f}''(x) = 6ax + 2b$

So, $\mathrm{f}'(2) = 0 \Rightarrow 12a + 4b + 4 = 0 \Rightarrow 3a + b = -1$

$\mathrm{f}''(2) = 10 \Rightarrow 12a + 2b = 10 \Rightarrow 6a + b = 5$

Solving simultaneously gives $a = 2$, $b = -7$.

2 Period of $\sin 4x$ is 90°, period of $\sin 6x$ is 60°.

Period of the overall function is the LCM of the constituent periods, i.e. 180°.

3 **a** $\overrightarrow{MP} = \overrightarrow{MB} + \overrightarrow{BP} = \frac{1}{3}\mathbf{a} + \frac{3}{4}\mathbf{b}$

$\overrightarrow{QN} = \overrightarrow{QD} + \overrightarrow{DN} = \frac{1}{2}\mathbf{b} + \frac{2}{9}\mathbf{a}$

b $\overrightarrow{MP} = \frac{1}{12}(4\mathbf{a} + 9\mathbf{b})$, $\overrightarrow{QN} = \frac{1}{18}(4\mathbf{a} + 9\mathbf{b})$

$\therefore \overrightarrow{MP} = \frac{3}{2}\overrightarrow{QN}$ and so they are parallel.

4 $\tan(2\theta) = \sqrt{3}$, $0 \leqslant 2\theta \leqslant 360^\circ$

$\arctan\sqrt{3} = 60^\circ$

$\Rightarrow 2\theta = 60^\circ$ or $60 + 180 = 240^\circ \Rightarrow \theta = 30^\circ, 120^\circ$

5 **a** $\mathrm{f}(-1) = (-1)^3 - 5(-1)^2 - (-1) + 5 = -1 - 5 + 1 + 5 = 0$.

Hence, by factor theorem, $(x + 1)$ is a factor of $\mathrm{f}(x)$.

You can now factorise $\mathrm{f}(x)$, using polynomial division or comparing coefficients:

$x^3 - 5x^2 - x + 5 = (x + 1)(x^2 - 6x + 5) = (x + 1)(x - 1)(x - 5)$

b The x-intercepts are -1, 1 and 5. You need to find the two parts of the area separately.

$$\int_{-1}^{1} \mathrm{f}(x)\mathrm{d}x = \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 - \frac{1}{2}x^2 + 5x\right]_{-1}^{1} = \frac{20}{3}$$

$$\int_{-1}^{5} \mathrm{f}(x)\mathrm{d}x = \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 - \frac{1}{2}x^2 + 5x\right]_{1}^{5} = \frac{128}{3}$$

So the total area is $\frac{20}{3} + \frac{128}{3} = \frac{148}{3}$.

6 **a** $\frac{(x-2)(x-6)}{\sqrt{x}} = \frac{x^2 - 8x + 12}{x^{\frac{1}{2}}} = x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + 12x^{-\frac{1}{2}}$ i.e. $a = \frac{3}{2}$, $b = \frac{1}{2}$, $c = -\frac{1}{2}$

b $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} - 6x^{-\frac{3}{2}}$

At $x = 4$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{4}$ $\therefore$ gradient of normal $= -4$

When $x = 4$, $y = -2$. so, $y - (-2) = -4(x - 4) \Rightarrow y = -4x + 14$

c i At the intersection with the x-axis, $y = 0 : 0 = -4x + 14 \Rightarrow x = \frac{7}{2}$.

At the intersection with the y-axis, $x = 0 : y = 14$.

So, $P\left(\frac{7}{2}, 0\right), Q(0, 14)$

ii Area $= \frac{1}{2} \times \frac{7}{2} \times 14 = \frac{49}{2}$

7 a $\frac{1}{\cos x} - \cos x = \frac{1 - \cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x} = \sin x \tan x$

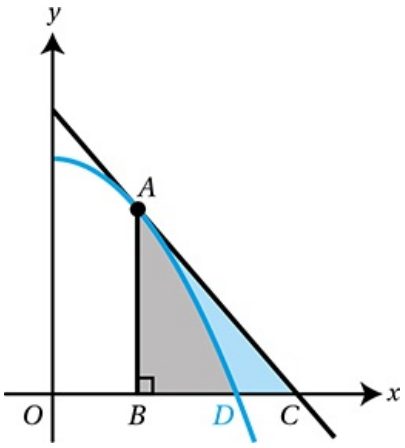
b i $\sin x - \cos x = \frac{1}{\sin x} - \frac{1}{\cos x} \Rightarrow \sin x - \frac{1}{\sin x} = \cos x - \frac{1}{\cos x}$

$\Rightarrow \frac{1 - \sin^2 x}{\sin x} = \sin x \tan x$ (from question 7a)

$\Rightarrow \cos^2 x = \sin^2 x \tan x \Rightarrow \tan^3 x = 1 \Rightarrow \tan x = 1$

ii $\tan x = 1 \Rightarrow x = 225^\circ$

8 The shaded area can be found as (area of triangle ABC) − (area S under the curve).



The coordinates of the marked points are $A(1, 8)$, $B(1, 0)$.

D is where the graph crosses the x-axis: $9 - x^2 = 0$, so D is the point $(3, 0)$.

To find coordinates of C you need the equation of the tangent at A:

$\frac{dy}{dx} = -2x = -2$ at $x = 1$

The equation of the tangent at A is: $y - 8 = -2(x - 1)$. Its x-intercept is when $y = 0$:

$-8 = -2(x - 1) \Rightarrow x = 5$

Hence C is the point $(5, 0)$.

Triangle ABC has base $BC = 4$ and height $AB = 8$, so its area is $\frac{4 \times 8}{2} = 16$.

The area marked S is: $\int_1^3 9 - x^2 dx = \left[9x - \frac{1}{3}x^3\right]_1^3 = \frac{28}{3}$

Hence the shaded area is $16 - \frac{28}{3} = \frac{20}{3}$.

9 Cosine rule: $(2x + 1)^2 = (x + 1)^2 + (x + 3)^2 - 2(x + 1)(x + 3)\left(-\frac{1}{2}\right)$

$\Rightarrow x^2 - 8x - 12 = 0 \therefore x = 9.29$

10 a $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$, so : $(x + h)^4 - x^4 = 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

b Let $f(x) = x^4$. Then:

$$\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}=\frac{(x+h)^4-x^4}{h}=4x^3+6x^2h+4xh^2+h^3$$

When h tends to 0, this expression tends to $4x^3 \therefore \mathrm{f}'(x)=4x^3$

11 a $\mathrm{f}(x)=x^2-6x+c$

Passes through $(2,\,0) \Rightarrow 0=4-12+c \Rightarrow c=8$

$\mathrm{f}(x)=x^2-6x+8$

b At $(2,\,0)$, $\mathrm{f}'(x)=-2$

So normal has gradient $\frac{1}{2}$ and therefore has equation $y=\frac{x}{2}-1$.

c Normal meets the curve when $\frac{x}{2}-1=x^2-6x+8$.

$\Rightarrow x^2-\frac{13}{2}x+9=0 \Rightarrow (x-2)\left(x-\frac{9}{2}\right)=0$

The second intersection point is $\left(\frac{9}{2},\,\frac{5}{4}\right)$.

12 a The coordinates of the vertex are $(4,\,5)$, so $p=4$, $q=5$.

Both curves pass through point M, which is where $x=0$:

$(0-3)^2=a(0-4)^2+5 \Rightarrow a=\frac{1}{4}$

b Solve the two equations simultaneously:

$(x-3)^2=\frac{1}{4}(x-4)^2+5 \Rightarrow 4(x^2-6x+9)=(x^2-8x+16)+20$

$\Rightarrow 3x^2-16x=0 \Rightarrow x=0$ or $\frac{16}{3}$

So the coordinates of the intersection points are $M(0,\,9)$ and $N\left(\frac{16}{3},\,\frac{49}{9}\right)$.

c The vertical distance is the difference between the y-coordinates when the x-coordinates are equal:

$d=\frac{1}{4}(x-4)^2+5-(x-3)^2=-\frac{3}{4}x^2+4x$

For the maximum distance, $\frac{\mathrm{d}d}{\mathrm{d}x}=0$:

$-\frac{3}{2}x+4=0 \Rightarrow x=\frac{8}{3} \therefore d_{\max}=-\frac{3}{4}\left(\frac{8}{3}\right)^2+4\left(\frac{8}{3}\right)=\frac{16}{3}$

13 The centre of the circle is $(0,\,0)$, so the tangent at A is perpendicular to OA, which has gradient $\frac{4}{3}$.

Hence the two tangents both have the gradient $-\frac{3}{4}$.

For a parabola $y=10-x^2$, the gradient is $\frac{\mathrm{d}y}{\mathrm{d}x}=2x$, so: $2x=-\frac{3}{4} \Rightarrow x=-\frac{3}{8}$

The coordinates of B are $\left(-\frac{3}{8},\,\frac{631}{64}\right)$.

14 a $\mathrm{f}(1)=1-1-3+3=0$ so $x=1$ is a root. It follows from the factor theorem that $(x-1)$ is a factor of $\mathrm{f}(x)$, so you can factorise: $\mathrm{f}(x)=(x-1)(x^2-3)$

So the roots of $\mathrm{f}(x)=0$ are $x=1,\pm\sqrt{3}$

b From question 14a, $\tan\theta=1,\pm\sqrt{3}$, so $\theta=45^\circ,\,60^\circ,\,120^\circ$

15 $\sin A=\frac{1}{3}$, so $\cos A=\sqrt{1-\left(\frac{1}{3}\right)^2}=\frac{\sqrt{8}}{3}$

16 a Since the circle passes through the origin, the coordinates of the centre are $(0,\,r)$, so the

equation of the circle is $x^2 + (y-r)^2 = r^2$.

Substitute x^2 from $y = ax^2$: $\frac{y}{a} + (y-r)^2 = r^2$

$\Leftrightarrow \frac{y}{a} + y^2 - 2yr + r^2 = r^2 \Leftrightarrow y^2 - 2yr + \frac{y}{a} = 0$

b The coordinates of any intersections satisfy the above equation, so:

$y\left(y - 2r + \frac{1}{a}\right) = 0 \Rightarrow y = 0$ or $2r - \frac{1}{a}$

The circle and the parabola both pass through the origin (where $y = 0$). Any other intersection points have a positive y-coordinate. So if there are no other intersections then:

$2r - \frac{1}{a} \leqslant 0 \Rightarrow r \leqslant \frac{1}{2a}$

17 a $(a^2 + b^2)^2 - 2a^2b^2 = (a^4 + 2a^2b^2 + b^4) - 2a^2b^2 = a^4 + b^4$

Hence, setting $a = \cos\theta$ and $b = \sin\theta$:

$\cos^4\theta + \sin^4\theta = (\cos^2\theta + \sin^2\theta)^2 - 2\cos^2\theta\sin^2\theta = 1 - 2\cos^2\theta\sin^2\theta$

b Write $a = \cos\theta$, $b = \sin\theta$. Using binomial expansion:

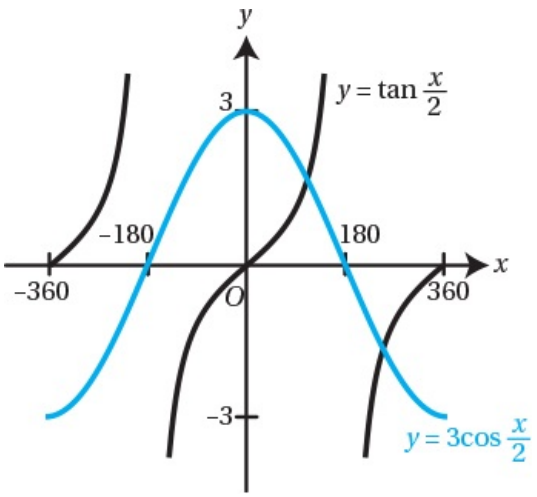
$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

Using $a^4 + b^4 = 1 - 2a^2b^2$ from question 17a:

$$\begin{aligned}(a-b)^4 &= 1 - 2a^2b^2 - 4a^3b + 6a^2b^2 - 4ab^3 \\ &= 1 + 4a^2b^2 - 4a^3b - 4ab^3 \\ &= 1 + 4a^2b^2 - 4ab\left(a^2 + b^2\right)\end{aligned}$$

But $a^2 + b^2 = 1$, so: $(a-b)^4 = 1 + 4a^2b^2 - 4ab = (1 - 2ab)^2$ as required.

18 a Both graphs have their period doubled. The cosine graph has amplitude 3, and y-intercept $(0, 3)$.



b Write $A = \frac{1}{2}x$ and use $\tan A = \frac{\sin A}{\cos A}$: $\tan(A) = 3\cos(A)$

$\Leftrightarrow \frac{\sin(A)}{\cos(A)} = 3\cos(A) \Leftrightarrow \sin(A) = 3\cos^2(A)$

$\Leftrightarrow \sin(A) = 3(1 - \sin^2(A)) \Leftrightarrow 3\sin^2 A + \sin A - 3 = 0$

Using the quadratic formula: $\sin A = \frac{-1 \pm \sqrt{37}}{6} = 0.847$ or -1.18

Only the first one of these is possible. Remembering that $A = \frac{1}{2}x$, and $-360^\circ \leqslant x \leqslant 360^\circ$ so $-180^\circ \leqslant A \leqslant 180^\circ$, the solutions are:

$A = 57.9^\circ$ or 122°, so $x = 116^\circ$ or 244°

Cross topic review exercise 3 worked solutions

Cross-topic review exercise 3

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

1 The number of possible outcomes is 36.

The average of 3 is equivalent to the sum of 6, and the outcomes that give this are $1+5, 2+4, 3+3, 4+2, 5+1$

Hence the probability is $\frac{5}{36}$.

2 All the probabilities add up to $1 : \ln k + \ln 2k + \ln 3k + \ln 4k = 1$

$\Rightarrow \ln(24k^4) = 1 \Rightarrow 24k^4 = \mathrm{e} \Rightarrow k = \sqrt[4]{\frac{\mathrm{e}}{24}}$

3 **a** All the probabilities add up to 1, so $p = 0.13$.

Hence the probability of rolling a 2 or a 3 is $0.18 + 0.13 = 0.31$.

b Let X be the number of times the die lands on a 2 or a 3. Then $X \sim \mathrm{B}(12, 0.31)$

$\mathrm{P}(X \geqslant 5) = 1 - \mathrm{P}(X \leqslant 5) = 0.303$

4 **a** **i** Opportunity sampling.

ii Stratified sampling would require knowing the proportion of men to women in the population and randomly sampling from the men and women in proportion to that ratio.

The proposed model only samples from the street (and therefore not from the whole population) with resultant possibility of bias related to the sampling location.

b Assuming that the sample is representative (which the bias involved in the sampling would argue against) and that the sample is also of independent individuals (again dubious):

Let X be the number of people in a random sample of size n who work from home.

$X \sim \mathrm{B}(n, p)$

$\mathrm{H}_0 : p = 0.11$ $H_1 : p > 0.11$ (one-tailed)

Data: $n = 40$, $x = 6$. Test at 10% significance.

$\mathrm{P}(X \geqslant 6 \,|\, \mathrm{H}_0) = 0.274 > 0.1$ so do not reject H_0; there is insufficient evidence to suggest that the proportion working from home in her home town is greater than the national average.

5 **a** **i** $128 \times 2 = 256$

ii The pie chart only shows proportions. The total number of students in year 13 may not be the same as in year 12.

b **i** The box-and-whisker plots do not show frequencies. They only show that the spread of results is greater for male students than for female. The statement is false

ii Females have higher median score.

Female results are more compact (a smaller IQR).

iii Advantage: box-and-whisker plot shows five key values, highest and lowest, lower and upper quartiles and median.

Disadvantage: Some information is lost so further calculations on the data are not possible.

c $(102 \times 51 + 26 \times 59) \div 128 = 56.625 \approx 56.6 (3\,\text{s.f.})$

6 a

Time in minutes (t)	**midpoint**	**Frequency**
$6 < t \leqslant 10$	8	6
$10 < t \leqslant 15$	12.5	6
$15 < t \leqslant 20$	17.5	10
$20 < t \leqslant 30$	25	16
$30 \leqslant t < 45$	37.5	7

b Add the midpoints to the table. $\text{Mean} = 21.3\,\text{min}$, $\text{variance} = 82.8\,\text{min}^2$

c Actual data values are not given, only groups.

7 The expression for the mean is

$$\frac{3 + x^2 + x}{3} = \frac{1}{3}x^2 + \frac{1}{3}x + 1$$

You can find the minimum value of this quadratic by completing the square:

$$\begin{aligned}\frac{1}{3}x^2 + \frac{1}{3}x + 1 &= \frac{1}{3}(x^2 + x + 3) \\ &= \frac{1}{3}\left(\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 3\right) \\ &= \frac{1}{3}\left(x + \frac{1}{2}\right)^2 + \frac{11}{12}\end{aligned}$$

Hence the minimum possible value is $\frac{11}{12}$.

Tip

You could also use differentiation to find the minimum point. However, completing the square gives you the minimum value without having to find the corresponding value of x first.

8 The mean is $\frac{3 + 7 + x}{3} = \frac{10 + x}{3}$, so the variance is:

$$\begin{aligned}\frac{9 + 49 + x^2}{3} - \left(\frac{10 + x}{3}\right)^2 &= \frac{58 + x^2}{3} - \frac{100 + 20x + x^2}{9} \\ &= \frac{74 - 20x + 2x^2}{9}\end{aligned}$$

You can find the minimum value by completing the square. It helps if you take out a factor of 2 first:

$$\frac{2}{9}(x^2 - 10x + 37) = \frac{2}{9}\left((x - 5)^2 + 12\right) = \frac{2}{9}(x - 5)^2 + \frac{8}{3}$$

Hence the smallest possible value of the variance is $\frac{8}{3}$.

Tip

This smallest value occurs when $x = 5$, which is half-way between 3 and 7.

IRANINOVIN

9 a The number of tram journeys does not equate directly to number of tram users because the same users might be travelling more often.

The proportion of the population using the trams relates to both the number using the trams and the number in the population. If the population were increasing at a greater rate than the number of tram users, the proportion of the population using the trams would actually be declining; no information is offered on the size of the population.

b Let X be the number of people in a random sample of size n who use the tram.

$X \sim \mathrm{B}(n, p)$

$\mathrm{H}_0 : p = 0.18\, \mathrm{H}_1 : p \neq 0.18$ (two-tailed test)

Data: $n = 230, x = 52$. Test at 5% significance.

$\mathrm{P}(X \geqslant 0.18) = 0.0445 > 0.025$ so do not reject H_0; there is insufficient evidence to suggest that the proportion using the tram has changed since 2003.

10 a For independent events, $\mathrm{P}(A \text{ and } B) = \mathrm{P}(A)\mathrm{P}(B)$. But $0.85 \times 0.60 \neq 0.55$, so A and B are *not* independent events.

b Let X be the number of buildings completed on time. Then $X \sim \mathrm{B}(25, 0.60)$.

i $\mathrm{P}(X = 20) = 0.0199$

ii The conditions required for the binomial distribution are: constant probability and independent trials. In this context: The probability of completion on time is the same for every building. Whether a building is completed on time is independent of any other building.

c Let Y be the number of buildings completed on time after the improvements. Then $Y \sim \mathrm{B}(25, p)$:

The hypotheses are $\mathrm{H}_0 : p = 0.6, \mathrm{H}_1 : p > 0.6$

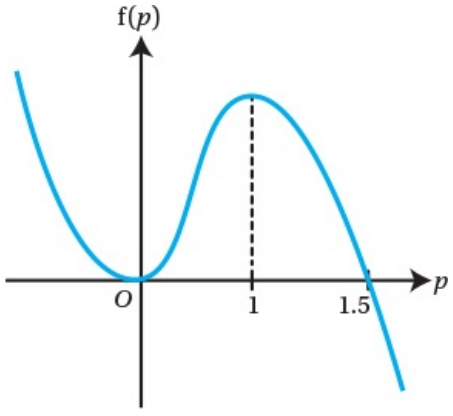
The observed value is $y = 20$. If H_0 is true, $\mathrm{P}(Y \geqslant 20) = 1 - \mathrm{P}(Y \leqslant 19) = 0.294 > 0.05$

Therefore you reject the null hypothesis. There is sufficient evidence that the probability of a building being completed on time has increased.

11 a $X \sim \mathrm{B}\big(3, p\big), \mathrm{P}(X = 2 \text{ or } 3) = 3p^2(1-p) + p^3 = 3p^2 - 2p^3$

b $\mathrm{f}(p) = p^2(3 - 2p)$. The x-intercepts are 0 and 1.5, and only the first one is in the required range.

For stationary points: $\mathrm{f}'(p) = 0$, so $6p - 6p^2 = 0 \Rightarrow p = 0$ or 1.



c $\mathrm{f}\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^2 \left(3 - \frac{2}{4}\right) = \frac{1}{16} \times \frac{10}{4} = \frac{5}{32}$. The graph shows that there is only one solution between 0 and 1.

12 Total score before new student $= 34.5 \times 8 = 276$

Total score after new student $= 276 + 38 = 314$

Mean score after new student $= \dfrac{314}{9} = 34.9$

IRANINOVIN

Sum of squared scores before new student $= 8 \times (5.75 + 34.5^2) = 9568$

Sum of squared scores after new student $= 9568 + 38^2 = 11012$

Variance after new student $= \dfrac{11012}{9} - 34.9^2 = 6.32$

13 Average monthly expenditure from Jan–Aug is £620 means

$$620 = \frac{\sum_{i=1}^{8} x_i}{8} \Rightarrow \sum_{i=1}^{8} x_i = 4960$$

Average monthly expenditure over the whole year is £586 means

$$586 = \frac{\sum_{i=1}^{12} x_i}{12} \Rightarrow \sum_{i=1}^{12} x_i = 7032$$

So, total expenditure for Sept–Dec $= 7032 - 4960 = 2072$

And therefore mean for Sept–Dec $\dfrac{2072}{4} = £518$

14 a The probabilities add up to: $1 : e^{-k} + e^{-2k} + 0.76 = 1$

This is a quadratic equation in: $\mathrm{e}^{-k} : \mathrm{e}^{-2k} + \mathrm{e}^{-k} - 0.24 = 0$

$(\mathrm{e}^{-k} - 0.2)(\mathrm{e}^{-k} + 1.2) = 0$

As e^{x} is always positive, the only solution is $\mathrm{e}^{-k} = 0.2 \Rightarrow \mathrm{e}^{k} = 5 \Rightarrow k = \ln 5$.

b Let Y be the number of ones observed. Then $Y \sim \mathrm{B}(70, 0.26)$:

$\mathrm{P}(Y \geqslant 30) = 1 - \mathrm{P}(Y \leqslant 29) = 0.001\,62$

15 a Binomial, $n = 12, p = 0.1$

The probability of a plate having a minor defect needs to be constant at 0.1, independently of any other plate.

b Using $X \sim \mathrm{B}(12, 0.1)$:

i $\mathrm{P}(X = 3) = 0.0852$

ii $\mathrm{P}(X \geqslant 1) = 1 - \mathrm{P}(X = 0) = 0.718$

c Let y be the number of batches that contain at least one second. You want $\mathrm{P}(Y < 3)$:

The probability of a batch containing at least one second is 0.718 (from question 15b). There are 4 batches, so $Y \sim \mathrm{B}(4, 0.718)$

$\mathrm{P}(Y < 3) = \mathrm{P}(Y \leqslant 2) = 0.317$

16 You use the formula for binomial probabilities.

a $\binom{6}{4} p^4(1-p)^2 = 3\binom{6}{3} p^3(1-p)^3$ so $15p^4(1-p)^2 = 60p^3(1-p)^3$

Dividing both sides by $p^3(1-p)^2 : 15p = 60(1\ -p)$

$75p = 60$ therefore $p = \dfrac{4}{5}$

b $15p^4(1-p)^2 = 3 \times 15p^2(1-p)^4$

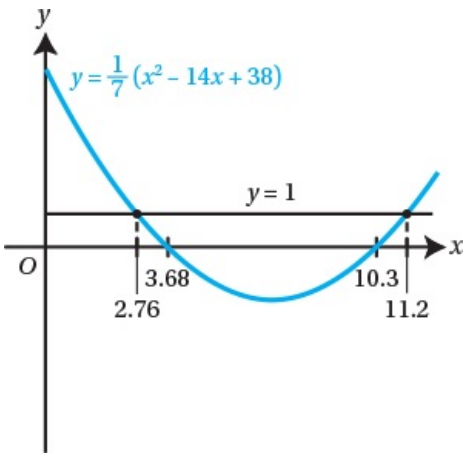
Dividing both sides by $p^2(1-p)^2 : 15p^2 = 45(1-p)^2$

$p^2 = 3 - 6p + 3p^2$ therefore $2p^2 - 6p + 3 = 0$

Only one of the solution is smaller than 1, so $p = 0.275$.

17 Probability has to be between 0 and 1, so you need all integers for which

$0 \leqslant \dfrac{1}{7}(x^2 - 14x + 38) \leqslant 1$



Using graph on GDC, the inequality is satisfied for $2.76 \leqslant x \leqslant 3.68$ and $10.3 \leqslant x \leqslant 11.2$ Hence the possible integer values of x are 3 and 11.

18 a The selections are not independent; for example, the probability that the first selected number is 500 or less is $\frac{500}{726}$, but the probability that the second one is also 500 or less depends on what the first number was: it is either $\frac{499}{725}$ or $\frac{500}{725}$.

However, these two numbers are very similar $(0.833 \text{ and } 0.832)$. Furthermore, you are only selecting 10 people out of 726; for the last person the probability is somewhere between $\frac{491}{717} = 0.645$ and $\frac{500}{717} = 0.697$ so if you do a calculation assuming independence, you'll get an answer close to the correct one.

Tip

It is tempting to say that, when the selection is done without replacement, the probability is not constant. However, the probability that the second selected number is 500 or less is (e.g. draw a tree diagram):

$$\frac{500}{726} \times \frac{499}{725} + \frac{226}{726} \times \frac{500}{725} = \frac{500}{726}.$$

So it is only the independence condition of the binomial distribution that is not satisfied.

b Since P (a selected number is 500 or less) $= \frac{500}{726}$, the probability that all 10 numbers are 500 or less is $\left(\frac{500}{726}\right)^{10} = 0.0240$.

Tip

The actual probability, when the selection is done without replacement, is 0.0233.

c The statement is *not* correct. This sample is as likely as any other.

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Cross topic review exercise 4 worked solutions

Cross-topic review exercise 4

Worked solutions are provided for all levelled practice and discussion questions, as well as Cross-topic review exercises. They are not provided for drill questions.

1 The net force is $\mathbf{F} = (18\mathbf{i} + 12\mathbf{j})\text{N}$.

Using $\mathbf{F} = \text{m}\boldsymbol{a}, \boldsymbol{a} = \dfrac{(18\text{i} + 12\text{j})}{3} = (6\mathbf{i} + 4\mathbf{j})\,\text{m s}^{-2}$

2 The average speed equals total distance divided by time.

The total distance is the total area of the two triangles: $\dfrac{8 \times 6}{2} + \dfrac{12 \times 4}{2} = 48\,\text{m}$.

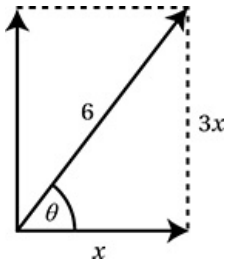
Hence the average speed is $\dfrac{48}{20} = 2.4\,\text{m s}^{-1}$.

3 $\dfrac{13.6}{4.2} = 3.238\ \text{m s}^{-2}$

$v^2 = u^2 + 2as = 0 + 2(3.238)(6.2) = 40.15$ therefore $v = \sqrt{40.15} = 6.34\,\text{m s}^{-1}$

4 **a** $x^2 + (3x)^2 = 6^2$ so $10x^2 = 36$ therefore $x = \sqrt{3.6} = 1.90$

b



$\tan\theta = \dfrac{3}{1} \Rightarrow \theta = \tan^{-1}3 = 71.6^\circ$

5 Use constant acceleration formulae with $v = 0, t = 0.5, g = -9.8$ (the direction of motion is upwards and at the highest point the particle is at rest).

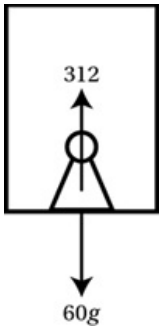
a $v = u + at : 0 = u - (9.8)(0.5)$, so $u = 4.9\,\text{m s}^{-1}$

b $s = ut + \dfrac{1}{2}at^2 = 4.9 \times 0.5 - \dfrac{1}{2} \times 9.8 \times 0.5^2 = 1.225\,\text{m}$

c The particle falls downwards from rest, from the height of $1.225 + 0.539 = 1.764\,\text{m}$

$v^2 = u^2 + 2as = 0 + 2 \times 9.8 \times 1.764 = 34.6$, so $v = 5.88\,\text{m s}^{-1}$.

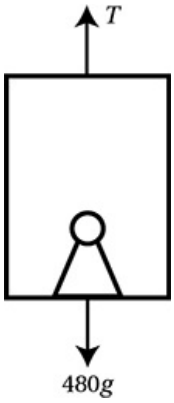
6 **a** Force diagram for the person:



Newton's second law, with positive direction downwards:

$60a = 60 \times 9.8 - 312$, so $a = 4.6\,\mathrm{m\,s^{-2}}$ direction downwards.

b Consider the lift and the person as a single particle, mass $= 60 + 420 = 480\,\mathrm{kg}$. Acceleration is downwards.



$480 \times 4.6 = 480g - T,\ \text{so}\ T = 2496\ \mathrm{N}$

c When you considered the lift and the person as a single particle, its mass did not include the mass of the cable.

7 **a** For the journey from A to B:

$t = 5,\ s = 95$: $s = ut + \frac{1}{2}at^2 \Rightarrow 5u + \frac{5^2}{2}a = 95$

Since you want the speed at A, consider the journey from A to C:

$t = 7, s = 175 : 7u + \frac{7^2}{2}a = 175$

You now have simultaneous equations: $\begin{cases} 5u + 12.5a = 95 \\ 7u + 24.5a = 175 \end{cases}$

Solving gives $a = 6, u = 4$.

b The net force acting on the car is $7200 - 2500 = 4700\ \mathrm{N}$

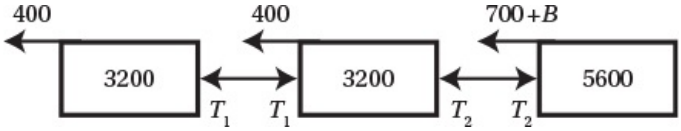
$F = ma : m = \frac{4700}{6} = 783.3\,\mathrm{kg}$

8 You first find the acceleration of the train (using constant acceleration formulae) and then use $F = ma$

$u = \frac{40\,000\ \mathrm{m}}{3600\ \mathrm{s}} = 11.1\,\mathrm{m\,s^{-1}}$, $v = 0, s = 360$:

$v^2 = u^2 + 2as \Rightarrow 0 = 11.1^2 + 720a \Rightarrow a = -0.171\mathrm{ms^{-2}}$

Draw a force diagram. B is the braking force applied by the engine. Since the train is braking, assume that the force in each coupling is a thrust, so directed towards the carriage. (If the force comes out as negative it would mean that it is a tension.)



You want to find the force T_2. The equation for the first carriage is:

$400 + T_2 - T_1 = 3200 \times 0.171 = 548.7$

You need T_1 first.

For the second carriage: $400 + T_1 = 3200 \times 0.171, \text{so} T_1 = 148.7$

So, $T_2 = 548.7 + 148.7 - 400 = 297\mathrm{N}$

This is positive, so the force is a thrust.

9 a i $v = \int a\,dt = 3t^2 - 22t + c$

$v = 35$ when $t = 0 \Rightarrow c = 35 \therefore v = 3t^2 - 22t + 35$

ii $3t^2 - 22t + 35 = 0 \Rightarrow (3t - 7)(t - 5) = 0 \Rightarrow t = \dfrac{7}{3}, 5$ seconds

iii For local max/min, $v' = 0$, so $6t - 22 = 0 \Rightarrow t = \dfrac{11}{3}$.

However, at $t = \dfrac{11}{3}$, $v = -\dfrac{16}{3}$, whereas at $t = 0$, $v = 35$.

$\therefore$ max speed $= 35\,\mathrm{m\,s^{-1}}$

b You know that P has negative velocity for $\dfrac{7}{3} < t < 5$, so P is moving back towards O here.

So find the displacement for $0 < t < \dfrac{7}{3}$ and $\dfrac{7}{3} < t < 5$ separately:

$$s_1 = \int_0^{\frac{7}{3}} v\,\mathrm{d}t = \int_0^{\frac{7}{3}} 3t^2 - 22t + 35\,\mathrm{d}t = \left[t^3 - 11t^2 + 35t\right]_0^{\frac{7}{3}}$$
$$= \frac{931}{27}\mathrm{m}$$
$$s_2 = \int_{\frac{7}{3}}^{5} v\,\mathrm{d}t = \int_{\frac{7}{3}}^{5} 3t^2 - 22t + 35\,\mathrm{d}t = \left[t^3 - 11t^2 + 35t\right]_{\frac{7}{3}}^{5}$$
$$= -\frac{256}{27}\mathrm{m}$$
$$\therefore s = \frac{931}{27} + \frac{256}{27} = \frac{1187}{27} \approx 44\,\mathrm{m}$$

10 The velocity is always positive, so distance travelled is the same as the displacement. Hence:

$$\frac{(\text{displacement from } t = 0 \text{ to } t = T)}{T} = 4 \Rightarrow \int_0^T t^2 + 1 dx = 4T$$
$$\Rightarrow \left[\frac{t^3}{3} + t\right]_0^T = 4T \Rightarrow \frac{T^3}{3} + T = 4T$$
$$\Rightarrow T^3 - 9T = 0 \Rightarrow T(T^2 - 9) = 0$$

As t has to be positive, $T = 3$.

11 a Integrate acceleration to get the velocity: $v = \int 0.8t\,\mathrm{dt} = 0.4t^2 + c$

When $t = 0$, $v = 13$, so $c = 13$. Then $v(6) = 0.4(6^2) + 13 = 27.4\,\mathrm{m\,s^{-1}}$

b Integrate velocity to get the displacement: $s = \int 0.4t^2 + 13\,\mathrm{dt} = \dfrac{0.4}{3}t^3 + 13t + k$

When $t = 0, s = 0$ so $k = 0$.

When $t = 6$: $s = \dfrac{0.4}{3}(6^3) + 13 \times 6 = 106.8\,\mathrm{m}$

c i The correct figure is Figure 2.

ii Figure 1 has zero gradient at the origin, so zero initial velocity, which is not correct.

Figure 3 does not have an increasing gradient, so the velocity is not increasing.

12 a Using $v^2 = u^2 + 2as$: the initial velocity is upwards, acceleration is downwards $(-g)$ and the final displacement is $-2.5\,\mathrm{m}$.

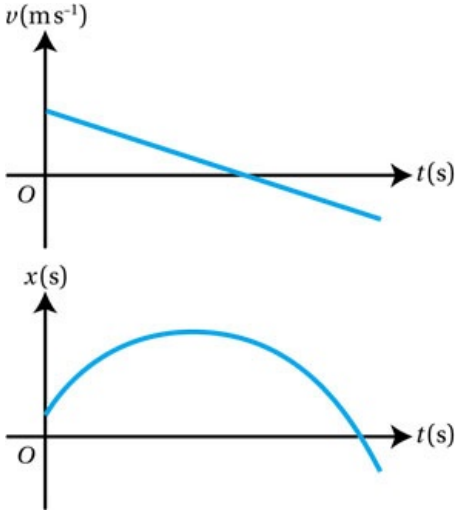
$v^2 = 5^2 + 2 \times (-9.8) \times (-2.5)$ so $v = 8.60\,\mathrm{m\,s^{-1}}$

b Using $v = u + at$ with $u = 5, g = -9.8, v = -8.60$ (as the initial velocity is upwards and the final is downwards).

$-8.60 = 5 - 9.8t$, so $t = 1.39\text{s}$

c **i** The velocity decreases with a constant gradient $(\text{of } -g)$.

ii The displacement graph is a parabola.



13 a Differentiate v to get a : $a = 0.012t - 0.18$

b The minimum speed is when $a = 0 : 0.012t - 0.18 = 0 \Rightarrow t = 15$

$v(15) = 0.006 \times 15^2 - 0.18 \times 15 + k = 0.65 \Rightarrow k = 2$

c Integrate v to get x. $x = 0.002t^3 - 0.09t^2 + 2t + c$

When $t = 0, x = 0, \text{ so } c = 0$. The distance equal to the length L of the pool is when $t = 28.4$.

$L = 0.002 \times 28.4^3 - 0.09 \times 28.4^2 + 2 \times 28.4 = 30\,\text{m}$

14 a Differentiate x to get velocity, and differentiate again to get acceleration.

$v = 0.3t^2 - 0.6t + 0.2$

$a = 0.6t - 0.6$

b You need to set $a = 0$, find t and then find x.

$0.6t - 0.6 = 0 \Rightarrow t = 1 \text{ so } x(1) = 0.1 \times 1 - 0.3 \times 1 + 0.2 \times 1 = 0$

c The particle is stationary when $v = 0$:

$0.3t^2 - 0.6t + 0.2 = 0$ so $t = 0.423$ and $1.58\,\text{s}$

d You need to find an expression for the displacement of Q from O and set it equal to the displacement of P.

$x = \int 0.2t^2 - 0.4\text{d}t = \frac{0.2}{3}t^3 - 0.4t + k$ so $x = 0$ when $t = 0$ so $k = 0$

Equating the two displacement equations: $0.1t^3 - 0.3t^2 + 0.2t = \frac{0.2}{3}t^3 - 0.4t$

Both particles are at O when $t = 0$. For the other collision, you can divide by t:

$0.1t^2 - 0.3t + 0.2 = \frac{0.2}{3}t^2 - 0.4$

This rearranges to $t^2 - 9t + 18 = 0$ and solving gives $T = 3\,\text{s}$.

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

1 Proof and mathematical communication

Support sheet

Fill in the blanks (from top left to bottom right).

$n+1$

$n+1$ $2n+1$

$+$ $2n+1$

1 Proof

2 Proof

3 Proof

4 Proof

5 **a** Proof

b Proof

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

2 Indices and surds

Support sheet

Fill in the blanks (from top left to bottom right).

$2\sqrt{3}$ $4\sqrt{3}$

2 4

1. $\frac{4}{11}$
2. $\frac{5}{8}$
3. $2\sqrt{3}$
4. $48 + 32\sqrt{2}$
5. $\frac{2}{3}$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

3 Quadratic functions

Support sheet

Fill in the blanks (from top left to bottom right).

$4ac$

$k + 6$ $+ 12k + 4$

$5k - 2$ $k - 2$ $\frac{2}{5}$ 2

$\frac{2}{5}$ 2

1 $k < -2,\ k > 2$

2 $0 < k < 8$

3 $k \leqslant 2,\ k \geqslant 6$

4 $\frac{7}{5} < k < 3$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

4 Polynomials

Support sheet

Fill in the blanks (from top left to bottom right).

-1

-1 -1 -1

2

2 2 2

2

1 $a = 7, b = 4$

2 $a = -1, b = 24$

3 $a = -3, b = -2$

4 $a = 2, b = -3$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

5 Using graphs

Support sheet

Fill in the blanks (from top left to bottom right).

$\frac{x}{2}$

horizontal 2

$(0, -3)$

$(1, -3)$

right

1 Horizontal stretch, scale factor $\frac{1}{3}$

2 Translation with vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

3 Reflection in the y-axis

4 Horizontal stretch, scale factor 1.5

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

6 Coordinate geometry

Support sheet

Fill in the blanks (from top left to bottom right).

$3^2 - 4(3) + (1)^2 + 2(1) = 9 - 12 + 1 + 2 = 0$

$2 \qquad 4 \qquad 1 \qquad 1$

$2 \qquad 1 \qquad 5 \qquad (2, -1)$

$1 \qquad -1 \qquad 3 \qquad 2$

$1 \qquad 3$

$x - 3$

1 Proof

2 **a** $C(2, 1), r = \sqrt{3}$

b $C\left(\frac{3}{2}, -2\right), r = \frac{5}{2}$

3 $y + 1 = \frac{5}{6}(x - 2)$

4 $y + 2 = \frac{5}{7}(x - 1)$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

7 Logarithms

Support sheet

Fill in the blanks (from top left to bottom right).

$\log_3\left(3x^2-2x-4\right)$

$(2-3x)^2$

$\dfrac{(2-3x)^2}{3x^2-2x-4}$

9

$4-12x+9x^2$

$3x^2-2x-4$

$18x^2-6x-40$

is not a solution of the original equation

$-\dfrac{4}{3}$

1 $x=\dfrac{1}{8}$

2 $x=\dfrac{7}{2}$

3 $x=\dfrac{1}{4},\ \dfrac{3}{2}$

4 $x=2,4$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

8 Exponential models

Support sheet

Fill in the blanks (from top left to bottom right).

ln ln

$\ln a$ $\ln (b^x)$

$\ln b$ $\ln a$

$\ln 3$ 3 2 e^2

1 **a** $\ln y = n \ln x + \ln a$

b $a = e^{-1}$, $n = \frac{1}{4}$

2 $y = 1000x^{-4}$

3 $y = 8 \times 2^{-\frac{1}{2}x}$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

9 Binomial expansion

Support sheet

Fill in the blanks (from top left to bottom right).

$-2x$ 10 1080

1080

1080 810 5130

5130

1 -48

2 6

3 322

4 432

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

10 Trigonometric functions and equations

Support sheet

Fill in the blanks (from top left to bottom right).

$\cos^2 x$

$4\cos^2 x$

$\cos x$ y^2 y

$\frac{3}{4}$

360

1 $41.8^\circ, 90^\circ, 138^\circ$

2 $-124^\circ, -18.4^\circ, 56.3^\circ, 162^\circ$

3 $233^\circ, 307^\circ$

4 $-90^\circ, -33.6^\circ, 33.6^\circ, 90^\circ$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

11 Triangle geometry

Support sheet

Fill in the blanks (from top left to bottom right).

4 7 B

14

150

$4^2 + 7^2 - 2 \times 4 \times 7 \times \cos B$

150

150°

1 10.3 cm

2 9.0 cm

3 84.9°

4 67.9 cm^2

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

12 Vectors

Support sheet

Fill in the blanks (from top left to bottom right).

2^2 $(-5)^2$

$\sqrt{29}$

$\dfrac{10}{\sqrt{29}}$

1 $\dfrac{1}{25}(-7\mathbf{i}+24\mathbf{j})$

2 $\pm\dfrac{1}{\sqrt{13}}(3\mathbf{i}-2\mathbf{j})$

3 $-\dfrac{9}{\sqrt{17}}(\mathbf{i}+4\mathbf{j})$

4 $\pm\dfrac{2}{\sqrt{10}}(-3\mathbf{i}+\mathbf{j})$

5 $\dfrac{20}{\sqrt{41}}(5\mathbf{i}-4\mathbf{j})\,\mathrm{m\,s^{-1}}$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

13 Differentiation

Support sheet

Fill in the blanks (from top left to bottom right).

$3x^2 + 18x + 24$

$>$ $>$

$>$

-2 -4

1 $2 < x < 3$

2 $x < -4$ or $x > 4$

3 $x < 0$

4 $x < 6$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

14 Applications of differentiation

Support sheet

Fill in the blanks (from top left to bottom right).

x^{-1}

$-16x^{-2} + 2x$

0

0 x^3

12 $(2, 12)$

minimum

1 min

2 max

3 min, $\left(-\frac{1}{3}, -6\right)$ max

4 min, $(-4, 80)$ max

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

15 Integration

Support sheet

Fill in the blanks (from top left to bottom right).

$2 \qquad 5 \qquad 0 \qquad 2 \qquad 5$

$2 \qquad \frac{7}{3}x^3 \qquad 2$

$\frac{16}{3}$

$2 \qquad \frac{1}{4}x^4 - \frac{7}{3}x^3 + 5x^2 \qquad 2$

$-\frac{125}{12} \qquad \frac{16}{3}$

$\frac{63}{4} \qquad \frac{16}{3}$

1 $\frac{142}{12} = \frac{71}{6}$

2 $\frac{131}{4}$

3 $\frac{152}{3}$

4 $\frac{72}{5}$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

16 Working with data

Support sheet

Fill in the blanks (from top left to bottom right).

120

142.5

1. mean $= 22.8$, $\sigma = 12.3$
2. mean $= 52.25$, $\sigma = 20.0$
3. mean $= 73.0$, $\sigma = 16.5$
4. mean $=3.54\ \text{kg}$, $\sigma = 1.06\ \text{kg}$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

17 Probability

Support sheet

Fill in the blanks (from top left to bottom right).

0.1746

$X \leqslant 14$ 0.1958 0.8042

1 0.558

2 **a** 0.150

b 0.719

3 0.174

4 **a** 0.231

b 0.894

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

18 Statistical hypothesis testing

Support sheet

Fill in the blanks (from top left to bottom right).

$p > 0.85$

$X \leqslant 63$

0.0838 0.05 insufficient

insufficient

1. $p = 0.0206 < 0.05$; there is sufficient evidence to suggest that the percentage of pupils who cycle to school has increased.
2. $p = 0.205 > 0.1$; there is insufficient evidence that the probability of the goods being damaged has decreased.
3. $p = 0.114 > 0.05$; there is insufficient evidence that the probability of rolling a ‘six’ is greater than $\frac{1}{6}$.
4. $p = 0.0151 < 0.025$; there is sufficient evidence that the percentage of female Maths students in the second school is different.

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

19 Introduction to kinematics

Support sheet

Fill in the blanks (from top left to bottom right).

2 5

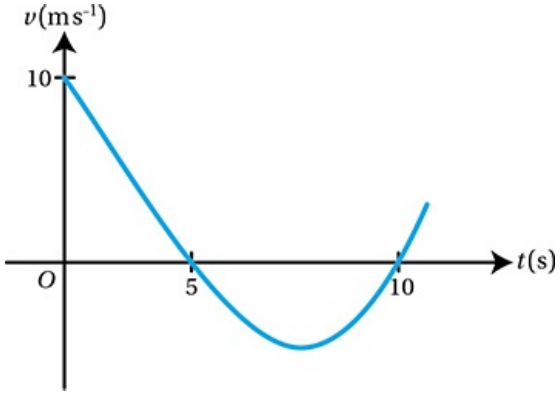
0 2 5

3.2

3.2

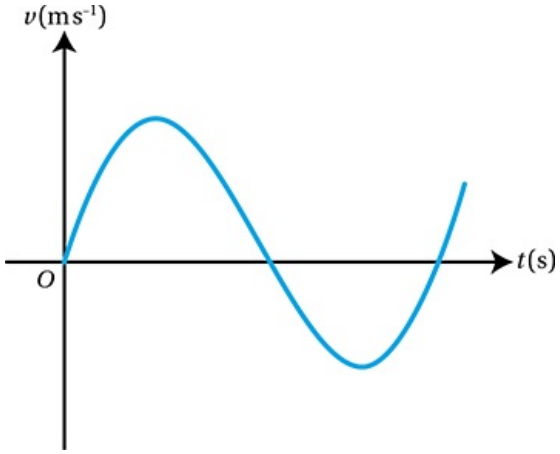
1 **a** $t = 5, 10$

b $25\,\text{m}$



2 **a** $t = 0, 3, 5$

b $0.506\,\text{m}$

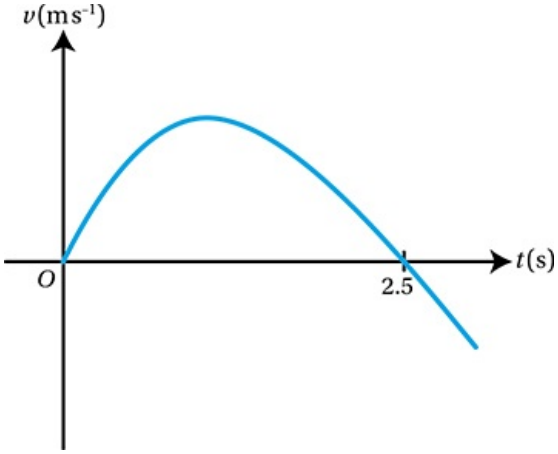


3 $98\,\text{m}$

4 **a** $v = 5t - 2t^2$

b $13.1\,\text{m}$

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v(ms⁻¹)
t(s)
O
2.5

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

20 Motion with constant acceleration

Support sheet

Fill in the blanks (from top left to bottom right).

$u + at$ $11.2 - 9.8t$ $\dfrac{11.2}{9.8}$

$11.2 \times 2 + \dfrac{1}{2} \times (-9.8) \times 2^2$

1 **a** $9.98\,\text{m}$

b $1.43\,\text{s}$

2 **a** $30.6\,\text{m}$

b $5\,\text{s}$

3 $12.7\,\text{m}\,\text{s}^{-1}$

4 $0.83\,\text{s}$, $3.46\,\text{s}$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

21 Force and motion

Support sheet

Fill in the blanks (from top left to bottom right).

$ut + \frac{1}{2}at^2$ $0 + \frac{1}{2} \times a \times 4.5^2$

$45 - P$

1. $44.3\,\text{N}$
2. $21.25\,\text{N}$
3. $1.56\,\text{N}$
4. $49\,\text{N}$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

22 Objects in contact

Support sheet

Fill in the blanks (from top left to bottom right).

T $2g$

g

$\frac{12g}{5}$

1 $28\,\mathrm{N}$

2 $3.92\,\mathrm{m\,s^{-2}}$

3 $3.33\,\mathrm{kg}$

4 $2.31\,\mathrm{kg}$

5 $3.86\,\mathrm{kg},\ 2.83\,\mathrm{kg}$

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

1 Proof and mathematical communication

Extension sheet

1 Proof

2 Proof

3 Proof

4 Proof

5 Proof

6 Proof

7 Proof

8 Proof

9 Proof

10 **a** Proof

b Proof

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

2 Indices and surds

Extension sheet

1 2^{3500}

2 $10^{10!}$

3 **a** $2 + \sqrt{3}$

b $\sqrt{5} - \sqrt{3}$

c $2\sqrt{5} - \sqrt{3}$

4 **a** $\sqrt{x+1} - \sqrt{x}$

b Proof

5 **a** Proof

b 2

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

3 Quadratic functions

Extension sheet

1 **a** $b \leqslant -2 \text{ or } b \geqslant 2$

b Proof

c $b < -2$

d

Value of b	**Number of real roots**	**Sign**
$b > 2$	two	positive
$b = 2$	one	positive
$b = -2$	one	negative

2 **a** Proof

b Proof

3 **a** No real solutions when $a < -\frac{1}{4}$, one negative solution when $a = -\frac{1}{4}$, two negative solutions when $-\frac{1}{4} < a < 0$, one negative and one zero solution when $a = 0$, one positive and one negative solution when $a > 0$.

b One solution when $a \geqslant 0$, no solutions when $a < 0$.

4 **a** $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

b $a = 2, b = 0$

c $2(x + y + 1)^2 + (2x - y)^2$

d $(1, -1), (-1, 1), \left(\frac{1}{3}, -\frac{7}{3}\right), \left(-\frac{5}{3}, -\frac{1}{3}\right)$

5 47

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

4 Polynomials

Extension sheet

1 **a** $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

b 8080

2 **a** Proof

b Proof

3 Proof

4 **a** $\mathrm{f}\left(x\right) = \frac{1}{10!}(x-1)\,(x-2)\ldots(x-10)$, $\mathrm{f}\,(12) = 11$

b 23

5 $\frac{1}{501}$

6 Proof

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

5 Using graphs

Extension sheet

1 $\mathrm{f}(-x) = -\mathrm{f}(x)$

2 **a** Even

b Odd

c Neither

d Odd

e Odd

3 Proof

4 $\mathrm{f}(a - x) = \mathrm{f}(a + x)$ or $\mathrm{f}(x) = \mathrm{f}(2a - x)$

5 $\mathrm{f}(a + x) - b = b - \mathrm{f}(a - x)$ or $\mathrm{f}(x) = 2b - \mathrm{f}(2a - x)$

6 **a** Proof

b Proof

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

6 Coordinate geometry

Extension sheet

1 Proof

2 Coordinates of $A(0, y_A)$. B and C will be on the x-axis. Let B be to the right of the origin and C to the left. Then coordinates of $B(x_B, 0)$ and of $C(-x_C, 0)$.

This gives equation of altitude from B:

$$y = \frac{-x_C}{y_A}(x - x_B)$$

and equation of altitude from C:

$$y = \frac{-x_B}{y_A}(x + x_C)$$

Intersect at $\left(0, \frac{x_B x_C}{y_A}\right)$

3 **a** $\frac{2y_a + y_b}{3}$

b Many possible answers, for example:

$$C(0,0), A(1,0), B(z,w), P\left(\frac{3z}{4}, \frac{3w}{4}\right), Q(3,0), R\left(\frac{2+z}{3}, \frac{w}{3}\right)$$

c Proof

4 Proof

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

7 Logarithms

Extension sheet

1 **a** Proof

b $x = 4$ or $\sqrt{2}$

c 9

d $x = ab$

2 **a** 151

b 6

c 3

d 2

3 Proof; x^y

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

8 Exponential models

Extension sheet

1 Differ by a factor of 10.

2 20.0

3 $\log 3 = 0.477$

4 Lower

5 8.11

6 Tomato juice – 5 times higher

7 $22.4 \mu \text{Pa}$

8 **a** $20 \log 2 = 6.02$

b $10 \log 2 = 3.01$

9 85 dB

10 $\text{F}\#$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

9 Binomial expansion

Extension sheet

1 Proof

2 Proof

3 Proof

4 Proof

5 Proof

6 Proof

7 Proof

8 Proof

9 Proof

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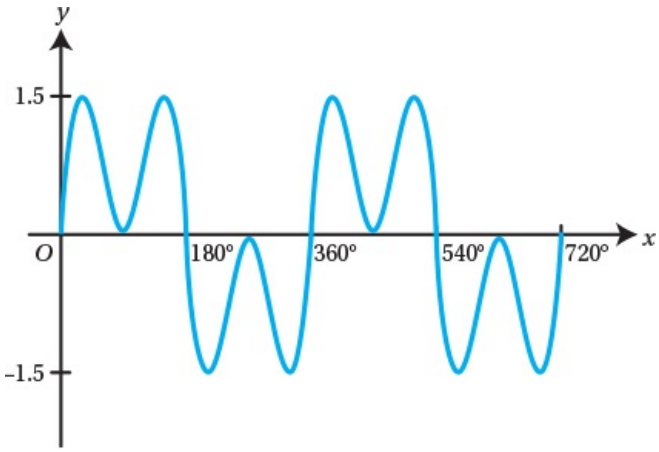
Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

10 Trigonometric functions and equations

Extension sheet

1 **a**



b 360°

c Proof

2 **a** 360°

b 180°

c 360°

d 180°

3 $360^\circ /$ (greatest common divisor of m, n)

4 **a** 1080°

b 1440°

c 2160°

d 4320°

5 $360^\circ \times$ (lowest common multiple of $m,\ n$)

6 **a** $\dfrac{360^\circ}{\pi}$

b No

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

11 Triangle geometry

Extension sheet

1 πp^2

2 **a** r

b Proof

c **i** 30°

ii $r^2\left(\frac{\sqrt{3}}{4}+\frac{\pi}{6}\right)$

3 $(\pi-2)r^2$

4 **a** 36°

b 115°

5 **a** $AM=BM=a$ because M lies on the circles with centres A and B and radius $AB=a$.

b 30°

c $\frac{1}{3}\pi a^2$

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

12 Vectors

Extension sheet

1 $\overrightarrow{MN} = \frac{1}{2}\mathbf{c} - \frac{1}{2}\mathbf{b} = \frac{1}{2}\overrightarrow{BC}$

2 **a** **i** $\overrightarrow{BE} = \frac{1}{2}\mathbf{c} - \mathbf{b}$

ii $\overrightarrow{BG} = \frac{1}{3}\mathbf{c} - \frac{2}{3}\mathbf{b}$

b **i** $\overrightarrow{CG} = \frac{1}{3}\mathbf{b} - \frac{2}{3}\mathbf{c}$

ii $\overrightarrow{CF} = \frac{1}{2}\mathbf{b} - \mathbf{c}$

c Proof

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

13 Differentiation

Extension sheet

1 **a** $1, n, \frac{n(n-1)}{2}$

b Proof

c Non-negative integers

2 Proof

3 **a** Proof

b Proof

4 **a** Proof

b Proof

5 **a** **i** $3y^2$

ii $\frac{1}{3}x^{-\frac{2}{3}}$

b $\frac{1}{n}x^{\left(\frac{1}{n}-1\right)}$

6 Discussion

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

14 Applications of differentiation

Extension sheet

1 Proof

2 $\sqrt[3]{\frac{1}{16}}$

3 **a** $S(3, 9)$ and $T(5, 25)$

b Proof

4 a

5 **a** $\left(-\frac{1}{a^3}, -a^3\right)$

b $y + a^6x = 2a^3$

6 **a** $(-a, -ab + c)$

b $a^2 = 4\left(\frac{b^2 \ - 1}{b}\right)$

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

15 Integration

Extension sheet

1 **a** Proof

b $\frac{a(a-1)}{2}$

c $\frac{[a]([a]-1)}{2} + a - [a]$

d $\frac{a(a-1)(2a-1)}{6}$

2 $1 : n$

3 **a** t^2

b **i** $\mathrm{f}(t)$

ii $\mathrm{f}(t) + \mathrm{f}(-t)$

iii $2t\mathrm{f}(t)$

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

16 Working with data

Extension sheet

1 a, b, c worked answer only

2 a, b worked answer only

3 worked answer only

4 worked answer only

5 worked answer only

6 worked answer only

7 worked answer only

8 worked answer only

9 worked answer only

10 worked answer only

11 worked answer only

12 worked answer only

13 worked answer only

14 worked answer only

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

17 Probability

Extension sheet

1 $\frac{\pi}{8}$

2 $\frac{7}{16}$

3 Proof

4 0.4

5 **a** 0.84

b 0.25

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

18 Statistical hypothesis testing

Extension sheet

1 **a** Leading question, pushing for answer ‘yes’ by giving only a positive argument.

b Leading question, pushing for answer ‘no’ by giving only a negative argument.

c Leading question, pushing for answer ‘no’ by presenting the positive as unreasonable.

d Leading question, pushing for answer ‘yes’ by presenting the negative as unreasonable.

2 **a** Self-selected sample population; only those with strong opinions on the subject (likely to be disproportionately those opposed) would attend.

b Majority of those surveyed will be people who already shop in town; also, a person’s willingness to participate in the survey (which suggests some free time while in town) may also be linked to opinion on the shopping centre proposal.

c Survey will focus on people who are accustomed to using internet, and therefore are more likely to shop online than the general population, which might affect opinions on ‘bricks and mortar’ outlets.

d People whose surnames begin with ‘rare’ letters (e.g. X) would be disproportionately likely to be included in the survey while those with more commonly encountered initial letters would be disproportionately unlikely to be included. Whether this renders the survey unrepresentative would depend on what the survey is intending to represent.

Tip

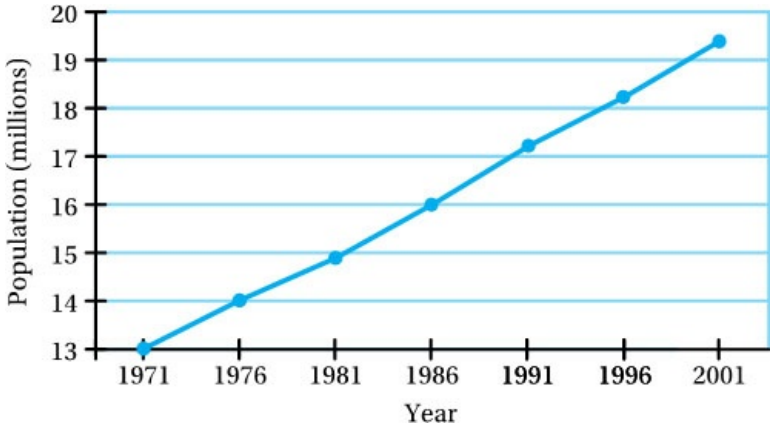
The word ‘unrepresentative’ is problematic – without actually specifying what body of population the survey is intended to represent, this description is potentially meaningless. For example, if you were deliberately seeking the opinion of people who frequent the current high street shops, option b may be acceptably representative of our target population.

3 From Study C, the statement ‘in a study, 80% of those surveyed said they recommended *Dentistat*’ would be favoured by the company.

An independent researcher (not knowing the basis for each of the studies) would do best to combine the three studies and state that 18 out of 30 (60%) of those surveyed recommended *Dentistat*, though some investigation into the rigour of the sampling would be appropriate first.

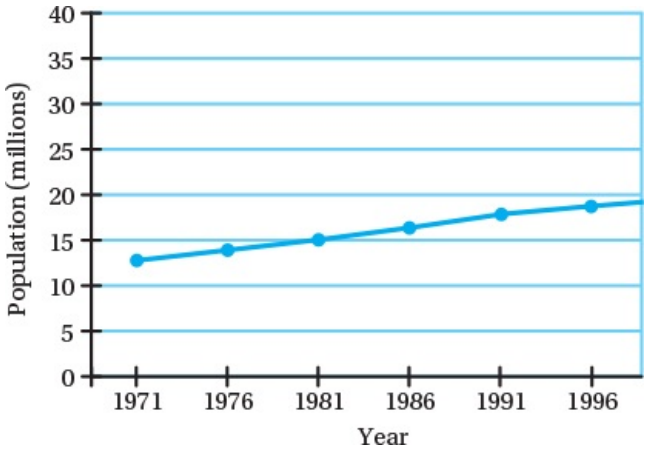
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4 **a**



Truncating the vertical axis scale in this way exaggerates the change; the population appears to rise rapidly (from almost nothing) so that, to the eye, it appears to double from 1976 to 1981, for example.

b



This representation almost does the reverse – by extending the y-axis so far beyond the data, the graph makes the line appear even flatter than necessary.

c The second graph is less visually misleading.

5 **a** From the bars, approximately 2.5 (i.e. 150% higher sales for *Willingham Voice*).

b The actual ratio is 1.09 (i.e. 9% higher sales for *Willingham Voice*).

6 **a** The relative value of the Canadian dollar has fallen from 1 USD to 0.46 USD, a fall of approximately 50%, so the value has reduced to approximately a half of its original.

b The visual representation, where radius is used to indicate size, means that the apparent coin *area* has reduced by 75%, so has reduced to a quarter of its original.

7 **a** Mode: £4000, Median £6000, Mean £$21\,300$

b The union representative would highlight the mode, perhaps citing the 'standard wage' of the workers in the factory as £4000. Median would also be a reasonable option.

c The factory owner could describe the 'average wage' as £$21\,300$.

8 **a** Neither statement is true, though **ii** is *less* untrue. The data refers to the *percentage* of internet users but the statements refer to *numbers* of internet users with cashback accounts. If the number of people using the internet changed (it will have increased, in all likelihood, but that information is not provided) then the absolute number will not have increased by a factor of $\frac{2.34}{0.98}$.

If you suppose that the number of internet users was the same in both years, then **ii** is a reasonable approximation. **i** should state that there has been a '1 percentage point' increase, not a 1% increase.

b As identified in question 8a, there simply is not the information to calculate this, but if you

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assume the same number of internet users in the two years, then the ratio is $\frac{2.34}{0.98} = 2.39$, so a 139% increase.

Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

19 Introduction to kinematics

Extension sheet

1 Yes

2 **a** $158\,\text{cm}$

b 6

3 **a** 6

b Proof

c Investigation

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

20 Motion with constant acceleration

Extension sheet

1. 0.0101 miles or about $16\,\mathrm{m}$
2. $20\,\mathrm{mph}$
3. $u > 2\sqrt{ad}$
4. $2\,\mathrm{p.m.}$
5. $\dfrac{y(z-x)}{x(y-z)}$ or its reciprocal
6. $3x - y\,\mathrm{m}$
7. About $43\,\mathrm{m}$

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

21 Force and motion

Extension sheet

1 **a** **i** $200\ \mathrm{N}$

ii $5000\ \mathrm{N}$

iii $200\ \mathrm{N}$

iv $5000\ \mathrm{N}$

b $6 \times 10^{12}\mathrm{N}$ – results in an acceleration of about $10^{-12}\ \mathrm{m\ s^{-1}}$ so insignificant.

2 **a** **i** $200\ \mathrm{N}$

ii $200\ \mathrm{N}$

b $200\ \mathrm{N\ cm^{-1}}$

c $4000\ \mathrm{m\ s^{-2}}$ to the left

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Worksheet answers

All answers to ‘fill in the blanks’ activities are provided in the order the blanks appear, reading from the top left line by line to the bottom right.

22 Objects in contact

Extension sheet

1 **a** $2g\dfrac{(M-2m)}{M+4m}$

b **i** $a=-g$; it is as if the block is in free fall.

ii $a=2g$; the pulley is in free fall. For every one unit it goes down it pulls the string up two units.

2 **a** **i** Proof

ii Proof

b Proof

c **i** The answer becomes $-g$; it is as if only mass m_1 is there.

ii The answer becomes $3g$; m_1 and m_3 are both in free fall. Then use the result in question 2a ii.

iii The answer becomes $\dfrac{-g}{3}$; the right pulley is falling but it is being slowed by m_1.

3 **a** Proof

b **i** The answer becomes $-g$; it is as if only mass m_1 is there.

ii The answer becomes $3g$; m_2 and m_3 are both in free fall. If they both fall down one unit, m_1 will move up 3 units.

iii The answer becomes 0; the system is in equilibrium.

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Working with the large data set

OCR have provided you with a data set looking at different methods for travelling to work and age structures in different local authorities across England. The data is taken from censuses in 2001 and 2011. You can find the large data set on the OCR website.

This chapter is meant to be used to complement the statistical chapters within the main book.

Learning about the data set

Every data set has its own peculiarities. These questions are designed to get you to think critically about exactly what is meant by some of the data given.

QUESTIONS

1 **a** Which local authority had the highest percentage of their workforce travelling to work by bicycle in 2011?

b What is the modal method of travel to work in 2011?

c Which local authority had the highest proportion of the working population driving to work?

d Which region showed the largest decrease in the proportion of people driving to work between 2001 and 2011?

e Which categories might you include in 'public transport'?

f Which local authority had the highest proportion of people commuting by public transport in 2011?

2 'All usual residents' of County Durham in 2011 has a value of $513\,242$.

a What do you think the term 'all usual residents' means?

b Find the total number of people from County Durham included in the 2011 travel to work survey.

c Explain why your answer to part **b** is not $513\,242$. By exploring the age structure data, can you suggest which age groups are included in the travel to work survey?

3 **a** Suggest some methods of travel which might be included in 'Other method of travel to work'.

b Consider Birmingham in 2011.

i Does the sum of all the methods of travel equal the number of people in employment?

ii What does this mean for people who have more than one method of travel to work?

4 **a** Estimate the number of people that can fit on a typical bus.

b Use the figures from the large data set to estimate the number of buses required each morning in Blackpool in 2011.

c Give two reasons why your answer is likely to be an underestimate.

5 The 'car occupancy rate' is defined as the sum of the number of car drivers and passengers, divided by the number of drivers.

a Calculate the car occupancy rate in 2001.

b Calculate the car occupancy rate in 2011.

c What do these two numbers tells you about the changing use of cars between these two

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periods?

6 **a** Estimate the number of people driving to work in Sheffield in a car with no passengers in 2011.

b Explain why your answer is only an estimate.

7 Cornwall has no underground, trams or metro services, but some people in Cornwall commute to work using these methods. Explain why this might be the case.

8 **a** Find the percentage increase in the number of the people working from home between 2001 and 2011.

b Find the percentage increase in the proportion of the working population who work from home.

c The *2011 Census Analysis – Method of Travel to Work in England and Wales Report* (Gower) states that: "between 2001 and 2011 the proportion of the working population that worked at or from home increased by 1.1 percentage points".

Explain how they found this figure.

(This document is available online from the Office of National Statistics: www.ons.gov.uk/ons/dcp171766_299766.pdf)

d Why might each of the comparisons in parts **a**, **b** and **c** be useful?

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A reminder of statistical diagrams

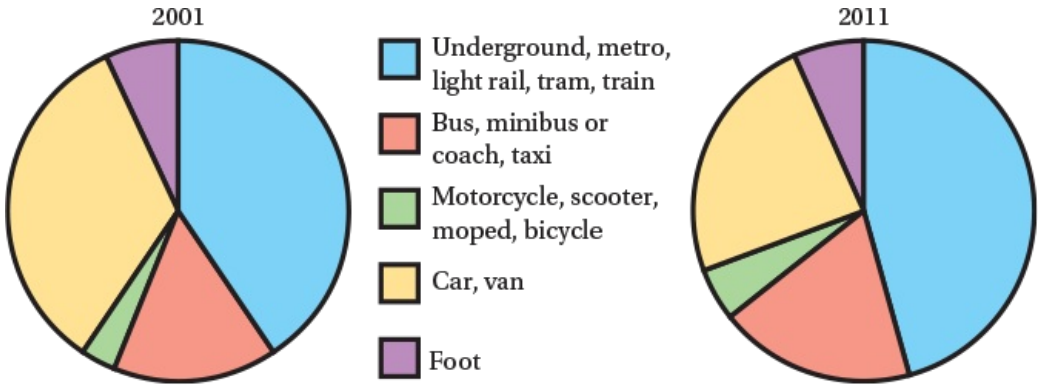
You need to be able to interpret a range of diagrams including:

- pie charts
- bar charts
- histograms
- cumulative frequency charts
- box-and-whisker plots.

You may find it helpful to explore how to create these (and other) diagrams using technology, and think about when each type of diagram is useful. The following questions might help you.

QUESTIONS

9 The pie charts show some of the methods of transport to work used in Lewisham.



a Summarise the main changes in travel to work illustrated in the pie chart.

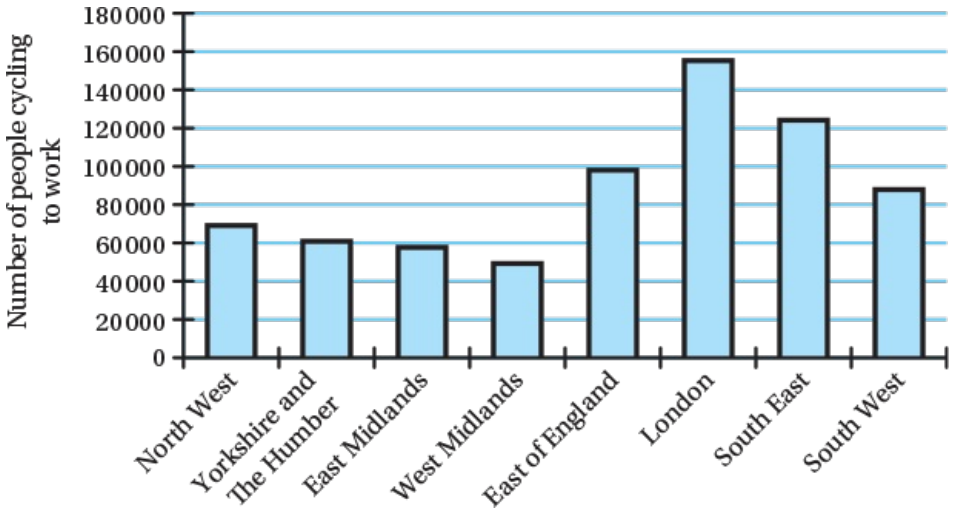
b What changes in method of transport are not illustrated by the pie chart?

10 Draw a pie chart to show the proportion of employed people who travel to work by car in Wales in 2011.

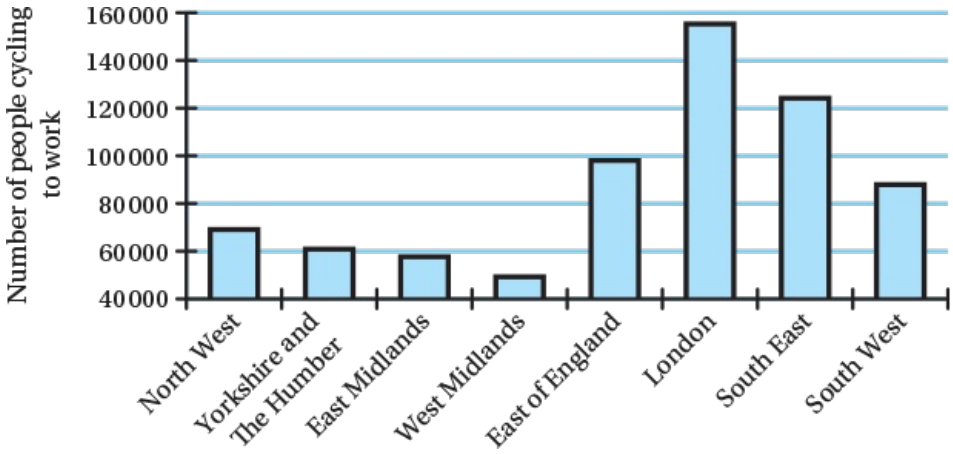
11 The following three bar charts all display information on the number of people who cycle to work in different areas of England in 2011.

Graph 1

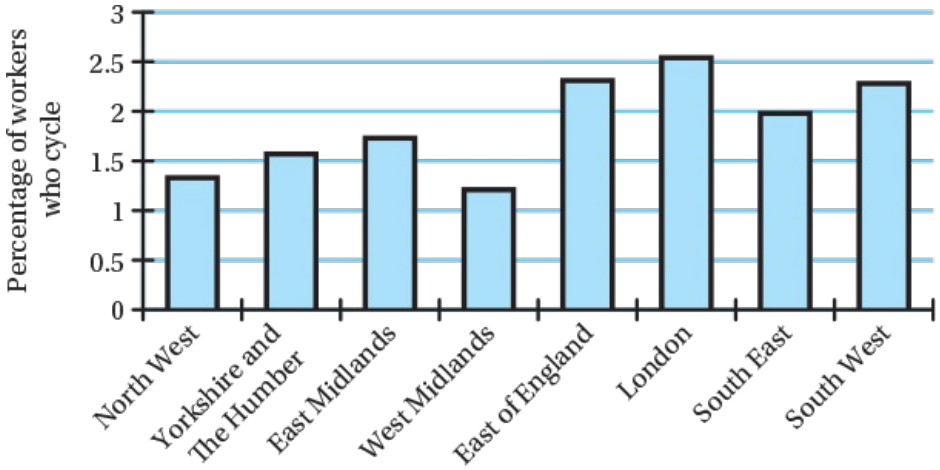
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Graph 2



Graph 3



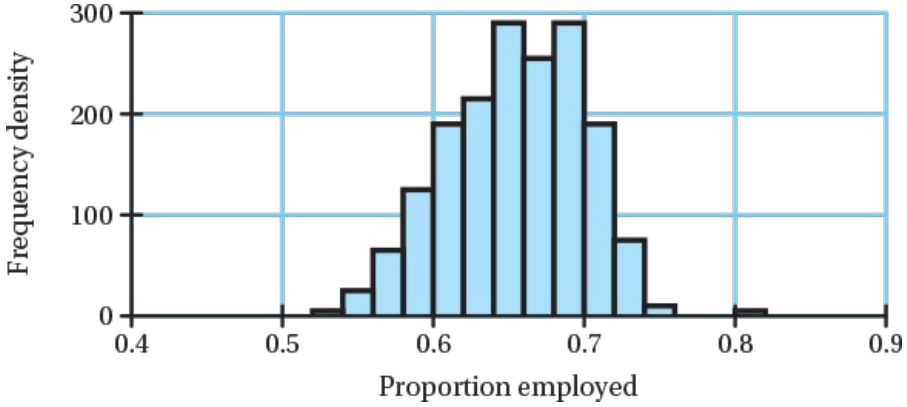
a Which area has the most cyclists? Which graph illustrates this best?

b Is cycling more popular in the North West or the East Midlands? Explain your answer.

c Why might the second graph be considered misleading?

12 The histogram shows the proportions of people of working age who are employed across the different local authorities.

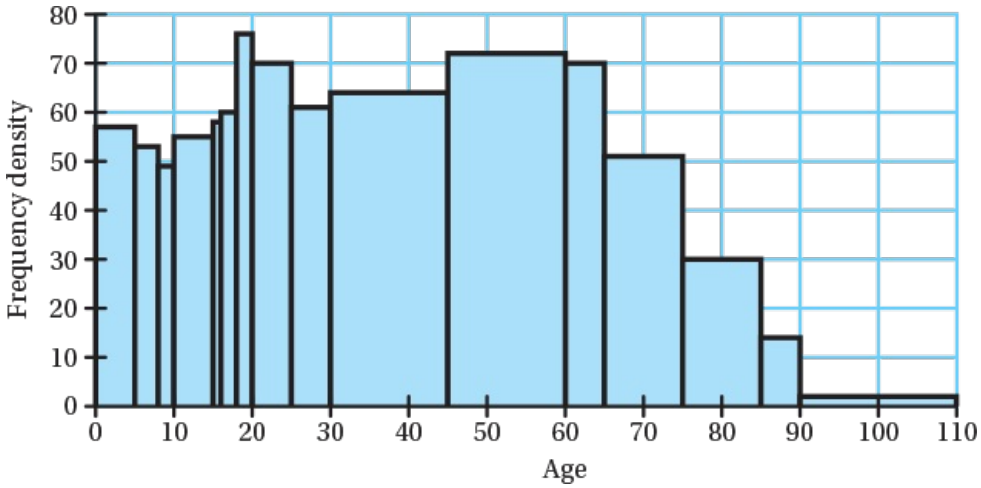
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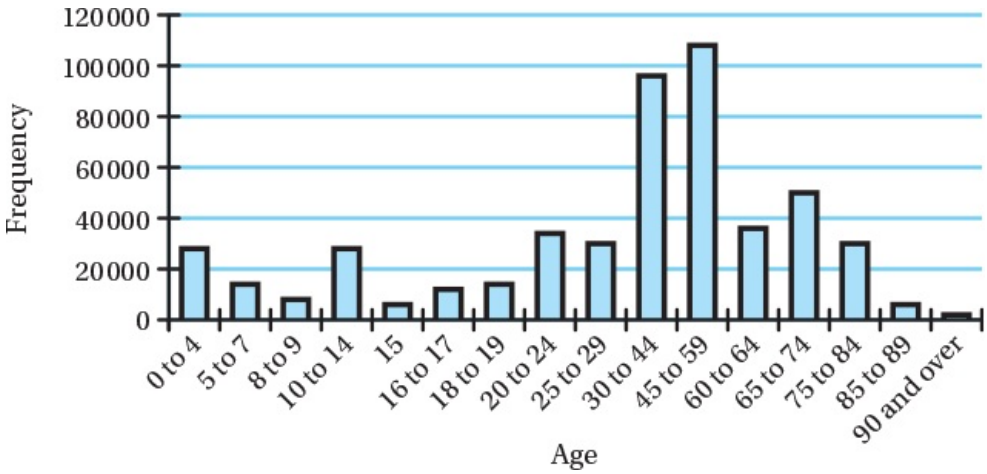
a Approximately how many of the local authorities have an employment rate between 0.6 and 0.64?

b Using technology, create a histogram of the same data with class widths of size 0.002. What are the advantages and disadvantages of this plot compared to the histogram with class widths of size 0.02?

13 This histogram shows the age distribution in County Durham in 2011.



It could also be represented using a bar chart:



a What is the advantage of the histogram compared to the bar chart?

b The first group is described as ages 0 to 4. Explain why this group is plotted over the region from 0 to 5.

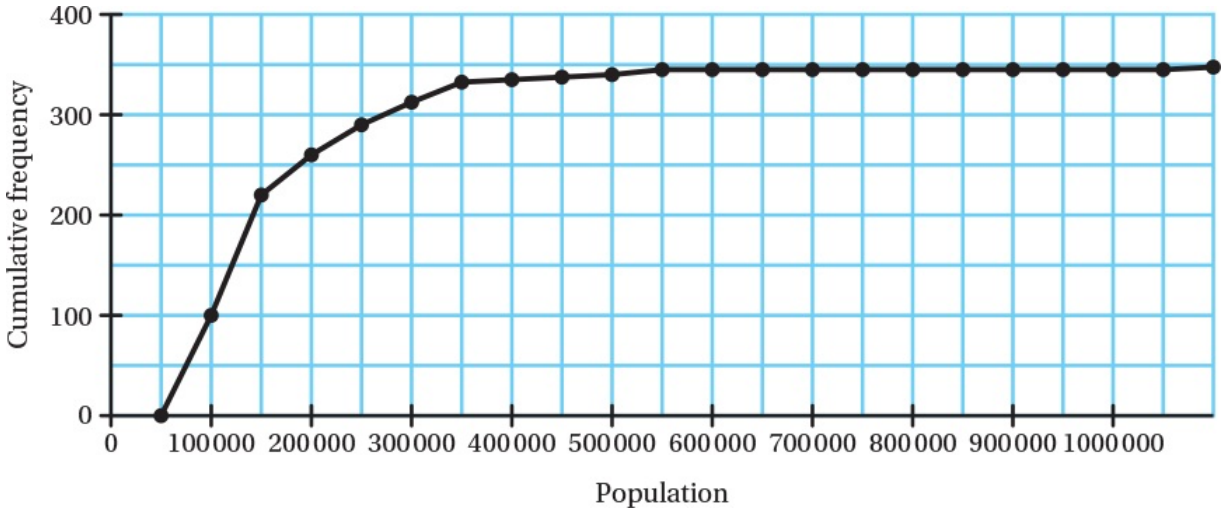
c The modal group is defined as the group with the highest frequency density.

i Why is this a better definition than the group with the highest frequency?

ii What is the modal age group for these data?

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14 The following cumulative frequency diagram shows the population of each local authority in 2011.

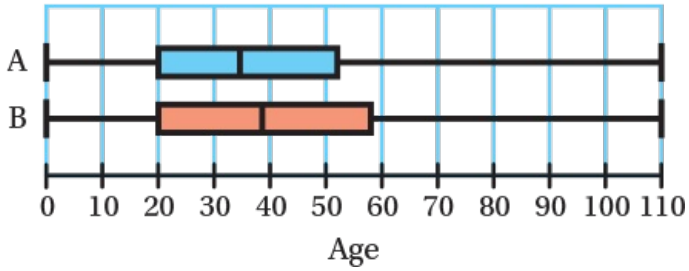


a Use the graph to estimate the number of local authorities in 2011.

b Use the graph to estimate the median size of a local authority.

c Given that the biggest local authority is Birmingham and the smallest local authority is the Isles of Scilly, sketch a box-and-whisker plot for these data.

15 The box-and-whisker plots show ages of populations in 2001 in two different areas, labelled A and B.

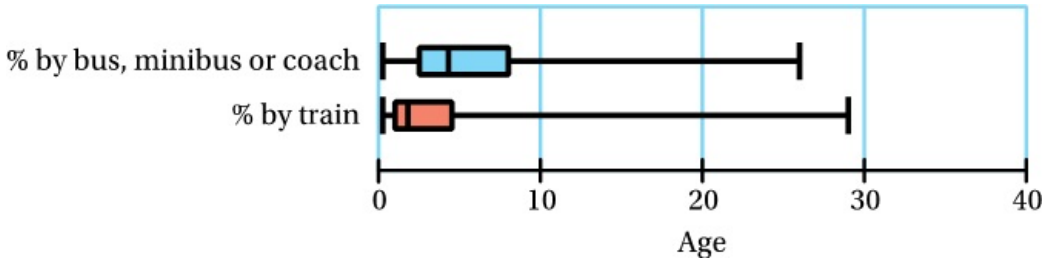


a Which region has the larger interquartile range of ages?

b Which region has the lower median age?

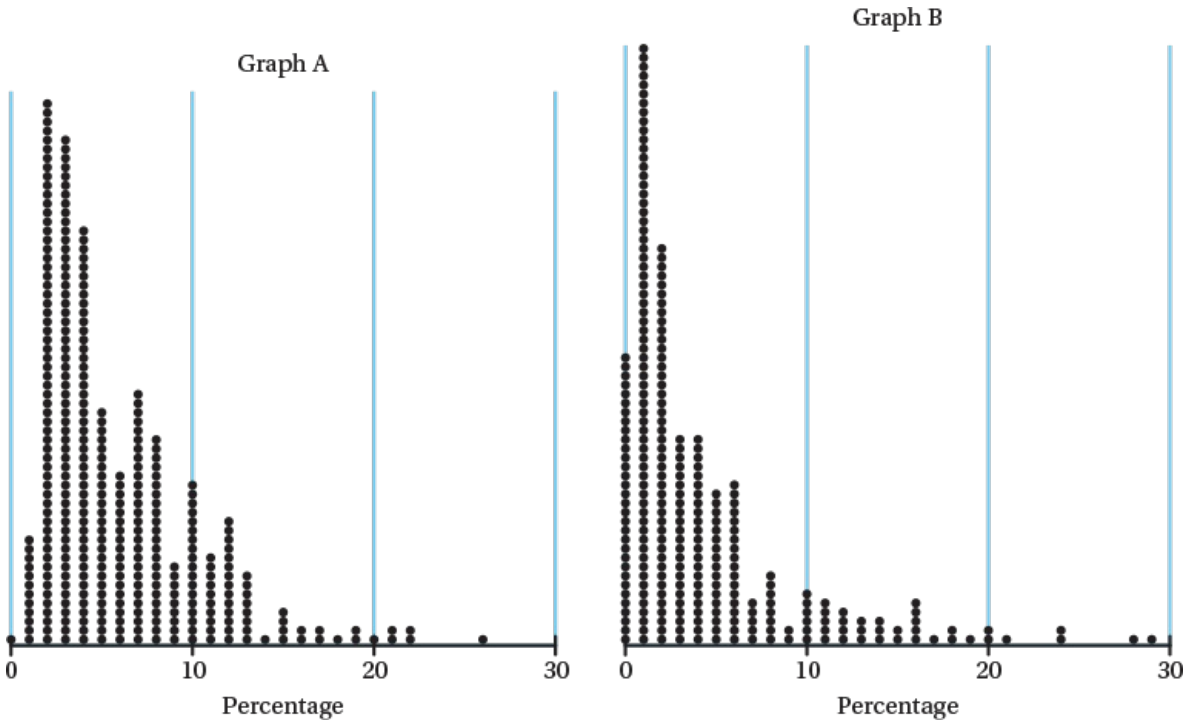
c One of the areas is London and the other is Wales. By looking at the original data in the large data set, decide which box-and-whisker plot corresponds to which area.

16 The box-and-whisker plots show the percentage of people in 2001 who travelled to work by bus, minibus or coach and the percentage who travelled by train.



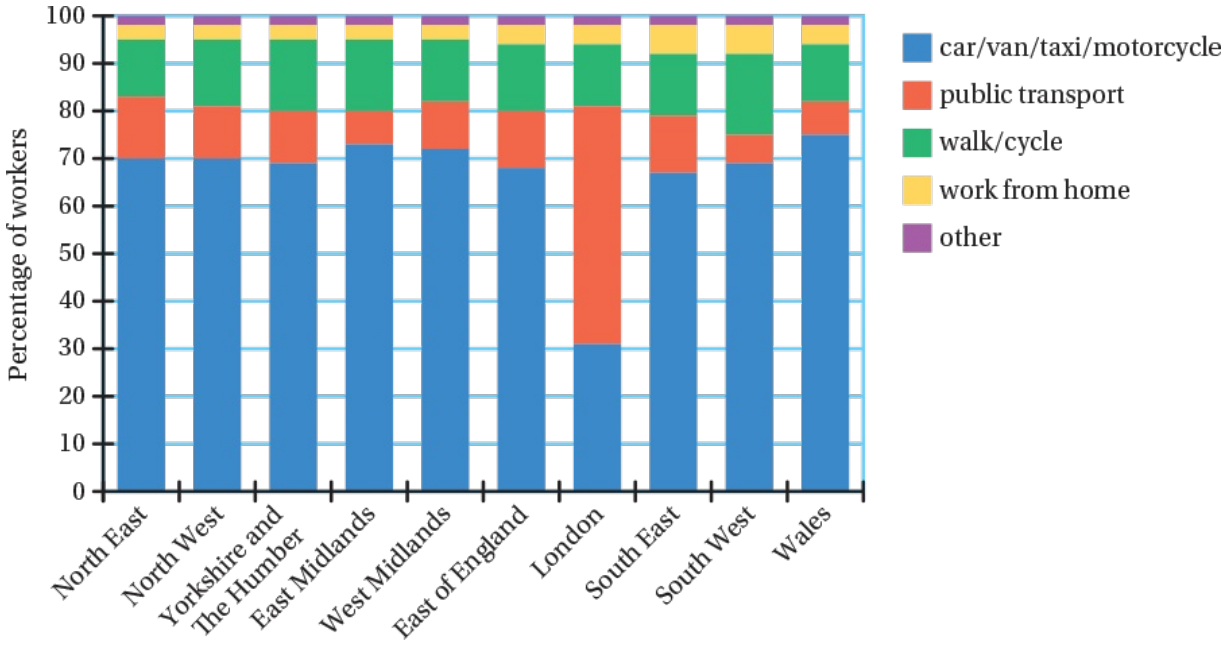
The data is also displayed as dot plots:

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Which of these graphs corresponds to travelling by train?

17 The *2011 Census Analysis – Method of Travel to Work in England and Wales Report* (Gower) shows the following bar chart:



a Give reasons why it is better to use:

i percentage of workers rather than number of workers

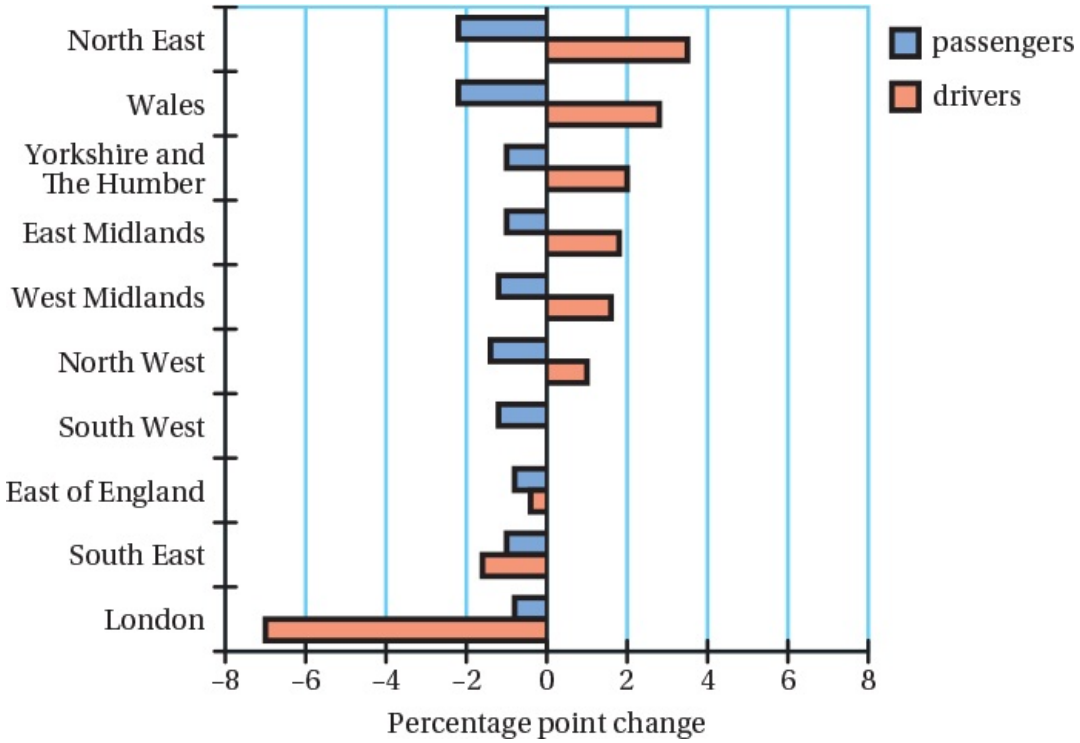
ii a stacked bar chart rather than a parallel bar chart.

b Summarise the main points highlighted by this chart.

c An environmental campaigner uses this chart to claim that London is less polluting than other areas of the country. State three further assumptions that the campaigner has made when making this claim.

18 The *2011 Census Analysis – Method of Travel to Work in England and Wales Report* (Gower) also shows this bar chart of percentage point change in drivers and passengers in cars between 2001 and 2011.

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a Why is a parallel bar chart better than a stacked bar chart to illustrate this information?

b A politician claims that the London congestion charge has caused a decrease in the number of Londoners travelling by car. Does this graph support this claim?

c Politicians are debating whether a campaign to increase car sharing on the way to work in London has been successful. Provide evidence to suggest that:

i the campaign failed

ii the campaign succeeded.

19 Suggest a reason to choose each of the following diagrams to illustrate data.

a pie chart

b bar chart

c cumulative frequency diagram

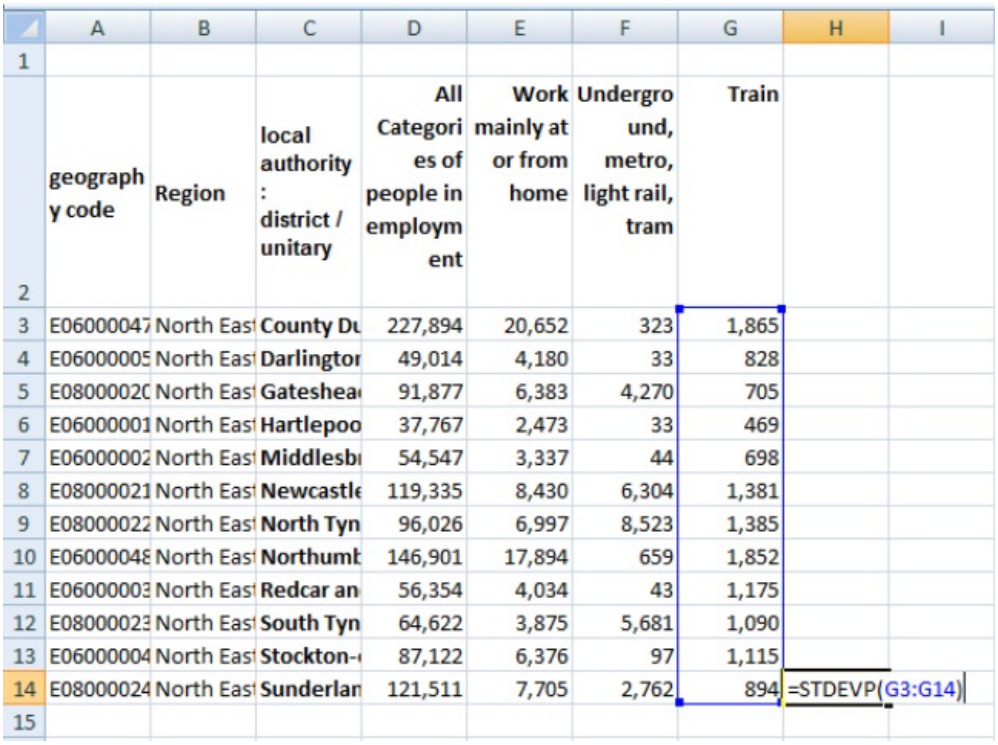
d histogram

e box-and-whisker plot

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Standard deviation, interquartile range and other statistical calculations

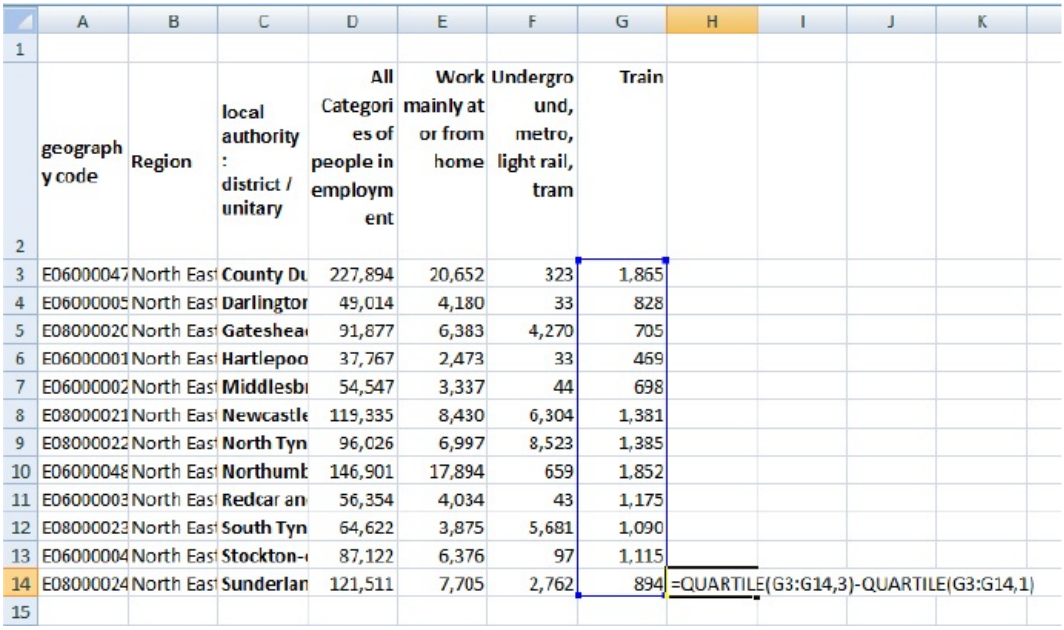
Spreadsheets can be used to find the standard deviation of data. A common syntax might look like this:



	A	B	C	D	E	F	G	H	I
1									
2	geography code	Region	local authority : district / unitary	All Categories of people in employment	Work mainly at or from home	Underground, metro, light rail, tram	Train		
3	E06000047	North East	County Du	227,894	20,652	323	1,865		
4	E06000005	North East	Darlington	49,014	4,180	33	828		
5	E0800002C	North East	Gateshea	91,877	6,383	4,270	705		
6	E06000001	North East	Hartlepoo	37,767	2,473	33	469		
7	E06000002	North East	Middlesbr	54,547	3,337	44	698		
8	E08000021	North East	Newcastle	119,335	8,430	6,304	1,381		
9	E08000022	North East	North Tyn	96,026	6,997	8,523	1,385		
10	E06000048	North East	Northumb	146,901	17,894	659	1,852		
11	E06000003	North East	Redcar an	56,354	4,034	43	1,175		
12	E08000023	North East	South Tyn	64,622	3,875	5,681	1,090		
13	E06000004	North East	Stockton-	87,122	6,376	97	1,115		
14	E08000024	North East	Sunderlan	121,511	7,705	2,762	894	=STDEVP(G3:G14)	
15									

Be careful! The command shown in Excel gives the standard deviation of the values presented. There is also a command STDEV which actually gives an estimate of the population's standard deviation based on the data presented being a sample. This idea comes up a lot in statistics and you will meet this more if you study the statistics option of Further Mathematics.

You can also use spreadsheet commands to find the range and the interquartile range. A common syntax might look like this:



	A	B	C	D	E	F	G	H	I	J	K
1											
2	geography code	Region	local authority : district / unitary	All Categories of people in employment	Work mainly at or from home	Underground, metro, light rail, tram	Train				
3	E06000047	North East	County Du	227,894	20,652	323	1,865				
4	E06000005	North East	Darlington	49,014	4,180	33	828				
5	E0800002C	North East	Gateshea	91,877	6,383	4,270	705				
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8	E08000021	North East	Newcastle	119,335	8,430	6,304	1,381				
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11	E06000003	North East	Redcar an	56,354	4,034	43	1,175				
12	E08000023	North East	South Tyn	64,622	3,875	5,681	1,090				
13	E06000004	North East	Stockton-	87,122	6,376	97	1,115				
14	E08000024	North East	Sunderlan	121,511	7,705	2,762	894	=QUARTILE(G3:G14,3)-QUARTILE(G3:G14,1)			
15											

QUESTIONS

20 a i Find the standard deviation of the number of people who take the train to work in the North East in 2011.

ii Find the standard deviation of the number of people who take the train to work in the North West in 2011.

b Hence compare the distribution of the number of people who take the train in the two areas.

21 a i Find the interquartile range for the number of people who travel to work on foot in the West Midlands in 2011.

ii Find the interquartile range for number of people who travel to work on foot in the East Midlands in 2011.

b Hence compare the distribution of the two populations.

22 a i Find the interquartile range for the number not in employment in the North East in 2011.

ii Find the standard deviation of the number not in employment in the South West in 2011.

b Hence compare the distribution of the two populations.

23 a i Find the standard deviation for the number of people who travel to work by driving a car or van in the entire country in 2011.

ii Find the standard deviation for the number of people who travel to work by motorcycle, moped or scooter in the entire country in 2011.

b Asher says:

> 'Since the standard deviation of the number of people who travel to work by driving a car or van is larger than the standard deviation of the number of people who travel to work by motorcycle, moped or scooter, there was greater variation in the number of people who travel to work by driving a car or van in this period.'

Do you agree?

c Elsa suggest that a better measure to compare the variation is to use:

$$\frac{\text{standard deviation}}{\text{mean}}$$

What are the advantages and disadvantages of this measure?

24 Daria and Leonie want to calculate the average percentage of the employable population not in employment in 2011.

a Daria calculates the percentage not in employment in each local authority and finds the mean across all local authorities.

i Show that the percentage of people not in employment in County Durham in 2011 is approximately 40.6%.

ii What answer does she get for the average of people not in employment across England and Wales?

b Leonie finds the total of all the people in employment and not in employment across England and Wales and uses that figure to calculate the average percentage not in employment. What answer does she get?

c Explain why your answers to parts **a ii** and **b** are different.

d Suggest why your answers to part **a** could differ from publicly reported statistics into the percentage of unemployed people.

25 a i Use an appropriate technological method to estimate the median age in County Durham in 2011. (Do not use the quoted value!)

ii Stating an appropriate assumption, estimate the mean age in County Durham in 2011. (Do

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not use the quoted value!)

iii Explain why these are slightly different from the values quoted.

b Estimate the standard deviation and interquartile range for the age in County Durham in 2011.

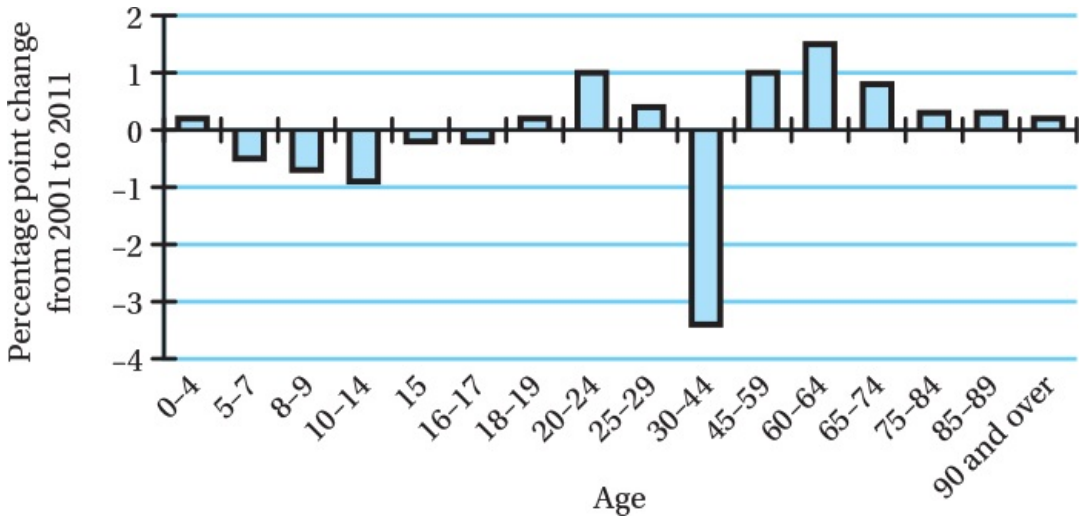
c In 2001 the local authority equivalent to County Durham is split up into seven local authorities. Use the information sheet on the large data set to find the names of these seven local authorities.

d Hence find the mean, median, standard deviation and interquartile range of the area equivalent to County Durham in 2001.

e Compare the age distributions in County Durham from 2001 to 2011.

f Has the change in County Durham reflected the change across the country?

g A geographer thinks that there may be some economic migration involved in the changing demographics of County Durham. He looks at the changes in the population in each age group and plots the following bar chart.



Does this provide evidence for the geographer's claim?

Outliers and cleaning data

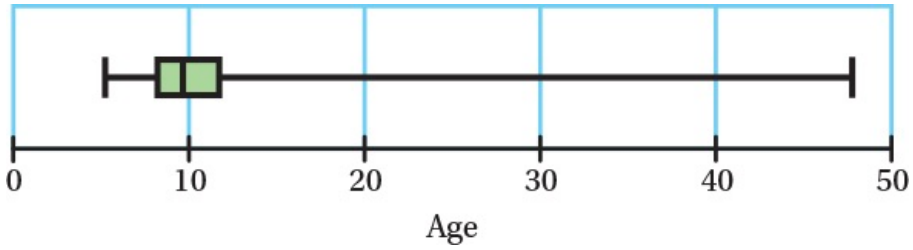
There are two standard definitions for deciding if something is an outlier.

- An outlier is any number more than 1.5 interquartile ranges away from the nearest quartile.
- An outlier is anything more than two standard deviations from the mean.

Once an outlier has been identified you must investigate and use your judgement to decide if it is an error (i.e. a clear mistake – such as putting a decimal point in the wrong place) or just an anomaly. If it is an anomaly then you can choose whether or not to include it, depending upon exactly what question you are trying to answer.

QUESTIONS

26 The following diagram shows the percentage of workers who walk to work in 2001.



a Using the first definition for an outlier, given above, show that there must be at least one outlier in this data.

b Is the largest outlier likely to be an error? Explain your answer.

c Using the original data set calculate the standard deviation and interquartile range of the percentage of workers who walk to work in 2001.

d Remove the outliers and recalculate the standard deviation and interquartile range.

e Use your calculations to explain whether it is better to calculate the standard deviation or interquartile range of data with outliers.

27 The following is an extract from the methodology used in the travel to work survey 2011:

> *The method of travel used for the longest part, by distance, of the usual journey to work. This topic is only applicable to people who were in employment in the week before the census.*
>
> *This table prioritises workplace address information over method of travel to identify home-workers, to allow a direct comparison with data from the 2001 Census.*
>
> *For example, a person who has indicated their place of work as their home address and said that they travel to work by driving a car or van (eg visiting clients) appears in the category 'Work mainly at or from home', as was the case in 2001.*
>
> (*Method of travel to work (2001 specification) by NS-SeC by sex*: www.nomisweb.co.uk/census/2011/dc7604ewla)

This is a method used to clean the data when someone has made a mistake in filling out the survey. Is this always appropriate?

28 a Explain why the mean **number** of people taking the underground, metro, light rail or tram across each local authority is not a useful statistic.

b Theo is trying to calculate the average of the percentages of employed people who take the underground, metro, light rail or tram in 2001. He uses a spreadsheet to find the percentage in each local authority and then finds the mean of all of these figures.

i What problem occurs if you try to follow Theo's procedure?

ii By suitably editing the data, find an appropriate value for the average of the percentages of employed people who take the underground, metro, light rail or tram in 2001.

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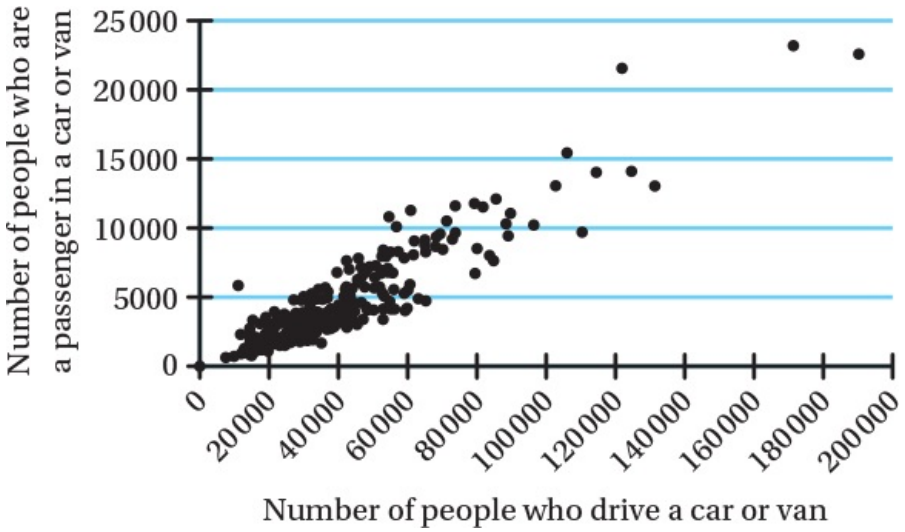
Scatter diagrams and correlation

You can use a spreadsheet to find the correlation coefficient between two variables. One common syntax is shown here:

	A	B	C	D	E	F	G	H	I
1									
2	geography code	Region	local authority : district / unitary	All Categories of people in employment	Work mainly at or from home	Underground, metro, light rail, tram	Train		
3	E06000047	North East	County Du	227,894	20,652	323	1,865	=correl(F3:F14,G3:G14)	
4	E06000005	North East	Darlington	49,014	4,180	33	828		
5	E08000020	North East	Gateshea	91,877	6,383	4,270	705		
6	E06000001	North East	Hartlepoo	37,767	2,473	33	469		
7	E06000002	North East	Middlesbr	54,547	3,337	44	698		

QUESTIONS

29 The 2001 Census gathered data on how people commuted to their workplace. This scatter graph shows, for each local authority surveyed, the number of people who drove a car or van to work against the number of people who said they were a passenger in a car or van.



a Which of the following is the correlation coefficient for these data?

A 1

B 0.92

C 0.16

D -0.68

b How would you describe the relationship between these two variables?

c There are three points that might be described as outliers. By looking at the original data identify the local authorities associated with these points. Should these results be discarded?

d Would the correlation coefficient be higher or lower with the three outliers discarded? Recalculate the correlation coefficient with the three outlying points discarded.

e Conduct a test at the 1% significance level to see if the underlying correlation coefficient is positive.

f Suggest a third factor that could explain the relationship between these two variables.

30 a Use technology to find the correlation coefficient between the percentage of workers who walk to work and the percentage of people who cycle to work in London in 2001.

b Test the above correlation coefficient at the 5% significance level to decide if there is underlying correlation.

c Use technology to plot a scatter diagram showing the relationship between the percentage of workers who walk to work and the percentage of workers who cycle to work in London in 2001.

d Identify the outlier from the scatter graph. Is it reasonable to remove this point?

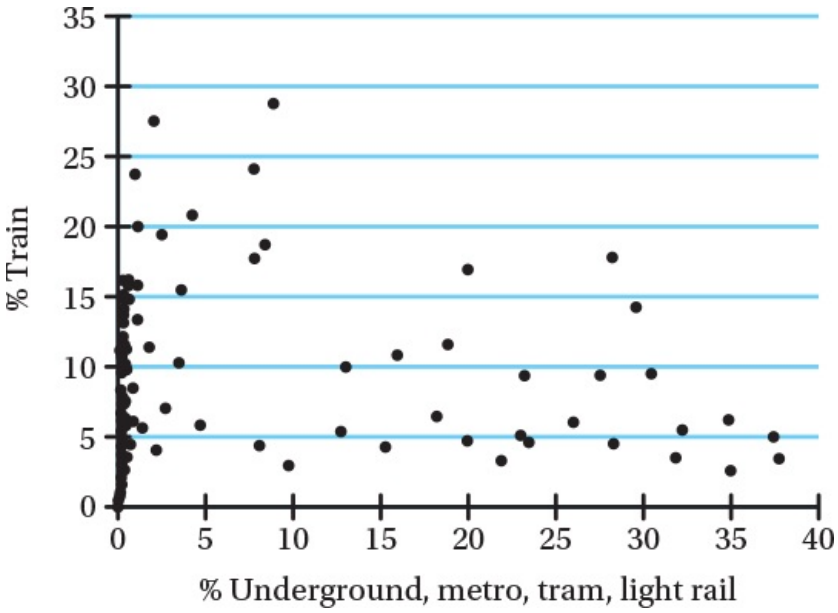
e Repeat parts **a** and **b** without the outlying point.

f Use technology to find the correlation coefficient between the percentage of workers who walk to work and the percentage who drive to work in a car or van in London in 2001. (Do not remove any outliers.)

g Test the above correlation coefficient at 5% significance to decide if there is underlying negative correlation.

h Use technology to plot a scatter diagram showing the relationship between the percentage of workers who walk to work and the percentage who drive to work in a car or van in London in 2001. Comment on your answer to part **f** in this context.

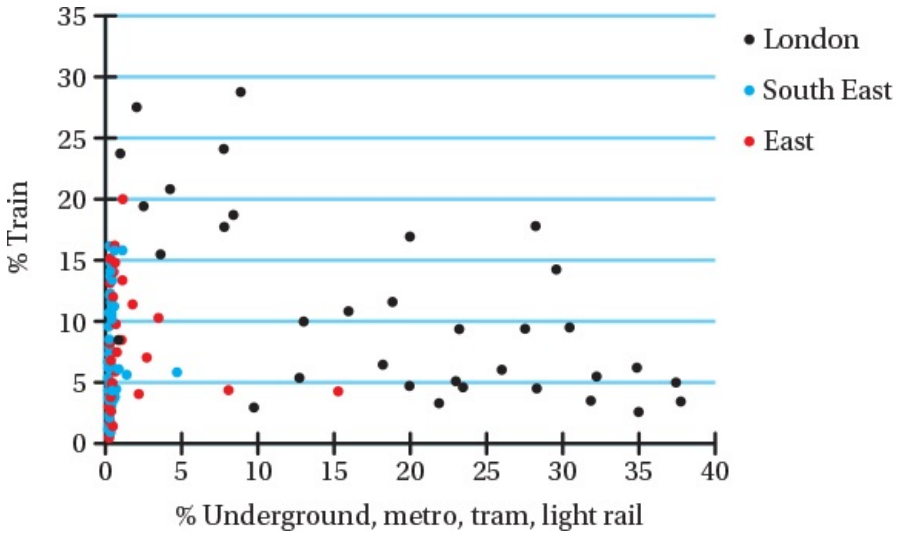
31 The following scatter graph shows the percentage of people who go to work by train against the percentage travelling by underground, metro, tram or light rail in the East of England, the South East and London.



a Describe the pattern observed.

b The graph is then changed to highlight which region each point comes from. Does this change your answer to part **a**?

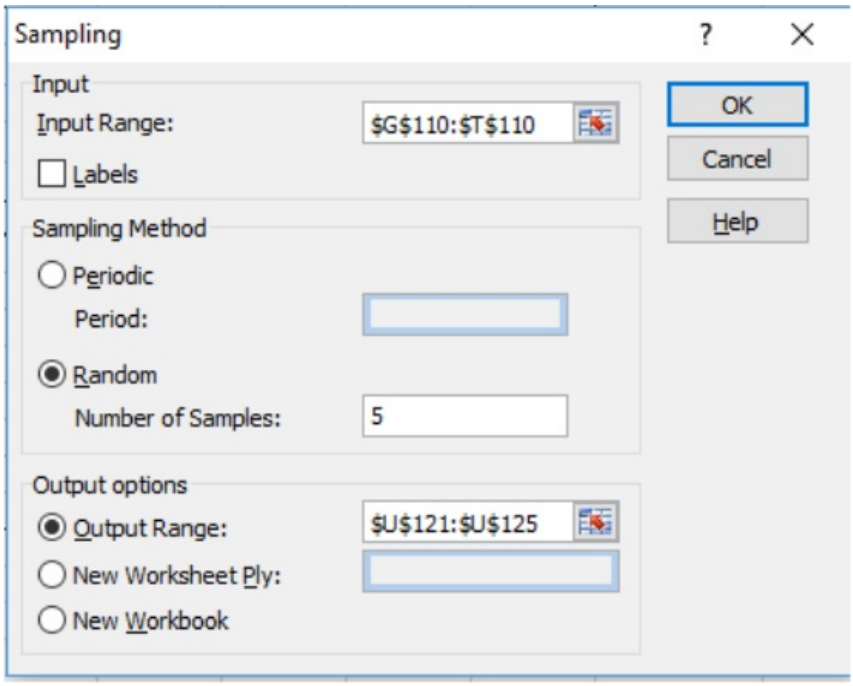
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35
30
25
20
15
10
5
0
% Train
0
5
10
15
20
25
30
35
40
% Underground, metro, tram, light rail
London
South East
East

Sampling and hypothesis testing

Whenever we are doing real statistics we have to take into account the fact that our data is only a sample of the whole population we are interested in. We need to understand the sampling method to understand what population the sample represents. To make any decisions about the population we also need to take into account the fact that we might just have an unusual sample. The method for doing this is hypothesis testing.

Many spreadsheets can take random samples from data. For example, you can use the data analysis ToolPak in Excel:



QUESTIONS

32 This data set comes from the 2001 and 2011 UK Censuses. These are sent out via post to every household in the country and answers refer to one week in the year. In 2011, 94% of households responded.

a Explain why this census is not a simple random sample of the population.

b The fact that the survey asked about only one week is also a sample. What is the population that it is drawn from? What sampling method has been chosen?

c Explain why the results of the census should not simply be multiplied by $100/94$ to estimate results for the entire population.

d To try to correct for people not returning their census forms a more detailed survey was undertaken in 350 areas. These areas were chosen at random, although it was ensured that the balance of regions and economic prosperity matched the country as a whole. Name the type of sampling method used.

33 It is thought that 50% of local authorities in 2011 had a total population of less than $126\,000$.

a Use a two-tailed binomial test at 10% significance to determine if this is the case using data from:

i the South West

ii all local authorities.

b Select 20 random samples of size 5 from the data. In how many of these random samples is there sufficient evidence to reject the hypothesis that 50% of local authorities in 2011 had a total population of less than $126\,000$?

34 In 2011 the mean age across local authorities is 40 years with a standard deviation of 2.91 years. It is thought that the average age is increasing.

a The average age in a mini-census of 100 local authorities is 40.8. What is the probability of this value or greater being observed if the statistics from 2011 are unchanged? Is this sufficient evidence at the 5% significance level to say that the average age is increasing?

b Create 20 random samples of size 5 from the data for local authorities in 2011. How many of these samples would provide sufficient evidence of an increase in the mean age from 40 using a 10% significance level?

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Statistical problem solving

In real life we can use data to answer questions. However, we rarely have the perfect data to answer our questions so we have to make inferences from the data we have. For each question below, suggest a statistical method that could be used to answer or illustrate it, and describe the assumptions you are making.

QUESTIONS

35 Are people in England and Wales getting healthier?

36 Is the increase in popularity of cycling reducing the use of cars purchased?

37 Are the transport preferences of Londoners very different to those of people in the rest of the country?

38 Do people drive more in 2011 than 2001?

39 Does prosperity affect the mode of transport people use?

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Working with the large data set

1 **a** Cambridge (29.0%).

b Driving by car or van.

c South Staffordshire (71.6%).

d London – about 7 percentage point decrease.

e 'Underground, metro, light rail, tram', 'bus, minibus or coach' and 'trains' should definitely be included. It is perhaps subjective as to whether taxis might be considered 'public transport'. (The official report did not do so.)

f Using the definition in part **e**, Newham (63.3%).

2 **a** The people who regularly live in County Durham – for example, if someone was just staying there with relatives during the census they would not count.

b $383\,796$

c Only working age people are included. It seems that this is people aged from 16 to 74.

3 **a** For example, helicopter, lorry, horse.

b **i** The sum does equal all people in employment. This means that people with more than one method of travel are only counted once.

ii They might have self-classified as 'Other method of travel' but the actual instruction on the survey was to put the method used for the longest part of the journey to work.

4 **a** About 50.

b $5174/50 \approx 100$

c Not every bus will be full to capacity. There may be non-workers using the bus.

5 **a** 1.11

b 1.09

c This might indicate that there is a decrease in the amount of 'car sharing' on the way to work, although the numbers are very similar so it might be down to normal statistical variation.

6 **a** An estimate might be to subtract the number of passengers from the number of drivers, getting $100\,297$.

b There are several reasons this is an estimate. It assumes that each car has at most one passenger. Giving 6 significant figures is hard to justify as on any given day there will be variation. Not everybody driving to work in Sheffield will live in Sheffield, so the figure might be larger owing to people commuting from nearby towns.

7 A resident of Cornwall might work in London during the week, possibly from a second home.

8 **a** 25.5% (from $2\,170\,547$ to $2\,724\,010$)

b 11.8% (from 0.092 to 0.103)

c A direct subtraction of 9.2% from 10.3%.

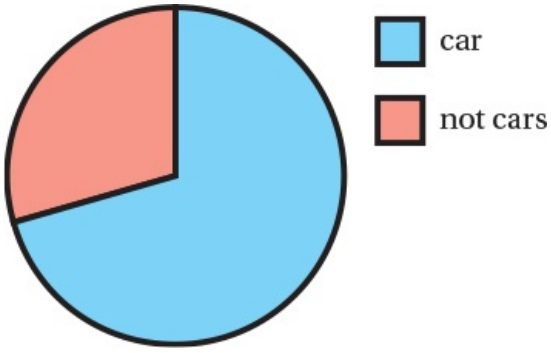
d A home office supplier might be interested in the actual number of people who might buy their products, so use the value in **a**. People interested in changing patterns of behaviour, such as economists, could use either **b** or **c**. The answer to part **b** emphasises the change relative to the starting proportion while part **c** emphasises the change relative to the whole population. Both are valid.

9 **a** The biggest change is that the proportion of car usage has decreased from 2001 to 2011. More people travel to work by bus, minibus or coach, taxi.

b The increase in population from 2001 to 2011. It also does not show the 'other' category, although this would be negligible.

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10

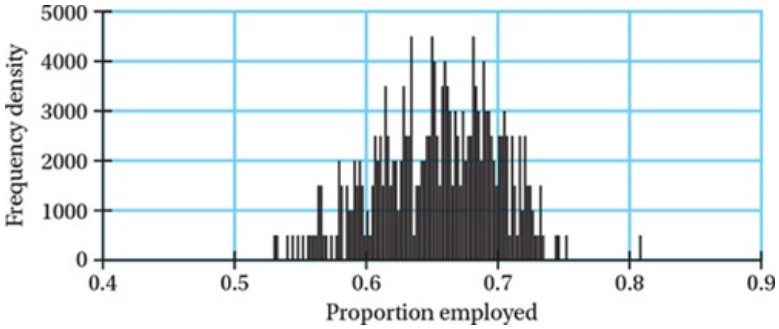


11 a London has the greatest number of people who cycle to work. Best shown by graph 1.

b Although more people cycle to work in the North West than in the East Midlands, a larger percentage cycle to work in the East Midlands.

c The vertical axis starting at 40000 distorts the impression given of London compared to the other areas.

12 a About 80.



b There is perhaps too much detail here, which makes the overall picture harder to see. However, if fine detail is needed this might be better – particularly at the edges.

13 a A bar chart does not take into account the different widths of the groups.

b Ages are rounded down so someone would be described as being 4 up until the day that they are 5.

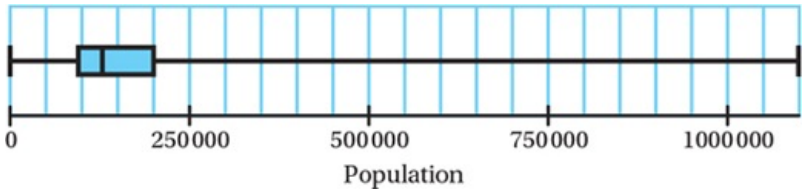
c **i** The total frequency depends on the arbitrary decision about the width of each group.

ii $18 - 20$

14 a About 350

b About 130000

c



15 a B

b A

c A = London, B = Wales

16 B

17 a **i** The number of workers would reflect the population of the areas more than the patterns of travel choices.

ii A stacked bar chart emphasises that the total proportion is constant.

b Most areas follow a similar pattern with the majority of people travelling by

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car/van/taxi/motorcycle. London shows a significantly different pattern with many more people using public transport.

c There are lots of possibilities – that the population of London is the same as the other areas; that public transport is less polluting per person than cars; that the average number of miles travelled to work is the same in each region; that the types of car used in each region are equally polluting; that transport is the main source of pollution; that commuting to work is the main type of journey (or is proportional to the total number of journeys) – many more!

18 a It allows the patterns in passengers and drivers to be compared to each other and amongst regions more easily.

b This graph does provide some evidence that supports the claim in so far as the change in the pattern of driving to work is very different in London compared to other areas. However, you might have expected the number of passengers in cars to increase under the congestion charge scheme. There may be other factors (such as improvements in public transport, rising costs of fuel) which caused the decrease in driver numbers, so it is hard to determine causality from this graph.

c **i** The percentage of passengers has decreased.

ii The decrease in London has been less than in other areas.

19 a Shows proportions of each category.

b Quickly shows a value associated with each category.

c Mainly used to read off median/quartiles of grouped data.

d Displays the frequencies of grouped continuous data.

e Illustrates key stats (median, quartiles, range) and allows distributions to be quickly compared.

20 a **i** 423

ii 2520

b At first sight there seems to be a greater variation in the number of train journeys in the North West. However, you might think that this variation is actually due to the different sizes of the local authorities in the two regions, so maybe it would be better to compare standard deviations in the proportions.

21 a **i** 6096

ii 2251

b It seems that there is a greater variation in the West Midlands, although this might be due to the different sizes of local authorities.

22 a **i** $40\,637$

ii $29\,055$

b We can't compare IQR with standard deviation.

23 a **i** $26\,129$

ii 448

b It is a valid comparison, but it might simply be a reflection of the different numbers of people who tend to take these modes of transport rather than a greater underlying variability.

c This allows quantities with different units to be compared, but it can cause a problem if the mean is very small – for example it is perfectly possible for the mean to be zero, which would make this undefined.

24 a **i** Use the formula '$= \mathrm{Q2} / (\mathrm{D2} + \mathrm{Q2}) * 100$'.

ii 34.5%

b 35.5%

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c The answer to **a ii** does not take into account the different sizes of the local authorities.

d People who are not employed includes students and perhaps people who are not looking for employment (e.g. people staying at home to look after children or people who have retired), who are not normally included in unemployment figures.

25 a i 41.8 (You might perhaps have to plot a cumulative frequency diagram if your spreadsheet does not do this automatically.)

ii Assuming that the maximum age is 110 , the mean is 41.6.

iii The values quoted come from the original data. By using grouped data some information is lost.

b $\sigma = 23.3$, $\mathrm{IQR} = 38.1$.

c Chester-le-Street, Derwentside, Durham, Easington, Sedgefield, Teesdale and Wear Valley.

d Median: 39.6 , mean: 40.2 , $\sigma = 23.0$, $\mathrm{IQR} = 37.1$

e The average age has increased and it has become more spread out.

f That data for all of England and Wales: In 2001: median: 38.2 , mean: 39.3 , $\sigma = 23.1$, $\mathrm{IQR} = 36.8$. In 2011: median: 39.1 , mean: 40.0 , $\sigma = 23.3$, $\mathrm{IQR} = 37.3$

So County Durham is slightly older on average than the rest of the country, but the whole country has generally seen an increase in average age and slight increase in spread of age from 2001 to 2011. This reflects a generally ageing population with more older people.

g It does provide some evidence. It seems that the biggest change in County Durham is a decrease in the 30–44 age group. This might be because working age people have migrated to other areas of the country. You have to be slightly careful here because the 30–44 age group in 2011 includes some people who were in the 20–24 and 25–29 age groups in 2001, so direct comparisons are tricky, but this graph is suggestive of emigration from the area.

26 a The UQ is 12 and the IQR is about 4 so anything above 18 is an outlier. The maximum value is about 48 so this is an outlier.

b This is by some way the largest value in any region, so it is suspicious but it is not a clear error – it is a plausible value (i.e. not a misplaced negative sign or a decimal point). So we can treat it as an anomaly, but it is not necessarily an error. You may use your geographical knowledge to realise that ‘City of London’ is an atypical region.

c $\mathrm{IQR} = 3.5$, $\sigma = 3.8$

d $\mathrm{IQR} = 3.4$, $\sigma = 2.64$

e The standard deviation is far more affected by the presence of the outlier, so the IQR is a more stable measure when there are outliers.

27 Someone might have a business based at their house, but they travel to clients at different addresses.

28 a Each local authority is a different size.

b i The Isles of Scilly has a ‘$-$’ instead of a value, resulting in an error when the percentage is calculated.

ii 2.19%

29 a B

b Strong linear correlation.

c Leeds, Birmingham and County Durham. These are amongst the largest local authorities, so we can explain why these numbers should be large so they should not really be discarded.

d It would be lower, since the three points are at the extreme of the line. The new value is 0.899.

e There is significant evidence of correlation.

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Tip

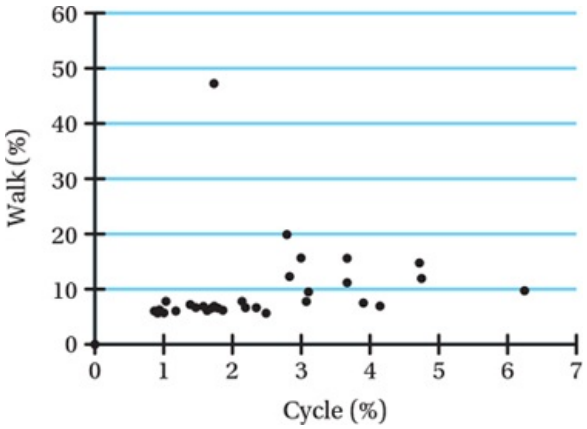
This hypothesis test assumes that the underlying data follows a bivariate normal distribution, which has not really been established here – in fact, it looks positively skewed.

f The size of the population in the local authority.

30 a 0.164

b No significant evidence.

c



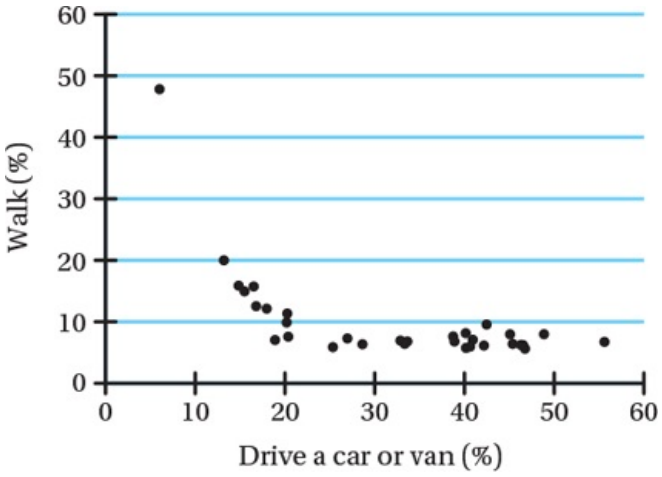
d City of London – this is a tiny, atypical borough of London, so it might be reasonable to remove it.

e New coefficient is 0.531 . There is significant evidence of correlation.

f correlation coefficient $= -0.621$

g Significant evidence of negative correlation.

h



There is a general negative trend, but it is not clear if it is linear. There might actually be two lines going on, or perhaps some sort of curve. Therefore looking for one linear correlation coefficient might not be appropriate.

31 a No real pattern.

b It seems that London is a very different area to the rest. It actually now looks more as if there are two distinct patterns within London (perhaps corresponding to inner and outer London). This data would have to be removed to get a better idea of the behaviour in the East of England and the South East.

32 a Not all households are equally likely to be included in the census, since those who decline are not represented.

b It is an opportunity sample of all weeks in the year.

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c Not all types of people are equally likely to have not responded.

d Stratified sampling.

33 a i Evidence that this is not the case.

ii Insufficient evidence that this is not the case.

b This is a powerful method called simulation. The hypothesis should be rejected in about 2 samples

34 a 0.00299 , which is sufficient evidence.

b About 2 of these samples should be extreme enough to say that the mean is above 40 (even though it really is 40!).

35 You might choose to look at the mean (or perhaps the median) age of all people in the country as a proxy variable for healthiness in 2001 and 2011 and compare them – but how big a difference would be big enough? You might look at the standard deviation to get a sense of whether the difference is significant. You could also compare a histogram or look at the percentage of people aged over 60. Of course, just because people are getting older doesn't mean that they are more healthy. You might instead want to look at the percentage who cycle or walk to work to infer something about attitudes towards fitness.

36 You might look at the correlation between percentage of cyclists and motorists in each local authority, but this does nothing to establish anything about changes. You might instead plot percentage change in cyclists from 2001 to 2011 against percentage change in motorists. However, this would not tell us about causality.

37 You might plot a stacked bar chart showing the percentages of different methods of travelling to work in different regions, but the problem is that travelling to work is different from general travel. You might also argue that the preference is different from their main method, which is down to availability and cost.

38 You could look at the total number of people who drive in these two years, but the increase might be due to increasing population. You could look at the percentage of people who drive to work, but that does not tell you about the distances they travel or the time they take.

39 You could plot the percentage in transport category against the percentage of the population in work on a scatter graph and look for a trend, maybe using a correlation coefficient.

However, the percentage of the population in work might be a poor proxy variable for prosperity as authorities with many students, babies or retired people might appear less prosperous.

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